Logical Relations for Fine-Grained Concurrency (Technical Appendix)

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Contents

1 Language

1.1 Syntax

$$
\begin{array}{rcl}
\tau & ::= & \mathbf{1} \mid \mathbf{B} \mid \mathbf{N} \mid \alpha \mid \tau + \tau \mid \mathbf{ref}(\overline{\tau}) \mid \mathbf{ref}_?(\overline{\tau}) \mid \mu\alpha.\tau \mid \forall\alpha.\tau \mid \tau \to \tau \\
\sigma & ::= & \mathbf{1} \mid \mathbf{B} \mid \mathbf{N} \mid \tau + \tau \mid \mathbf{ref}(\overline{\tau}) \mid \mathbf{ref}_?(\overline{\tau}) \mid \mu\alpha.\sigma \\
e & ::= & () \mid \mathbf{true} \mid \mathbf{false} \mid \mathbf{if}\ e\ \mathbf{then}\ e\ \mathbf{else}\ e \mid n \mid e + e \mid x \mid \mathbf{rec}\ f(x).e \mid e\ e \mid \Lambda.e \mid e\ \_ \\
 & \mid & \mathbf{null} \mid \mathbf{some}(e) \mid \mathbf{case}(e, \mathbf{null} \Rightarrow e, \mathbf{some}(x) \Rightarrow e) \mid \mathbf{inj}_i\ e \mid \mathbf{case}(e, \mathbf{inj}_1\ x \Rightarrow e, \mathbf{inj}_2\ y \Rightarrow e) \\
 & \mid & \mathbf{new}\ \overline{e} \mid e[i] \mid e[i] := e \mid \mathsf{CAS}(e[i], e, e) \mid \ell \mid \mathbf{fork}\ e \\
v & ::= & \mathbf{rec}\ f(x).e \mid \Lambda.e \mid () \mid n \mid \mathbf{true} \mid \mathbf{false} \mid \ell \mid x \\
\Gamma & ::= & \cdot \mid \Gamma, x : \tau \\
\Delta & ::= & \cdot \mid \Delta, \alpha \\
\Omega & ::= & \Delta; \Gamma
\end{array}
$$

Note: recursive types are required to be *productive*: the type variable must appear under a non-μ type constructor.

1.2 Type rules

$$\Omega \vdash () : \mathbf{1} \qquad \Omega \vdash \mathbf{true} : \mathbf{B} \qquad \Omega \vdash \mathbf{false} : \mathbf{B} \qquad \Omega \vdash n : \mathbf{N} \qquad \Omega, x : \tau \vdash x : \tau$$

$$\frac{\Omega \vdash e : \mathbf{B} \qquad \Omega \vdash e_i : \tau}{\Omega \vdash \mathbf{if}\ e\ \mathbf{then}\ e_1\ \mathbf{else}\ e_2 : \tau} \qquad \frac{\Omega, f : \tau' \to \tau, x : \tau' \vdash e : \tau}{\Omega \vdash \mathbf{rec}\ f(x).e : \tau' \to \tau}$$

$$\frac{\Omega \vdash e : \tau' \to \tau \qquad \Omega \vdash e' : \tau'}{\Omega \vdash e\ e' : \tau} \qquad \Omega \vdash \mathbf{null} : \mathbf{ref}_?(\overline{\tau}) \qquad \frac{\Omega \vdash e : \mathbf{ref}(\overline{\tau})}{\Omega \vdash \mathbf{some}(e) : \mathbf{ref}_?(\overline{\tau})}$$

$$\frac{\Omega \vdash e : \mathbf{ref}_?(\overline{\tau}) \qquad \Omega \vdash e_1 : \tau \qquad \Omega, x : \mathbf{ref}(\overline{\tau}) \vdash e_2 : \tau}{\Omega \vdash \mathbf{case}(e, \mathbf{null} \Rightarrow e_1, \mathbf{some}(x) \Rightarrow e_2) : \tau} \qquad \frac{\Omega \vdash e : \tau_i}{\Omega \vdash \mathbf{inj}_i\ e : \tau_1 + \tau_2}$$

$$\frac{\Omega \vdash e : \tau_1 + \tau_2 \qquad \Omega, x : \tau_i \vdash e_i : \tau}{\Omega \vdash \mathbf{case}(e, \mathbf{inj}_1\ x \Rightarrow e_1, \mathbf{inj}_2\ x \Rightarrow e_2) : \tau} \qquad \frac{\Omega, \alpha \vdash e : \tau}{\Omega \vdash \Lambda.e : \forall\alpha.\tau} \qquad \frac{\Omega \vdash e : \forall\alpha.\tau}{\Omega \vdash e\ \_ : \tau[\tau'/\alpha]}$$

$$\frac{\Omega \vdash e_i : \tau_i}{\Omega \vdash \mathbf{new}\ (\overline{e}) : \mathbf{ref}(\overline{\tau})} \qquad \frac{\Omega \vdash e : \mathbf{ref}(\overline{\tau})}{\Omega \vdash e[i] : \tau_i} \qquad \frac{\Omega \vdash e : \mathbf{ref}(\overline{\tau}) \qquad \Omega \vdash e' : \tau_i}{\Omega \vdash e[i] := e' : \mathbf{1}}$$

$$\frac{\Omega \vdash e : \mathbf{ref}(\overline{\tau}) \qquad \tau_i = \sigma \qquad \Omega \vdash e_o : \sigma \qquad \Omega \vdash e_n : \sigma}{\Omega \vdash \mathsf{CAS}(e[i], e_o, e_n) : \mathbf{B}} \qquad \frac{\Omega \vdash e : \mathbf{1}}{\Omega \vdash \mathbf{fork}\ e : \mathbf{1}}$$

$$\frac{\Omega \vdash e : \mu\alpha.\tau}{\Omega \vdash e : \tau[\mu\alpha.\tau/\alpha]} \qquad \frac{\Omega \vdash e : \tau[\mu\alpha.\tau/\alpha]}{\Omega \vdash e : \mu\alpha.\tau}$$

1.3 Operational semantics

$$
\begin{array}{rcl}
K & ::= & [\,] \mid \mathsf{if}\ K\ \mathsf{then}\ e\ \mathsf{else}\ e \mid K + e \mid v + K \mid K\ e \mid v\ K \\
& \mid & \mathsf{inj}_i\ K \mid \mathsf{case}(K, \mathsf{inj}_1\ x \Rightarrow e, \mathsf{inj}_2\ x \Rightarrow e) \mid \mathsf{some}(K) \mid \mathsf{case}(K, \mathsf{null} \Rightarrow e, \mathsf{some}(x) \Rightarrow e) \mid K\ \_ \\
& \mid & \mathsf{new}\ (\overline{v}, K, \overline{e}) \mid K[i] \mid K[i]\ :=\ e \mid v[i]\ :=\ K \mid \mathsf{CAS}(K[i], e, e) \mid \mathsf{CAS}(v[i], K, e) \mid \mathsf{CAS}(v[i], v, K) \\
T & \in & \text{ThreadPool} \triangleq \mathbb{N} \xrightarrow{\text{fin}} \text{Exp} \\
u & ::= & (\overline{v}) \mid \mathsf{inj}_i\ v \\
h & ::= & \cdot \mid h, \ell \mapsto u
\end{array}
$$

