Existence Theorems and Approximation Algorithms for Generalized Network Security Games

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Outline

1. Introduction
   - Motivation and examples
   - Model and definitions

2. Our results
   - Existence of pure NE
   - Approximating the social optimum
   - Simulation study

3. Concluding remarks
   - Future work
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Motivation

Computer network
Motivation

Human contact network
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Existence Thm and Approx Algo for Gen Netwk Security Games
Game-theoretic model

- Contact graph: $G(V,E)$.
- Strategies: install anti-virus software or not, $a_i \in \{0, 1\}$.
- Security cost/infection cost: $C_i, L_i$.
- Individual cost: $cost_i(\bar{a}) = a_i C_i + (1 - a_i) L_i p_i(\bar{a})$.
- Social cost: $\sum_i cost_i(\bar{a})$.
- Infection model: we assume infection is initiated at a node (picked with probability proportional to its weight $w_i$), and transmits over at most $d$ (which is a parameter) hops in the contact graph.
- Generalized Network Security Game: $GNS(d)$. 
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### List of results

<table>
<thead>
<tr>
<th></th>
<th>$d = 1$</th>
<th>$1 &lt; d &lt; \infty$</th>
<th>$d = \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>existence of pure NE</td>
<td>Yes</td>
<td>No/NP-complete</td>
<td>Yes</td>
</tr>
<tr>
<td>price of anarchy</td>
<td>$\Delta + 1$</td>
<td>$O(1/\alpha(G))$</td>
<td></td>
</tr>
<tr>
<td>approximating social</td>
<td>2</td>
<td>2d</td>
<td>$O(\log n)$</td>
</tr>
</tbody>
</table>

- $\Delta$ is the max degree in the contact graph.
- $\alpha(G)$ is the vertex expansion of the contact graph.
Aspnes et al 2006 introduced a basic model for \( d = \infty \) case that we have generalized here.
  - Show existence of pure NE in a uniform version.
  - Give an \( O(\log^{3/2} n) \)-approximation for social optimum.

Interdependent security games [Kearns-Ortiz 2004].
  - Similar to our model for special case of \( d = 1 \).
  - Crucial difference in assumption about initial infection.

\( n \)-intertwined games [Omic et al 2009].
  - Based on SIS model for worm spread.

Considerable work in SIR and SIS models in epidemiology.
Existence of pure NE when \( d = \infty \)

**Theorem**

*Every GNS(\( \infty \)) instance has a pure NE.*

- The existence proof is a potential function argument.
- Define Threshold of a node, \( t_i \): Bound on number of reachable nodes that would make the node want to secure itself.
- w.l.o.g., assume \( t_1 \geq t_2 \geq \cdots \geq t_m \).
- Define potential function: \( \hat{\Phi}(\vec{a}) = (\Phi_1(\vec{a}), \Phi_2(\vec{a}), \ldots, \Phi_m(\vec{a})) \)
  where \( \Phi_i(\vec{a}) \) is 0 if \( i \) is secure, \(-1\) if \( i \) is insecure and happy, and 1 otherwise.
Example of potential function

- $t_1 = 5$, $t_2 = 4$, $t_3 = 3$, $t_4 = 2$, $t_5 = 1$, $t_6 = 1$, $t_7 = 0$.
- 5 is secured.
- Potential function for this configuration is $(-1, -1, -1, 1, 0, -1, 1)$.
Case 1: unhappy insecure $\rightarrow$ happy secure. One component decreases by 1, while none of the other components increases.

Case 2: unhappy secure $\rightarrow$ happy insecure. All the happy insecure nodes with bigger thresholds are still happy. Happy insecure nodes with smaller thresholds may become unhappy. But the function still decreases lexicographically.
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Approximation algorithm for social optimum

**LP formulation**

- Let $P_{ij}^d$ denote the set of all simple paths from $i$ to $j$ of length at most $d$.
- $\forall v \in V$, $x_v = 1$ if $v$ is secure; $x_v = 0$ otherwise.
- $\forall i, j \in V$, $y_{ij} = 1$ is there is no $p \in P_{ij}^d$ consisting entirely of insecure nodes.

$$\min \sum_v C_v \cdot x_v + \sum_{j \in V} L_j \sum_{i \in V} w_i (1 - y_{ij})$$

s.t.

