Lecture Outline:

- Introduction to game theory

- Define Nash equilibrium

- Prove the existance of Nash equilibrium

1 Introduction to game theory

In this section, we will first define what are the games studied in game theory field. And then give two examples to illustrate this concept.

1.1 What is game

The games we are going to study in this course have the following crucial elements:

- Players: there is a finite set of palyers (who may be people, groups of people, or more abstract entities).

- Actions: at different points in the game, each player has a range of actions or choices. This set of actions is finite.

- Payoffs: at the end of each round, each player receives a numerical payoff, which is decided by the all players' actions. This number may be negative, in which case it is interpreted as a loss of the absolute value of the number.

1.2 Prisoner's dilemma

In this game the two players are partners in a crime who have been captured by the police. Each suspect is placed in a separate cell, and offered the opportunity to confess to the crime. The game can be represented by the following matrix of payoffs which are quantified in prison years

Table 1: Prisoner's dilemma

	Confess	Hold out
Confess	(-3, -3)	(0, -10)
Hold out	(-10, 0)	(-1, -1)

The idea solution for these two players would be they both hold out, in which case they will stay in prison only for 1 year. But this state is stable, because they both have the incentive to confess. So the state that they both confess is the stable state, which is an equilibrium.

1.3   Rock, paper, and scissors

Every one is familar with the game rock, paper, and scissors. Similarly, the game can be represented by the following matrix of payoffs

Table 2: Rock, paper, scissors

	Paper	Rock	Scissors
Paper	(0, 0)	(2, 0)	(0, 2)
Rock	(0, 2)	(0, 0)	(2, 0)
Scissors	(2, 0)	(0, 2)	(0, 0)

The difference between this game and Prisoner's delemma is, there is no pure strategy equilibrium in this game. But there is an equilibrium for mixed (randomized) strategy, in which each player has a probability distribution over his action set. The payoffs in mixed strategy are measured by expectation values. In this case, there is an equilibrium, which is both players choose rock, paper, and scissor equally likely.

2   Nash equilibrium

Definition 1. Nash equilibrium is the point where no player has incentive to change his strategy, assuming other players strateties stay fixed.

2.1   Existance of Nash equilibrium

Sperner's lemma $\Rightarrow$ Brouwer's fixed point theorem $\Rightarrow$ Kakutani's fixed point theorm $\Rightarrow$ Nash equilibrium.

Theorem 1. *Sperner's lemma is setting up as follows:*

- *Start with a triangle in the plane.*

- *Label the vertices with three different numbers. Let's use 0, 1, and 2 here.*

- *Now pick any finite number of points inside and on the edges of the triangle. It doesn't matter how many.*

- *Triangulate the interior of the triangle using these points. In other words, draw segments connecting the points chosen in the previous step so that each point is a vertex of a triangle.*

- *Now label each point with one of the three numbers, 0, 1 or 2. One caveat: any points on an edge of the big triangle can only be labeled with the number with which the endpoints of that edge are labeled. So if a point is on the edge of the big triangle between 0 and 1 vertices, it can only be labeled with 0 or 1.*

- *Sperner's lemma claims that there exists at least one small triangle (i.e. not the big triangle we started with) whose vertices are labeled with three different numbers.*

Sperner's lemma can be generalized to n-dimensional simplex.

Proof: We first proof Sperner's lemma is correct in 1-dimensional case using induction. The 1-dimensional simplex is just interval $(0, 1)$. For base case, if you put a point in $(0, 1)$, the number of intervals with two different numbers is 1. For inductive case, when you add a new point to $(0, 1)$, you can only increase the intervals with two different numbers by 0 or 2, which is alway an even number. So the overall number of intervals with two different numbers is odd number.