Primitive reductions $\boxed{h; e \hookrightarrow h'; e'}$

$$
\begin{array}{rcll}
h; n + m & \hookrightarrow & h; k & \text{when } k = n + m \\
h; \ell[i] & \hookrightarrow & h; v_i & \text{when } h(\ell) = (\overline{v}) \\
h; \ell[i]\ :=\ v & \hookrightarrow & h[\ell[i] = v]; () & \text{when } \ell \in \mathrm{dom}(h) \\
h; \mathsf{CAS}(\ell[i], v_o, v_n) & \hookrightarrow & h[\ell[i] = v_n]; \mathsf{true} & \text{when } h(\ell)[i] = v_o \\
h; \mathsf{CAS}(\ell[i], v_o, v_n) & \hookrightarrow & h; \mathsf{false} & \text{when } h(\ell)[i] \neq v_o \\
h; \mathsf{case}(\ell, \mathsf{inj}_1\ x \Rightarrow e_1, \mathsf{inj}_2\ x \Rightarrow e_2) & \hookrightarrow & h; e_i[v/x] & \text{when } h(\ell) = \mathsf{inj}_i\ v
\end{array}
$$

$$
\begin{array}{rcl}
h; \mathsf{if\ true\ then}\ e_1\ \mathsf{else}\ e_2 & \hookrightarrow & h; e_1 \\
h; \mathsf{if\ false\ then}\ e_1\ \mathsf{else}\ e_2 & \hookrightarrow & h; e_1 \\
h; \mathsf{null} & \hookrightarrow & h; () \\
h; \mathsf{some}(\ell) & \hookrightarrow & h; \ell \\
h; \mathsf{case}((), \mathsf{null} \Rightarrow e_1, \mathsf{some}(x) \Rightarrow e_2) & \hookrightarrow & h; e_1 \\
h; \mathsf{case}(\ell, \mathsf{null} \Rightarrow e_1, \mathsf{some}(x) \Rightarrow e_2) & \hookrightarrow & h; e_2[\ell/x] \\
h; \mathsf{rec}\ f(x).e\ v & \hookrightarrow & h; e[\mathsf{rec}\ f(x).e/f, v/x] \\
h; \mathsf{inj}_i\ v & \hookrightarrow & h \uplus [\ell \mapsto \mathsf{inj}_i\ v]; \ell \\
h; \Lambda.e\ \_ & \hookrightarrow & h; e \\
h; \mathsf{new}\ (\overline{v}) & \hookrightarrow & h \uplus [\ell \mapsto (\overline{v})]; \ell
\end{array}
$$

Program reduction $\boxed{h; T \to h'; T'}$

$$
\frac{h; e \hookrightarrow h'; e'}{h; T \uplus [i \mapsto K[e]] \to h'; T \uplus [i \mapsto K[e']]}
$$

$$
h; T \uplus [i \mapsto K[\mathsf{fork}\ e]] \to h; T \uplus [i \mapsto K[()]] \uplus [j \mapsto e]
$$

1.4 Contextual approximation (refinement)

$$
\begin{array}{rcll}
\Omega \models e_1 \preceq e_2 : \tau & \triangleq & \forall C : (\Omega, \tau) \rightsquigarrow (\emptyset, \mathbf{N}).\ \forall i, j. & \\
& & \text{if} \quad \emptyset; [i \mapsto C[e_1]] \to^* h_1; [i \mapsto n] \uplus T_1 & \\
& & \text{then} \quad \emptyset; [j \mapsto C[e_2]] \to^* h_2; [j \mapsto n] \uplus T_2 &
\end{array}
$$

1.5 Derived forms

$$
\begin{array}{rcl}
\mathsf{getVal}(e) & \triangleq & \textbf{case}(e, \textbf{null} \Rightarrow \mathsf{diverge}, \textbf{some}(x) \Rightarrow x) \\
\\
\mathsf{list}(\tau) & \triangleq & \mu\alpha.\textbf{ref}_?(\alpha, \tau) \\
\mathsf{cons}(e, e') & \triangleq & \textbf{some}(\textbf{new}\ (e, e')) \\
\\
\textbf{let}\ x = e\ \textbf{in}\ e' & \triangleq & (\lambda\_.e')\ e \\
e; e' & \triangleq & \textbf{let}\ \_ = e\ \textbf{in}\ e' \\
\\
\mathsf{forkID}\ e & \triangleq & \textbf{let}\ x = \textbf{new none in fork}\ x\ :=\ \textbf{some}(e);\ x \\
\mathsf{join} & \triangleq & \textbf{rec}\ f(x).\ \textbf{case}(x[0], \_ \Rightarrow f(x), r \Rightarrow r) \\
\\
\mathsf{acq} & \triangleq & \textbf{rec}\ f(x).\ \textbf{if}\ \mathsf{CAS}(x[1], \textbf{false}, \textbf{true})\ \textbf{then}\ ()\ \textbf{else}\ f(x) \\
\mathsf{rel} & \triangleq & \lambda x.x[1]\ :=\ \textbf{false} \\
\mathsf{sync}(e)\ \{\ e'\ \} & \triangleq & \textbf{let}\ x = e\ \textbf{in}\ \mathsf{acq}(x);\ \textbf{let}\ r = e'\ \textbf{in}\ \mathsf{rel}(x);\ r
\end{array}
$$

2 The model

2.1 Standard resources

$$\begin{aligned}\Sigma \in \text{StateSet} &\triangleq \overline{\wp}(\text{Heap} \times \text{ThreadPool}) \quad \text{(nonempty, finite subsets)}\\ \eta \in \text{ImplSpec} &\triangleq \text{Heap} \times \text{StateSet}\end{aligned}$$

Composition on state sets is strong:

$$\begin{aligned}\Sigma_1 \otimes \Sigma_2 &\triangleq \{\ h_1 \uplus h_2; T_1 \uplus T_2 \ \mid\ h_i; T_i \in \Sigma_i\ \} \quad \text{(defined only when } \textit{all} \text{ compositions are defined)}\\ (h_1, \Sigma_1) \otimes (h_2, \Sigma_2) &\triangleq (h_1 \uplus h_2, \Sigma_1 \otimes \Sigma_2)\end{aligned}$$

