1 Multicommodity flow

Demands multicommodity flow: Given graph $G = (V, E)$, edge capacity function $C : E \rightarrow \mathbb{Z}^+$. There are $k \geq 1$ commodities, each with its own source $s_i$, sink $t_i$, and demand $\text{dem}(i)$. The objective is to maximize $f$ such that we can send $f \cdot \text{dem}(i)$ units of commodity $i$ from $s_i$ to $t_i$ for each $i$ simultaneously, without violating the capacity constraint of any edge.

Sum-flow multicommodity flow: Given graph $G = (V, E)$, edge capacity function $C : E \rightarrow \mathbb{Z}^+$. There are $k \geq 1$ commodities, each with its own source $s_i$, sink $t_i$. The objective is to maximize the sum of the flow sent from $s_i$ to $t_i$, over all $i$, without violating the capacity constraint for any edge.

2 Two examples where Max-Flow is not equal to Min-Cut

It is well known that Max-Flow is equal to Min-Cut in Single Commodity Flow problem. But this is not true for Multicommodity Flow when the number of commodities is greater than 2.

First we give the definition of Min-Cut in multicommodity flow problem.

**Definition 1.** For any cut $(S, \bar{S})$ of the graph, let $C(S, \bar{S}) = \sum_{e \in (S, \bar{S})} C(e)$ which is the total capacities across this cut, and $D(S, \bar{S}) = \sum_{\{i | s_i \in S \land t_i \in \bar{S} \text{ or } s_i \in \bar{S} \land t_i \in S\}} \text{dem}(i)$ which is the total demand across this cut. Define the Min-Cut as $\eta = \min_{S \subseteq V} \frac{C(S, \bar{S})}{D(S, \bar{S})}$. We refer to $\frac{C(S, \bar{S})}{D(S, \bar{S})}$ as the ratio of cut $(S, \bar{S})$.

Let $f^*$ be the optimal value for demands multicommodity flow. It is clear that $f^* \leq \eta$. The first example (Figure 1, taken from Jon Kleinberg’s lecture notes) shows $f^*$ could be strictly smaller than $\eta$ in multicommodity flow problem.

In the graph, there are 4 flow pairs, each with a demand of 1, and the shortest path between each pair is 2 hops. So the total capacity consumed when we send $f^*$ flow for each commodity is $8f^*$. And there are only 6 edges in the graph. So we have $f^* \leq 3/4$.

The second example gives an even worse ratio between Max-Flow and Min-Cut, where $f^* \leq O\left(\frac{\eta}{\log n}\right)$. This example makes use of Uniform Multicommodity Flow and 3-regular expander graph.

**3-regular expanders:** 3-regular expander graph has the following properties:
Figure 1: Example of Max-Flow and Min-Cut in multicommodity flow

- degree of every vertex is equal to 3
- \( \exists c > 0 \) (\( c \) is a constant), \( \forall S \subseteq V \) if \( |S| \leq \frac{|V|}{2} \) then \( \delta(S) \geq c|S| \). Here \( \delta(S) \) is the number of edges that cross cut \( (S, \bar{S}) \).

Now construct the multicommodity flow problem in the following way. Given a 3-regular expander graph, set the cost of each edge to one, \( C(e) = 1 \). For each pair of vertices \( (u, v) \) set a source and sink pair \( (s_i, t_i) \). The demand of each \( (s_i, t_i) \) is equal to one, \( d_i = 1 \).

**Theorem 1.** \( f^* \leq O \left( \frac{\eta}{\log n} \right) \)

**Proof.** We first show that \( \eta = \Omega(1/n) \). Consider any cut \((S, \bar{S})\). Without loss of generality, we assume \( |S| \leq n/2 \). Owing to the expansion property, the number of edges crossing the cut is at least \( c|S| \). Therefore, the ratio for \((S, \bar{S})\) is at least \( c|S|/(|S| \cdot |\bar{S}|) \), which is at least \( c/n = \Omega(1/n) \).

For each vertex \( u \in V \), the number of vertices that are 1-hop away from \( u \) is 3 (this is a 3-regular expander graph), the number of vertices that are 2-hop away from \( u \) is at most 9, the number of vertices that are 3-hop away from \( u \) is at most 27 . . . . So there are at least \( 2n^2 / 3 \) vertices that are more than \( \lfloor \log_3 n \rfloor - 1 \) hops away from \( u \). And the number of pairs that are separated by more than \( \lfloor \log_3 n \rfloor - 1 \) hops is at least \( n \times \frac{2n^2}{3} = \frac{2n^3}{3} \). So the total capacity consumed by flows is at least \( 2n^2 \times \log_3 n \times f^* \). The total number of edges in this graph is \( \frac{3n^2}{2} \). From

\[
\frac{3n}{2} \geq \frac{2n^2}{3} \times \log_3 n \times f^*
\]

we have

\[
f^* \leq \frac{9}{4n \log_3 n} = O \left( \frac{\eta}{\log n} \right)
\]

\( \square \)
In other words, the Max-Flow for the Uniform Multicommodity Flow problem is at least a $O\left(\frac{\eta}{\log n}\right)$-factor smaller than the min-cut.

3 LP of Demands Multicommodity Flow

Let $P_i$ be the set of paths between pair $(s_i, t_i)$, $p^i_j$ be the $j^{th}$ path in $P_i$, $dem(i)$ be the demands of pair $(s_i, t_i)$, and $f^i_j$ be the amount of commodity $i$ sent on path $p^i_j$. Then we get the following LP for Demands Multicommodity Flow problem.

$$\max f$$
$$\text{s.t. } \sum_{p^i_j} f^i_j \geq f \cdot dem(i) \quad \forall i$$
$$\sum_{p^i_j: e \in p^i_j} f^i_j \leq c_e \quad \forall e$$
$$f^i_j \geq 0 \quad \forall i, j$$

And its dual is

$$\min \sum_e c_e d_e$$
$$\text{s.t. } \sum_{e \in p^i_j} d_e \geq l_i \quad \forall i, j$$
$$\sum_i l_i \cdot dem(i) \geq 1$$
$$d_e \geq 0 \quad \forall e$$

In the dual, $d_e$ can be viewed as the distance assigned to the edge. And $l_i$ is the length of the shortest path between $s_i$ and $t_i$, according to the distances given by $d_e$. A feasible solution to the dual yields a lower bound on $f^*$ (by weak duality). In fact, the proof of Theorem ?? can be seen as one based on weak duality. Take the 3-regular expander graph. We saw that $\eta \geq \frac{c}{n}$ where $c$ is a constant. We now show that $f^* \leq O\left(\frac{1}{n \log n}\right)$, using the dual LP defined above. Set $d_e = \frac{2}{n^2 \log n}$.

Then

$$\sum_i l_i \geq \frac{2}{3} n^2 \log n \cdot \frac{2}{n^2 \log n} \geq 1$$

because for each vertex, there are at least $\frac{2n}{3}$ vertices that are more than $\lfloor \log_3 n \rfloor - 1$ hops away from it, and the demand for each pair is 1, $dem(i) = 1$.

By weak duality, we thus have

$$f^* \leq \sum_e d_e = \frac{2}{n^2 \log n} \cdot \frac{3n}{2} = \frac{3}{n \log n} = O\left(\frac{1}{n \log n}\right)$$