$$\sum_{v \in p} x_v \geq y_{ij} \quad p \in P_{ij}^d$$

$$x_v \in \{0, 1\} \quad \forall v \in V$$

$$y_{ij} \in \{0, 1\} \quad \forall i, j \in V$$
Objective function of LP

\[
\begin{align*}
\min & \quad \sum_v C_v \cdot x_v + \sum_{j \in V} L_j \sum_{i \in V} w_i (1 - y_{ij}) \\
\text{s.t.} & \quad \sum_{v \in p} x_v \geq y_{ij} \quad p \in P_{ij}^d \\
& \quad x_v \in \{0, 1\} \quad \forall v \in V \\
& \quad y_{ij} \in \{0, 1\} \quad \forall i, j \in V
\end{align*}
\]

- First part of the objective function corresponds to the cost of securing nodes.
- Second part corresponds to the infection cost. For node \(j\), its infection cost is \(L_j\) times the sum of the probabilities of all nodes that have a path to \(j\) of length at most \(d\) consisting entirely of insecure nodes.
Constraints of LP

\[ \begin{align*}
\text{min} \quad & \sum_v C_v \cdot x_v + \sum_{j \in V} L_j \sum_{i \in V} w_i (1 - y_{ij}) \\
\text{s.t.} \quad & \sum_{v \in p} x_v \geq y_{ij} \quad p \in P_{ij}^d \\
& x_v \in \{0, 1\} \quad \forall v \in V \\
& y_{ij} \in \{0, 1\} \quad \forall i, j \in V
\end{align*} \]

- Constraint says, in order to separate a pair of nodes \( i \) and \( j \), we need to secure at least one node in every path between these two.
Solving LP

- $d$ is a constant:
  - Number of paths of length at most $d$ is polynomial.
  - So LP is poly-size and can be solved in poly-time.

- $d$ is not a constant:
  - Number of paths superpolynomial; still LP solvable using ellipsoid method.
  - Can also solve an equivalent LP of polynomial size.
LP objective

\[
\begin{align*}
\min & \sum_v C_v \cdot x_v + \sum_{j \in V} L_j \sum_{i \in V} w_i (1 - y_{ij}) \\
\end{align*}
\]

- Let \((x, y)\) denote an optimal solution.
- Round each \(y_{ij}\) to nearest integer.
  - So values at least 1/2 are rounded up to 1 and less than 1/2 rounded down to 0.
- Scale up each \(x_{ij}\) by a factor of 2.
  - If scaled value exceeds 1, set it to 1.
- New solution \((x, y)\) is still feasible and new cost at most twice that before.
It remains to round the $x$-values.

Simple approach: Each $x_{ij}$ that is at least $1/d$ is rounded up to 1, other $x_{ij}$s rounded down to 0.

- Yields $2d$-approximation.
- Perhaps acceptable for small $d$.

For $d = \infty$:

- Need to select a set of nodes to secure such that all pairs of nodes $i, j$ with $y_{ij} = 1$ are separated.
- This is precisely a vertex multicut problem for which $x$-values give a fractional optimum.
- Use algorithm of Garg-Vazirani-Yannakakis to round the $x$-values and obtain an $O(\log n)$-approximation.
• Study the convergence time for best response strategies.
• Study the performance of our approximation algorithms.
• Simulate on 2 types of graphs.
  • Random geometric graphs: distributing $n^2$ nodes uniformly at random in an $n \times n$ square, and add an edge between a pair of nodes if their distance is no more than 1.
  • Preferential attachment graphs.
Convergence time

$d = 1$

$d = \infty$

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Performance of approx algorithms

Results for preferential attachment graphs.

\( d = 1 \)

\( d = \infty \)
Performance of approx algorithms

Results for random geometric graphs.

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Future work

- Incorporate unreliable anti-virus software/vaccination.
  - For a natural probabilistic model, we can show pure NE may not exist.

- Consider other virus/disease transmission models.
  - Again, pure NE may not exist in a probabilistic transmission model.

- In probabilistic models, approximating social optimum is challenging since even estimating the epidemic size given secure nodes is \#P-hard.

- Stackelberg equilibrium: Given a budget of $k$ secure nodes, find low-cost pure NE

- For $d = \infty$, bridge gap between upper ($O(\log n)$) and lower bound ($O(1)$) on approximation ratio achievable in poly-time.