If N_1 and N_2 are subsets of ImplSpec we have:

$$N_1 * N_2 \triangleq \{\ \eta_1 \otimes \eta_2 \ \mid\ \eta_i \in N_i\ \}$$

2.2 Worlds and islands

$$\begin{aligned}\text{STS} &\triangleq \left\{\ \theta = (S, A, \rightsquigarrow, F) \ \middle|\ \rightsquigarrow \subseteq S \times S,\ F \in S \to \wp(A)\ \right\}\\ \text{World}_n &\triangleq \left\{\ W = (k, \omega) \ \middle|\ k < n,\ \omega \in \mathbb{N} \xrightarrow{\text{fin}} \text{Island}_k\ \right\}\\ \text{Island}_n &\triangleq \{\ \iota = (\theta, J, s, A) \ \mid\ \theta \in \text{STS},\ J \in \theta.S \to \text{UPred}_n,\ s \in \theta.S,\ A \subseteq \theta.A,\ A \# \theta.F(s)\ \}\\ \text{UPred}_n &\triangleq \{\ \Phi \subseteq \text{UWorld}_n \times \text{ImplSpec} \ \mid\ \forall (U, \eta) \in \Phi,\ U' \sqsupseteq U.\ (U', \eta) \in \Phi\ \}\\ \text{VRel}_n &\triangleq \{\ V \subseteq \text{UWorld}_n \times \text{Val} \times \text{Val} \ \mid\ \forall (U, v_1, v_2) \in V,\ U' \sqsupseteq U.\ (U', v_1, v_2) \in V\ \}\\ \text{UWorld}_n &\triangleq \{\ U \in \text{World}_n \ \mid\ U = |U|\ \}\end{aligned}$$

$$\begin{aligned}\lfloor \omega \rfloor_k &\triangleq i \mapsto \lfloor \omega(i) \rfloor_k\\ \lfloor (\theta, J, s_0, A) \rfloor_k &\triangleq (\theta, (s \mapsto I(s) \upharpoonright \text{UWorld}_k), s_0, A)\\ \theta \vdash (s, A) \rightsquigarrow (s', A') &\triangleq s \rightsquigarrow_\theta s' \quad \wedge \quad \theta.F(s) \uplus A = \theta.F(s') \uplus A'\\ (\theta, J, s, A) \overset{\text{guar}}{\sqsubseteq} (\theta', J', s', A') &\triangleq \theta = \theta',\ J = J',\ \theta \vdash (s, A) \rightsquigarrow^* (s', A')\\ \omega \overset{\text{guar}}{\sqsubseteq} \omega' &\triangleq \forall i \in \text{dom}(\omega).\ \omega(i) \overset{\text{guar}}{\sqsubseteq} \omega'(i)\\ (k, \omega) \overset{\text{guar}}{\sqsubseteq} (k', \omega') &\triangleq k \geq k' \quad \wedge \quad \lfloor \omega \rfloor_{k'} \overset{\text{guar}}{\sqsubseteq} \omega'\\ \text{frame}(k, \omega) &\triangleq (k,\ i \mapsto \text{frame}(\omega(i)))\\ \text{frame}(\theta, J, s, A) &\triangleq (\theta,\ J,\ s,\ \theta.A - (\theta.F(s) \cup A))\\ W \overset{\text{rely}}{\sqsubseteq} W' &\triangleq \text{frame}(W) \overset{\text{guar}}{\sqsubseteq} \text{frame}(W')\\ |(\theta, J, s, A)| &\triangleq (\theta, J, s, \emptyset)\\ |(k, \omega)| &\triangleq (k, i \mapsto |\omega(i)|)\\ (\theta, J, s, A) \otimes (\theta', J', s', A') &\triangleq \begin{cases}(\theta, J, s, A \cup A') & \theta = \theta', s' = s, J' = J, A \pitchfork A'\\ \text{undefined} & \text{otherwise}\end{cases}\\ (k, \omega) \otimes (k', \omega') &\triangleq \begin{cases}(k, \omega \otimes \omega') & k = k',\ \text{dom}(\omega) = \text{dom}(\omega'),\\ \text{undefined} & \text{otherwise}\end{cases}\\ W \# W' &\triangleq W \otimes W' \text{ defined}\\ \mathcal{I}[\![(\theta, J, s, A)]\!]_W &\triangleq J(s)(\triangleright |W|)\\ h, \Sigma : W, \eta &\triangleq W.k > 0 \implies (h, \Sigma) \in \{\eta\} * \textstyle\bigstar_{i \in \text{dom}(\omega)} \mathcal{I}[\![W.\omega(i)]\!]_W\end{aligned}$$

2.3 Assertions

$$
\begin{array}{rcl}
m & ::= & i \mid \mathsf{none} \\
\iota & ::= & (\theta, I, s, A) \text{ where } I \in \theta.S \to \text{Assert},\ s \in \theta.S,\ A \subseteq \theta.A,\ \theta.F(s) \# A \\
P & ::= & \mathsf{emp} \mid \triangleright P \mid \iota \mid v \mapsto_{\mathrm{I}} u \mid v \mapsto_{\mathrm{S}} u \mid i \rightarrowtail_{\mathrm{S}} e \mid T@m\ \langle x.\ P \rangle \\
 & \mid & P \Rightarrow P \mid P \wedge P \mid P \vee P \mid \exists x.P \mid \forall x.P \mid P * P \mid P \oplus P \mid \diamond P \mid \varphi \\
\varphi & ::= & v = v \mid \langle P \rangle\ e\ \langle x.\ Q \rangle \mid \Omega \vdash e \preceq^{\mathcal{E}} e : \tau \mid v \preceq^{\mathcal{V}} v : \tau \\
\mathrm{inv}(P) & \triangleq & ((\{1\}, \emptyset, \emptyset, \lambda\_.\emptyset), \lambda\_.P, 1, \emptyset) \\
\Sigma \rightrightarrows \Sigma' & \triangleq & \forall \varsigma' \in \Sigma'.\ \exists \varsigma \in \Sigma.\ \varsigma \to^* \varsigma'
\end{array}
$$

$$
[\![I]\!]^{\rho}_{k} \triangleq s \mapsto \{\ (U, \eta) \models^{\rho} I(s) \ \mid\ U \in \mathrm{UWorld}_k\ \}
$$