Then we can use induction on the number of dimensions to prove the general case. For n dimension simplex we label with numbers $0, 1, 2, \ldots, n$. And each n dimension simplex has $n + 1$ $(n-1)$-dimensional faces. Let $F(a_1, a_2, \ldots, a_n)$ be the number of cells (n-dimensional) labeled with numbers $a_1, a_2, \ldots, a_n$, $F_i(a_1, a_2, \ldots, a_{n-1})$ be the number of faces $((n-1)$-dimensional) labeled with $a_1, a_2, \ldots, a_{n-1}$ and inside the original n dimension simplex, and $F_b(a_1, a_2, \ldots, a_{n-1})$ be the number of faces $((n-1)$-dimensional) labeled with with $a_1, a_2, \ldots, a_{n-1}$ and on the boundary of the original n dimension simplex. By counting the number of face-cell pairs where face is labeled with $0, 1, 2, \ldots, n - 1$, we could have the following equality

$$F(0, 1, 2, \ldots, n) + \sum_{i=0}^{n-1} 2F(0, 1, 2, \ldots, n - 1, i) = 2F_i(0, 1, 2, \ldots, n - 1) + F_b(0, 1, 2, \ldots, n - 1)$$

By induction hypothesis, we know that $F_b(0, 1, 2, \ldots, n - 1)$ is an odd number. This means $F(0, 1, 2, \ldots, n)$ has to be odd. ◻

Theorem 2. *Brouwer's fixed point theorem: let $B = \{x \in \Re^n : \|x\| \leq 1\}$ be the closed unit ball in $\Re^n$. Any continuous function $f : B \to B$ has a fixed point $a = f(a)$.*

Proof: Let's first look at the Brouwer's theorem in one dimension. Let f be a continuous function in the interval $f : [-1, 1] \to [-1, 1]$. We want to show $\exists x \in [-1, 1]$ such that $x = f(x)$. Assume that $f(-1)$ and $f(1)$ are not equal to -1 and 1, respectively, otherwise we already have a fix point. Let $g(x) = f(x) - x$. Then $g(-1) > 0$ and $g(1) < 0$. Because $g(x)$ is continuous, there must exist an $x^* \in (-1, 1)$ such that $g(x^*) = 0$, which means $x^* = f(x^*)$.

Proof in two and higher dimensions uses Sperner's Lemma. Since there is a bijection between any n dimension simplex and n dimension convex body, it is sufficient to prove that n dimension simplex has a fix point. Any point inside the simplex, $\overrightarrow{p}$, could be represented as $p_0 \overrightarrow{0} + p_1 \overrightarrow{1} + \cdots + p_n \overrightarrow{n}$ such that $\sum_i p_i = 1$. Subdivide the simplex, as shown in Figure 1.

Let f be the continous function. Label point $\overrightarrow{p}$ by the 1st coordinate such that $\overrightarrow{p}$'s value $\geq f(\overrightarrow{p})$'s value. It is easy to verify that this labeling approach satisfies the rules in Sperner's lemma. This means we can at least have one simplex labeled with $0, 1, 2, \ldots, n$. As the subdivision goes on (i increases), we view this kind of simplex as a sequence. Then from Bolzano-Weierstrass theorem, we know there is a subsequence that is convergent. And function f is continuous. So this subsequence's f values also converge. Let $x^* = (x_1^*, x_2^*, x_3^*)$ be the point that the subsequence converges to, and $y^* = (y_1^*, y_2^*, y_3^*)$ is $f(x^*)$. Then we have $x_1^* \geq y_1^*, x_2^* \geq y_2^*, x_3^* \geq y_3^*$, which means $x^* = y^*$. We find the fix point.
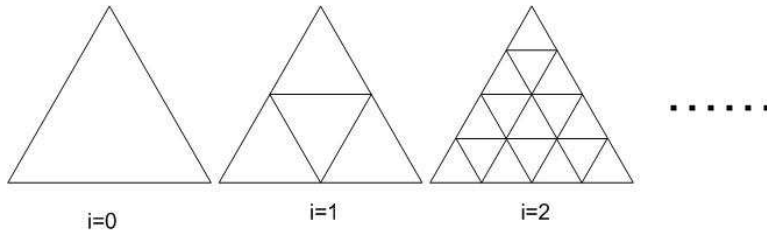
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Figure 1: Simplex division, 2 dimension case