Semantics of assertions (closed on term variables, may have free type variables closed by ρ)
$W, \eta \models^{\rho} P$:

$$
\begin{array}{lcl}
W, \eta \models^{\rho} \mathsf{emp} & \triangleq & W = |W|, \eta = (\emptyset, \{\emptyset; \emptyset\}) \\
W, \eta \models^{\rho} \triangleright P & \triangleq & W.k > 0 \implies \triangleright W, \eta \models^{\rho} P \\
W, \eta \models^{\rho} (\theta, I, s, A) & \triangleq & \exists i.\ W \overset{\text{rely}}{\sqsupseteq} (W.k, [i \mapsto (\theta, [\![I]\!]^{\rho}_{W.k}, s, A)]) \\
W, \eta \models^{\rho} v \mapsto_{\mathrm{I}} u & \triangleq & \eta = ([v \mapsto u], \{\emptyset; \emptyset\}) \\
W, \eta \models^{\rho} v \mapsto_{\mathrm{S}} u & \triangleq & \eta = (\emptyset, \{[v \mapsto u]; \emptyset\}) \\
W, \eta \models^{\rho} i \rightarrowtail_{\mathrm{S}} e & \triangleq & \eta = (\emptyset, \{\emptyset; [i \mapsto e]\}) \\
W, \eta \models^{\rho} P \Rightarrow P' & \triangleq & \forall W' \overset{\text{rely}}{\sqsupseteq} W.\ W', \eta \models^{\rho} P \text{ implies } W', \eta \models^{\rho} P' \\
W, \eta \models^{\rho} P \wedge P' & \triangleq & W, \eta \models^{\rho} P \text{ and } W, \eta \models^{\rho} P' \\
W, \eta \models^{\rho} P \vee P' & \triangleq & W, \eta \models^{\rho} P \text{ or } W, \eta \models^{\rho} P' \\
W, \eta \models^{\rho} \exists x.P & \triangleq & \exists v.\ W, \eta \models^{\rho} P[v/x] \\
W, \eta \models^{\rho} \forall x.P & \triangleq & \forall v.\ W, \eta \models^{\rho} P[v/x] \\
W, \eta \models^{\rho} P_1 * P_2 & \triangleq & W = W_1 \otimes W_2,\ \eta = \eta_1 \otimes \eta_2,\ W_i, \eta_i \models^{\rho} P_i \\
W, \eta \models^{\rho} P_1 \oplus P_2 & \triangleq & \eta.\Sigma = \Sigma_1 \cup \Sigma_2,\ W, (\eta.h, \Sigma_i) \models^{\rho} P_i \\
W, \eta \models^{\rho} \diamond P & \triangleq & \eta = (h, \Sigma),\ \forall \Sigma_F \# \Sigma.\ \exists \Sigma'.\ (\Sigma \otimes \Sigma_F) \rightrightarrows (\Sigma' \otimes \Sigma_F) \text{ and } W, (h, \Sigma') \models^{\rho} P \\
W, \eta \models^{\rho} \varphi & \triangleq & |W| \models^{\rho} \varphi \\
W_0, \eta \models^{\rho} T@m\ \langle x.\ Q \rangle & \triangleq & \forall W \overset{\text{rely}}{\sqsupseteq} W_0,\ \eta_F \# \eta. \\
\quad \text{if } W.k > 0 \text{ and } h, \Sigma : W, \eta \otimes \eta_F \text{ then} & & \\
\quad\quad \text{if } h; T \to h'; T' \text{ then } \exists \Sigma', \eta', W' \overset{\text{guar}}{\sqsupseteq} W. & & \\
\quad\quad\quad \Sigma \rightrightarrows \Sigma', \quad h', \Sigma' : W', \eta' \otimes \eta_F, \quad W'.k = W.k - 1, \quad W', \eta' \models^{\rho} T'@m\ \langle x.\ Q \rangle & & \\
\quad\quad \text{if } T = T_0 \uplus [m \mapsto v] \text{ then } \exists \Sigma', \eta', W' \overset{\text{guar}}{\sqsupseteq} W. & & \\
\quad\quad\quad \Sigma \rightrightarrows \Sigma', \quad h, \Sigma' : W', \eta' \otimes \eta_F, \quad W'.k = W.k, \quad W', \eta' \models^{\rho} Q[v/x] * T_0@\mathsf{none}\ \langle x.\ \mathsf{tt} \rangle &
\end{array}
$$

Semantics of pure assertions: $U \models^\rho \varphi$. If P is impure, $U \models^\rho P$ is short for $\forall \eta.\, U, \eta \models^\rho P$.

$$
\begin{array}{lcl}
U \models^\rho v_1 = v_2 & \triangleq & v_1 = v_2 \\
U \models^\rho \cdot \overset{\text{rely}}{\sqsupseteq} U_0 & \triangleq & U \overset{\text{rely}}{\sqsupseteq} U_0 \\
U \models^\rho \langle P \rangle\ e\ \langle x.\, Q \rangle & \triangleq & \forall i.\, U \models^\rho P \Rightarrow [i \mapsto e]@i\ \langle x.\, Q \rangle \\
U \models^\rho v_1 \preceq^{\mathcal{V}} v_2 : \tau_0 & \triangleq &
\end{array}
$$

τ_0	v_1, v_2	Requirements
τ_b	v, v	$\vdash v : \tau_b$ for $\tau_b \in \{\mathbf{1}, \mathbf{B}, \mathbf{N}\}$
α	v_1, v_2	$(U, v_1, v_2) \in \rho(\alpha)$
$\tau \to \tau'$	$\mathbf{rec}\ f(x).e_1, \mathbf{rec}\ f(x).e_2$	$U \models^\rho \triangleright(x : \tau \vdash e_1[\mathbf{rec}\ f(x).e_1/f] \preceq^{\mathcal{E}} e_2[\mathbf{rec}\ f(x).e_2/f] : \tau')$
$\forall \alpha.\tau$	$\Lambda.e_1, \Lambda.e_2$	$U \models^\rho \triangleright(\alpha \vdash e_1 \preceq^{\mathcal{E}} e_2 : \tau)$
$\mu\alpha.\tau$	v_1, v_2	$U \models^\rho v_1 \preceq^{\mathcal{V}} v_2 : \tau[\mu\alpha.\tau/\alpha]$
$\mathbf{ref}_?(\overline{\tau})$	v_1, v_2	$U \models^\rho v_1 \preceq^{\mathcal{V}} v_2 : \mathbf{1} \ \vee\ v_1 \preceq^{\mathcal{V}} v_2 : \mathbf{ref}(\overline{\tau})$
$\mathbf{ref}(\overline{\tau})$	ℓ_1, ℓ_2	$U \models^\rho \mathrm{inv}(\exists \overline{x}, \overline{y}.\ \bigwedge \overline{x \preceq^{\mathcal{V}} y : \tau} \wedge (\ell_1 \mapsto_{\mathrm{I}} (\overline{x}) * \ell_2 \mapsto_{\mathrm{S}} (\overline{y})))$
$\tau_1 + \tau_2$	ℓ_1, ℓ_2	$U \models^\rho \exists x, y.\ x \preceq^{\mathcal{V}} y : \tau_i \wedge \mathrm{inv}(\ell_1 \mapsto_{\mathrm{I}} \mathbf{inj}_i\ x * \ell_2 \mapsto_{\mathrm{S}} \mathbf{inj}_i\ y)$

$$
\begin{array}{lcl}
U \models^\rho \cdot ; \cdot \vdash e_1 \preceq^{\mathcal{E}} e_2 : \tau & \triangleq & \forall K, j.\, U \models^\rho \langle j \rightarrowtail_{\mathrm{S}} K[e_2] \rangle\ e_1\ \langle x_1.\, \exists x_2.\, x_1 \preceq^{\mathcal{V}} x_2 : \tau \wedge j \rightarrowtail_{\mathrm{S}} K[x_2] \rangle \\
U \models^\rho \cdot ; x : \tau', \Gamma \vdash e_1 \preceq^{\mathcal{E}} e_2 : \tau & \triangleq & U \models^\rho \forall x_1, x_2.\, x_1 \preceq^{\mathcal{V}} x_2 : \tau' \Rightarrow \Gamma \vdash e_1[x_1/x] \preceq^{\mathcal{E}} e_2[x_2/x] : \tau \\
U \models^\rho \alpha, \Delta; \Gamma \vdash e_1 \preceq^{\mathcal{E}} e_2 : \tau & \triangleq & \forall V.\, U \models^{\rho[\alpha \mapsto V]} \Delta; \Gamma \vdash e_1 \preceq^{\mathcal{E}} e_2 : \tau
\end{array}
$$

The set World $\times$ ImplSpec is a preorder when ordered by $(w, \eta) \sqsupseteq (w', \eta')$ iff $w \overset{\text{rely}}{\sqsupseteq} w' \wedge \eta = \eta'$. Since ImplSpec is a partial commutative monoid and World is a commutative semi-group with a unit element for each element, we have:

Lemma 1. The set $P^\uparrow(\text{World} \times \text{ImplSpec})$ of upwards-closed subsets of World $\times$ ImplSpec is a complete BI-algebra.

Hence we get a model of intuitionistic BI logic and, indeed, the interpretation of the logical connectives is as detailed in the above table showing the semantics of assertions.

3 Soundness

3.1 Basic Properties

Lemma 2 (Rely-guarantee Preorders)**.** The relations $\overset{\text{rely}}{\sqsubseteq}$ and $\overset{\text{guar}}{\sqsubseteq}$ are preorders.

Lemma 3 (Rely-closure of Assertions)**.** $W, \eta \models^\rho P$ and $W' \overset{\text{rely}}{\sqsupseteq} W$ implies $W', \eta \models^\rho P$.

Lemma 4. $|W| \otimes W = W$.

Lemma 5. If $W \overset{\text{rely}}{\sqsubseteq} W'$ then $|W| \overset{\text{rely}}{\sqsubseteq} |W'|$.

Lemma 6 (Rely Decomposition)**.** If $W_1 \otimes W_2 \overset{\text{rely}}{\sqsubseteq} W'$ then there are W_1' and W_2' with $W' = W_1' \otimes W_2'$, $W_1 \overset{\text{rely}}{\sqsubseteq} W_1'$ and $W_2 \overset{\text{rely}}{\sqsubseteq} W_2'$.

Lemma 7 (Token Framing)**.** If $W \overset{\text{guar}}{\sqsubseteq} W'$ and $W \otimes W_f$ is defined then there exists some $W_f' \overset{\text{rely}}{\sqsupseteq} W_f$ such that $W' \otimes W_f'$ is defined and $W \otimes W_f \overset{\text{guar}}{\sqsubseteq} W' \otimes W_f'$.

Lemma 8. If $h, \Sigma : W, \eta$ then $h, \Sigma : W \otimes W', \eta$.

Lemma 9. If $h, \Sigma : W \otimes W', \eta$ then $h, \Sigma : W, \eta$.

Lemma 10. If $W.k > 0$ then $\triangleright W \overset{\text{guar}}{\sqsupseteq} W$ and $\triangleright W \overset{\text{rely}}{\sqsupseteq} W$.

Lemma 11. If $W.k > 0$ then $|\triangleright W| = \triangleright |W|$.

Lemma 12 (Later Satisfaction)**.** If $W.k > 0$ and $h, \Sigma : W, \eta$ then $h, \Sigma : \triangleright W, \eta$.

Lemma 13. $\rightrightarrows$ is transitive.

3.2 Constructions with Threadpool Triples

Lemma 14 (Framing)**.** $W, \eta \models^\rho T@m \langle x.\, Q \rangle$ and $W_f, \eta_f \models^\rho R$ with $W \# W_f$, $\eta \# \eta_f$ gives

$$W \otimes W_f, \eta \otimes \eta_f \models^\rho T@m \langle x.\, Q * R \rangle.$$

Proof. The proof proceeds by induction on $W.k$.

Case $\boxed{W.k = 0}$

1. $(W \otimes W_f).k = W.k = 0$.

Case $\boxed{W.k > 0}$

2. Let $W' \overset{\text{rely}}{\sqsupseteq} W \otimes W_f, \eta_f' \# \eta \otimes \eta_f$.
3. Write $W' = W_1' \otimes W_2'$ with $W_1' \overset{\text{rely}}{\sqsupseteq} W, W_2' \overset{\text{rely}}{\sqsupseteq} W_f$ by Lem. 6.
4. Suppose $(W_1' \otimes W_2').k > 0$.
5. $W_1'.k = (W_1' \otimes W_2').k > 0$.
6. Suppose $h, \Sigma : W_1' \otimes W_2', \eta \otimes \eta_f \otimes \eta_f'$.
7. $h, \Sigma : W_1', \eta \otimes \eta_f \otimes \eta_f'$ by Lem. 9.

Case $\boxed{h; T \to h'; T'}$

8. Pick $\Sigma', \eta',$ and W_1'' with $W_1'' \overset{\text{guar}}{\sqsupseteq} W_1'$, $\Sigma \rightrightarrows \Sigma'$, $h', \Sigma' : W_1'', \eta' \otimes \eta_f \otimes \eta_f'$, $W_1''.k = W_1'.k - 1$, $W_1'', \eta' \models^\rho T'@m \left\langle x.\ Q \right\rangle$ — by assumption.
9. Pick $W_2'' \overset{\text{rely}}{\sqsupseteq} W_2'$ with $W_1'' \otimes W_2'' \overset{\text{guar}}{\sqsupseteq} W_1' \otimes W_2'$ — by Lem. 7.
10. $h', \Sigma' : W_1'' \otimes W_2'', \eta' \otimes \eta_f \otimes \eta_f'$ — by Lem. 8.
11. $(W_1'' \otimes W_2'').k = W_1''.k = W_1'.k - 1 = (W_1' \otimes W_2').k - 1$.
12. $W_2'', \eta_f \models^\rho R$.
13. $W_1'' \otimes W_2'', \eta' \otimes \eta_f \models^\rho T'@m \left\langle x.\ Q * R \right\rangle$ — by induction hypothesis.