Theorem 3. *Kakutani's fixed point theorem: let X be a convex closed bounded body in $\Re^n$. $\forall x \in X$, $f(x)$ is a nonempty convex subst of X. $\{\langle x, f(x) \rangle\}$ is closed. Then $\exists x^*$ such that $x^* \in f(x^*)$.*

Proof: The proof of Kakutani's fixed point theorem makes use of Brouwer's fixed point theorem. Since there is a bijection between any n dimension simplex and n dimension convex body, it is sufficient to prove n dimension simplex case. Subdivide the simplex as shown in Figure 1. Define function g_i as follows. If $\overrightarrow{x}$ is a vertex of a cell (n-dimensional simplex), then $g_i(\overrightarrow{x})$ is defined as any point that is in $f(\overrightarrow{x})$. If $\overrightarrow{x}$ is inside a cell, then define $g_i(\overrightarrow{x})$ as the average over all the $g_i(\overrightarrow{y})$ such that $\overrightarrow{y}$ is a vertex of the cell that contains $\overrightarrow{x}$. Not hard to see that function g_i is continuous. According to Brouwer's fixed point theorem, there is a fix point in the simplex. If this fix point, $\overrightarrow{x}$, is a vertex of a cell, then we are done. If it is not, as the vertices of a cell goes to one point, their g_i's values lie in one convex body. And $g_i(\overrightarrow{x})$ is in the convex-hull formed by vertices' g_i values. Hence it is inside the convex body $f(\overrightarrow{x})$. $\square$

Theorem 4. *Every finite (both players and actions) game has a mixed strategy Nash equilibrium.*

Proof: The proof of Nash makes use of Kakutani's fixed point theorem. Suppose there are 3 players, A, B, and C. Let $\bar{p}$, $\bar{q}$, and $\bar{r}$ be their probability distributions over the action sets. And α, β, and γ are their payoff functions. $P(\bar{q}, \bar{r})$ denotes the set of best-play $\bar{p}'s$. Not hard to see $P(\bar{q}, \bar{r})$ is a convex closed set. Similarly, we define $Q(\bar{r}, \bar{p})$ and $R(\bar{p}, \bar{q})$. Define function

$$f : \begin{pmatrix} \bar{p} \\ \bar{q} \\ \bar{r} \end{pmatrix} \longrightarrow \begin{pmatrix} P(\bar{q}, \bar{r}) \\ Q(\bar{r}, \bar{p}) \\ R(\bar{p}, \bar{q}) \end{pmatrix}$$

then if we could apply Kakutani's fix point theorem, we have

$$\begin{pmatrix} \bar{p} \\ \bar{q} \\ \bar{r} \end{pmatrix} \in f \begin{pmatrix} \bar{p} \\ \bar{q} \\ \bar{r} \end{pmatrix}$$

which shows the existance of Nash equilibrium. Now what we need to do is just verify our setup satisfies the conditions of Kakutani's fix point theorem. The first two conditions are trivially satisfied. We only need to show $\{\langle x, f(x) \rangle\}$ is closed, which means we need to show if $\langle x, f(x) \rangle \to \langle x^*, y^* \rangle$ then $\langle x^*, y^* \rangle \in f(x^*)$. Let $x = (\bar{p}, \bar{q}, \bar{r})$ and $y = (\bar{u}, \bar{v}, \bar{w})$.

$$\langle x^*, y^* \rangle \in f(x^*) \Leftrightarrow y^* \in f(x^*) \Leftrightarrow \begin{array}{l} \alpha(\bar{u}^*, \bar{q}^*, \bar{r}^*) \geq \alpha(\bar{p}', \bar{q}^*, \bar{r}^*) \; \forall \bar{p}' \\ \beta(\bar{p}^*, \bar{v}^*, \bar{r}^*) \geq \beta(\bar{p}^*, \bar{q}', \bar{r}^*) \; \forall \bar{q}' \\ \gamma(\bar{p}^*, \bar{q}^*, \bar{w}^*) \geq \gamma(\bar{p}^*, \bar{q}^*, \bar{r}') \; \forall \bar{r}' \end{array}$$

which shows $\{\langle x, f(x) \rangle\}$ is closed. And the above argument can easily apply to any finite number players. $\square$

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