Case $\boxed{m = i \text{ and } T = T_0 \uplus [i \mapsto v]}$

14. Pick $\Sigma', \eta',$ and W_1'' with $W_1'' \overset{\text{guar}}{\sqsupseteq} W_1'$, $\Sigma \rightrightarrows \Sigma'$, $h, \Sigma' : W_1'', \eta' \otimes \eta_f \otimes \eta_f'$, $W_1''.k = W_1'.k$, $W_1'', \eta' \models^\rho Q[v/x] * T_0@\mathsf{none} \left\langle x.\ \mathsf{tt} \right\rangle$ — by assumption.
15. Pick $W_2'' \overset{\text{rely}}{\sqsupseteq} W_2'$ with $W_1'' \otimes W_2'' \overset{\text{guar}}{\sqsupseteq} W_1' \otimes W_2'$ — by Lem. 7.
16. $h', \Sigma' : W_1'' \otimes W_2'', \eta' \otimes \eta_f \otimes \eta_f'$ — by Lem. 8.
17. $(W_1'' \otimes W_2'').k = W_1''.k = W_1'.k = (W_1' \otimes W_2').k$.
18. $W_2'', \eta_f \models^\rho R$.
19. $W_1'' \otimes W_2'', \eta' \otimes \eta_f \models^\rho Q[v/x] * R * T_0@\mathsf{none} \left\langle x.\ \mathsf{tt} \right\rangle$.

□

Corollary 1 (Precondition Extension)**.** $W, \eta \models^\rho T@m \left\langle x_1.\ \exists x_2.\ x_1 \preceq^{\mathcal{V}} x_2 : \tau \wedge j \rightarrowtail_{\mathrm{S}} K[x_2] \right\rangle$ together with $W_f \# W$ gives $W \otimes W_f, \eta \models^\rho T@m \left\langle x_1.\ \exists x_2.\ x_1 \preceq^{\mathcal{V}} x_2 : \tau \wedge j \rightarrowtail_{\mathrm{S}} K[x_2] \right\rangle$.

Corollary 2 (Postcondition Strengthening)**.** If we have $W, \eta \models^\rho T@m \left\langle x.\ Q \right\rangle$ then $W, \eta \models^\rho T@m \left\langle x.\ Q \wedge \cdot \overset{\text{rely}}{\sqsupseteq} |W| \right\rangle$ holds too.

Lemma 15 (Parallel Composition)**.** If we have $W_1, \eta_1 \models^\rho T_1@m_1 \left\langle x.\ Q_1 \right\rangle$ and $W_2, \eta_2 \models^\rho T_2@m_2 \left\langle x.\ Q_2 \right\rangle$ with $W_1 \# W_2$, $\eta_1 \# \eta_2$, $T_1 \# T_2$ and $m_1 \neq \mathsf{none} \Rightarrow m_1 \in \mathrm{dom}(T_1)$ then we also have

$$W_1 \otimes W_2, \eta_1 \otimes \eta_2 \models^\rho T_1 \uplus T_2@m_1 \left\langle x.\ Q_1 \right\rangle.$$

Proof. The proof proceeds by induction on the measure $M(W_1, m_1)$ defined by

$$M(W, m) = \begin{cases} W.k & m = \mathsf{none} \\ W.k + 1 & m \neq \mathsf{none}. \end{cases}$$

Case $\boxed{M(W_1, m_1) = 0}$

1. $(W_1 \otimes W_2).k = W_1.k = 0$.

Case $\boxed{M(W_1, m_1) > 0}$

2. Let $W' \overset{\text{rely}}{\sqsupseteq} W_1 \otimes W_2, \eta_f \# \eta_1 \otimes \eta_2$.
3. Write $W' = W_1' \otimes W_2'$ with $W_1' \overset{\text{rely}}{\sqsupseteq} W_1, W_2' \overset{\text{rely}}{\sqsupseteq} W_2$ by Lem. 6.
4. Suppose $(W_1' \otimes W_2').k > 0$.
5. $W_1'.k = (W_1' \otimes W_2').k > 0$.
6. Suppose $h, \Sigma : W_1' \otimes W_2', \eta_1 \otimes \eta_2 \otimes \eta_f$.
7. $h, \Sigma : W_1', \eta_1 \otimes \eta_2 \otimes \eta_f$ by Lem. 9.

Case $\boxed{h; T_1 \uplus T_2 \to h'; T'}$

8. Write $T' = T_1' \uplus T_2$ WLOG.
9. $h; T_1 \to h'; T_1'$ by nondeterminism of fork.
10. Pick $\Sigma', \eta_1',$ and W_1'' with $W_1'' \overset{\text{guar}}{\sqsupseteq} W_1'$, $\Sigma \rightrightarrows \Sigma'$, $W_1''.k = W_1'.k - 1$, $h', \Sigma' : W_1'', \eta_1' \otimes \eta_2 \otimes \eta_f$, $W_1'', \eta_1' \models^\rho T_1'@m_1 \langle x_1.\, Q_1 \rangle$ by assumption.
11. Pick $W_2'' \overset{\text{rely}}{\sqsupseteq} W_2'$ with $W_1'' \otimes W_2'' \overset{\text{guar}}{\sqsupseteq} W_1' \otimes W_2'$ by Lem. 7.
12. $(W_1'' \otimes W_2'').k = W_1''.k = W_1'.k - 1 = (W_1' \otimes W_2').k - 1$.
13. $h', \Sigma' : W_1'' \otimes W_2'', \eta_1' \otimes \eta_2 \otimes \eta_f$ by Lem. 8.
14. $W_2'', \eta_2 \models^\rho T_2@m_2 \langle x_2.\, Q_2 \rangle$ by assumption.
15. $W_1'' \otimes W_2'', \eta_1' \otimes \eta_2 \models^\rho T_1' \uplus T_2@m_1 \langle x_1.\, Q_1 \rangle$ by induction hypothesis.

Case $\boxed{T_1 * T_2 = T_0 \uplus [m_1 \mapsto v_1]}$

16. $m_1 \in \mathrm{dom}(T_1)$ by assumption.
17. Write $T_1 = T_1' \uplus [m_1 \mapsto v_1]$.
18. Pick $\Sigma', \eta_1',$ and W_1'' with $W_1'' \overset{\text{guar}}{\sqsupseteq} W_1'$, $\Sigma \rightrightarrows \Sigma'$, $W_1''.k = W_1'.k$, $h, \Sigma' : W_1'', \eta_1' \otimes \eta_2 \otimes \eta_f$, $W_1'', \eta_1' \models^\rho Q_1[v_1/x_1] * T_1'@\mathsf{none} \langle x_1.\, \mathsf{tt} \rangle$ by assumption.
19. Pick $W_2'' \overset{\text{rely}}{\sqsupseteq} W_2'$ with $W_1'' * W_2'' \overset{\text{guar}}{\sqsupseteq} W_1' * W_2'$ by Lem. 7.
20. $(W_1'' * W_2'').k = W_1''.k = W_1'.k = (W_1' * W_2').k$.
21. $h, \Sigma' : W_1'' * W_2'', \eta_1' * \eta_2 * \eta_f$ by Lem. 8.
22. $W_2'', \eta_2 \models^\rho T_2@m_2 \langle x_2.\, Q_2 \rangle$. by assumption.
23. $W_1'' * W_2'', \eta_1' * \eta_2 \models^\rho Q[v_1/x_1] * T_1' \uplus T_2@\mathsf{none} \langle x_1.\, \mathsf{tt} \rangle$. by induction hypothesis.

□

Lemma 16 (Sequential Composition)**.** If we have $W, \eta \models^\rho [i \mapsto e] \uplus T@i \left\langle x.\ Q \right\rangle$ and for all v and any W', η' with $W', \eta' \models Q[v/x]$ we have $W', \eta' \models^\rho [i \mapsto K[v]]@i \left\langle x.\ R \right\rangle$ then

$$W, \eta \models^\rho [i \mapsto K[e]] \uplus T@i \left\langle x.\ R \right\rangle.$$

Proof. The proof proceeds by induction on $W.k$; the case $W.k = 0$ is trivial so we assume $W.k > 0$. We branch on the structure of e:

1. Let $W' \overset{\text{rely}}{\sqsupseteq} W, \eta_f \# \eta$.
2. Suppose $W'.k > 0$.
3. Suppose $h, \Sigma : W', \eta \otimes \eta_f$.

Case $\boxed{e = v}$

4. Pick $\Sigma', \eta',$ and W'' with — by assumption.
   $W'' \overset{\text{guar}}{\sqsupseteq} W'$,
   $\Sigma \rightrightarrows \Sigma'$,
   $h, \Sigma' : W'', \eta' \otimes \eta_f$,
   $W''.k = W'.k$,
   $W'', \eta' \models^\rho Q[v/x] * T@\mathsf{none} \left\langle x.\ \mathsf{tt} \right\rangle$
5. Write $W'' = W_1'' \otimes W_2''$ and $\eta' = \eta_1' \otimes \eta_2'$. with
   $W_1'', \eta_1' \models^\rho Q[v/x]$,
   $W_2'', \eta_2' \models^\rho T@\mathsf{none} \left\langle x.\ \mathsf{tt} \right\rangle$.
6. $W_1'', \eta_1' \models^\rho [i \mapsto K[v]]@i \left\langle x.\ R \right\rangle$ — by assumption.
7. $W'', \eta' \models^\rho [i \mapsto K[v]] \uplus T@i \left\langle x.\ R \right\rangle$ — by Lem. 15.

Case $\boxed{K[v] = v'}$

8. Pick $\Sigma'', \eta'',$ and W''' with
   $W''' \overset{\text{guar}}{\sqsupseteq} W''$,
   $\Sigma' \rightrightarrows \Sigma''$,
   $h, \Sigma'' : W''', \eta'' \otimes \eta_f$,
   $W'''.k = W''.k$,
   $W''', \eta'' \models^\rho R[v'/x] * T@\mathsf{none} \left\langle x.\ \mathsf{tt} \right\rangle$
   by (7).
9. $\Sigma \rightrightarrows \Sigma''$.
10. $W'''.k = W''.k = W'.k$.

Case $\boxed{h; [i \mapsto K[v]] \uplus T \to h'; T'}$

11. Pick $\Sigma'', \eta'',$ and W''' with — by (7).
    $W''' \overset{\text{guar}}{\sqsupseteq} W''$,
    $\Sigma' \rightrightarrows \Sigma''$,
    $h', \Sigma'' : W''', \eta'' \otimes \eta_f$,
    $W'''.k = W''.k - 1$,
    $W''', \eta'' \models^\rho T'@i \left\langle x.\ R \right\rangle$
12. $\Sigma \rightrightarrows \Sigma''$.
13. $W'''.k = W''.k - 1 = W'.k - 1$.

Case $\boxed{e \neq v}$

14. Suppose $h; [i \mapsto K[e]] \uplus T \to h'; [i \mapsto K[e']] \uplus T'$.
15. $h; [i \mapsto e] \uplus T \to h'; [i \mapsto e'] \uplus T'$.
16. Pick $\Sigma', \eta',$ and W'' with — by assumption.
    $W'' \overset{\text{guar}}{\sqsupseteq} W'$,
    $\Sigma \rightrightarrows \Sigma'$,
    $h', \Sigma' : W'', \eta' \otimes \eta_f$,
    $W''.k = W'.k - 1$,
    $W'', \eta' \models^\rho [i \mapsto e'] \uplus T' @ i \left\langle x.\ Q \right\rangle$
17. $W'', \eta' \models^\rho [i \mapsto K[e']] \uplus T' @ i \left\langle x.\ R \right\rangle$ — by induction hypothesis.

□

3.3 Congruence

Lemma 17 (Soundness Shortcut)**.** $\Delta; \Gamma \models e_1 \preceq e_2 : \tau$ is equivalent to

$$\begin{aligned}
&\forall U\ \forall \rho : \Delta \to \mathrm{VRel}\ \forall \gamma_1, \gamma_2 : \mathrm{dom}(\Gamma) \to \mathrm{Val}. \\
&\quad \big[\forall x \in \mathrm{dom}(\Gamma).\ U \models^\rho \gamma_1(x) \preceq^{\mathcal{V}} \gamma_2(x) : \Gamma(x)\big] \\
&\qquad \Longrightarrow \\
&\quad \big[\forall K, j, i.\ U, (\emptyset, \{\emptyset; [j \mapsto K[e_2[\gamma_2/\Gamma]]]\}) \models^\rho \\
&\qquad [i \mapsto e_1[\gamma_1/\Gamma]] @ i \left\langle x_1.\ \exists x_2.\ x_1 \preceq^{\mathcal{V}} x_2 : \tau \wedge j \rightarrowtail_{\mathrm{S}} K[x_2] \right\rangle\big].
\end{aligned}$$

3.3.1 New

Lemma 18.

$$\frac{\Delta; \Gamma \models e_i \preceq f_i : \tau_i}{\Delta; \Gamma \models \mathsf{new}\ \overline{e} \preceq \mathsf{new}\ \overline{f} : \mathsf{ref}(\overline{\tau})}$$

Proof.

1. Let $U, \rho : \Delta \to \mathrm{VRel}, \gamma_1, \gamma_2 : \mathrm{dom}(\Gamma) \to \mathrm{Val}$.
2. Suppose $\forall x \in \mathrm{dom}(\Gamma).\ U \models^\rho \gamma_1(x) \preceq^{\mathcal{V}} \gamma_2(x) : \Gamma(x)$.
3. Write $e_i' = e_i[\gamma_1/\Gamma], f_i' = f_i[\gamma_2/\Gamma]$.
4. Let K, i, j.
5. Write $\eta = (\emptyset, \{\emptyset; [j \mapsto K[\mathsf{new}\ \overline{f'}]]\})$.
6. Write $Q = \exists y.\ x \preceq^{\mathcal{V}} y : \mathsf{ref}(\overline{\tau}) \wedge j \rightarrowtail_{\mathrm{S}} K[y]$.
7. Suffices to show $U, \eta \models^\rho [i \mapsto \mathsf{new}\ \overline{e'}] @ i \left\langle x.\ Q \right\rangle$. — by Lem. 17.

Let $M = |\mathsf{ref}(\overline{\tau})|$. We now proceed to make a claim: for any $0 \leq m \leq M$ it sufices to prove

$$U', \eta_m \models^\rho [i \mapsto \mathsf{new}\ v_1, \ldots, v_m, e'_{m+1}, \ldots, e'_M] @ i \left\langle x.\ Q \right\rangle,$$

for all $U' \overset{\text{rely}}{\sqsupseteq} U$ and all $U' \models^\rho v_1 \preceq^{\mathcal{V}} w_1 : \tau_1, \ldots, U' \models^\rho v_m \preceq^{\mathcal{V}} w_m : \tau_m$, where $\eta_m = (\emptyset, \{\emptyset; [j \mapsto K[\mathsf{new}\ w_1, \ldots, w_m, f'_{m+1}, \ldots, f'_M]]\})$. We prove the claim by induction on m;

the case $m = 0$ was proved above. So, suppose the claim holds for $0 \leq m < M$ and assume that we know that

$$U'', \eta_{m+1} \models^{\rho} [i \mapsto \textbf{new}\ v_1, \ldots, v_{m+1}, e'_{m+2}, \ldots, e'_M]@i\ \langle x.\ Q \rangle,$$

holds for all for all $U'' \overset{\text{rely}}{\sqsupseteq} U$ and all $U'' \models^{\rho} v_1 \preceq^{\mathcal{V}} w_1 : \tau_1, \ldots, U'' \models^{\rho} v_{m+1} \preceq^{\mathcal{V}} w_{m+1} : \tau_{m+1}$, where $\eta_{m+1} = (\emptyset, \{\emptyset; [j \mapsto K[\textbf{new}\ w_1, \ldots, w_{m+1}, f'_{m+2}, \ldots, f'_M]]\})$. In the interest of applying the induction hypothesis, we pick arbitrary $U' \overset{\text{rely}}{\sqsupseteq} U$ and $U' \models^{\rho} v_1 \preceq^{\mathcal{V}} w_1 : \tau_1, \ldots, U' \models^{\rho} v_m \preceq^{\mathcal{V}} w_m : \tau_m$. By induction, it will suffice for the claim to prove that

$$U', \eta_m \models^{\rho} [i \mapsto \textbf{new}\ v_1, \ldots, v_m, e'_{m+1}, \ldots, e'_M]@i\ \langle x.\ Q \rangle,$$

holds, where $\eta_m = (\emptyset, \{\emptyset; [j \mapsto K[\textbf{new}\ w_1, \ldots, w_m, f'_{m+1}, \ldots, f'_M]]\})$. Now, by assumption and Lemma 17 and Corollary 2 we have

$$U', \eta_m \models^{\rho} [i \mapsto e'_{m+1}]@i\ \langle x_{m+1}.\ Q_{m+1} \wedge \cdot \overset{\text{rely}}{\sqsupseteq} U' \rangle,$$

where $Q_{m+1} = \exists y_{m+1}.\ x_{m+1} \preceq^{\mathcal{V}} y_{m+1} : \tau_{m+1} \wedge j \rightarrowtail_{\mathrm{S}} K[\textbf{new}\ w_1, \ldots, w_m, y'_{m+1}, \ldots, f'_M]$. Now, let v_{m+1} be arbitrary and take $W'', \eta'' \models^{\rho} Q_{m+1}[v_{m+1}/x_{m+1}] \wedge \cdot \overset{\text{rely}}{\sqsupseteq} U'$ and by an application of Lemma 16 we have the claim if we can show

$$W'', \eta'' \models^{\rho} [i \mapsto \textbf{new}\ v_1, \ldots, v_{m+1}, e'_{m+2}, \ldots, e'_M]@i\ \langle x.\ Q \rangle.$$

Luckily, we can pick w_{n+1} such that we have $|W''| \models^{\rho} v_{m+1} \preceq^{\mathcal{V}} w_{m+1} : \tau_{m+1}$, $\eta'' = (\emptyset, \{\emptyset; [j \mapsto K[\textbf{new}\ w_1, \ldots, w_{m+1}, f'_{m+2}, \ldots, f'_M]]\})$ and $|W''| \overset{\text{rely}}{\sqsupseteq} U'$ and we can apply our original assumption. After this detour, we proceed with the proof proper:

8. Let $U' \overset{\text{rely}}{\sqsupseteq} U$.
9. Let $\bigwedge \overline{U' \models^{\rho} v \preceq^{\mathcal{V}} w : \tau}$.
10. Write $\eta' = (\emptyset, \{\emptyset; [j \mapsto K[\textbf{new}\ \overline{w}]]\})$.
11. Suffices to show $U', \eta' \models^{\rho} [i \mapsto \textbf{new}\ \overline{v}]@i\ \langle x.\ Q \rangle$.
12. Let $W' \overset{\text{rely}}{\sqsupseteq} U', \eta_f \# \eta'$.
13. Suppose $W'.k > 0$ and $h, \Sigma : W', \eta' \otimes \eta_f$.
14. $h; [i \mapsto \textbf{new}\ \overline{v}] \rightarrow h \uplus [\ell_1 \mapsto \overline{v}]; [i \mapsto \ell_1]$.
15. Pick ℓ_2 with $\forall h_2; T_2 \in \Sigma.\ \ell_2 \notin \mathrm{dom}(h_2)$.
16. Write $\Sigma = \{\emptyset; [j \mapsto K[\textbf{new}\ \overline{w}]]\} \otimes \Sigma_0$.
17. Write $\Sigma' = \{[\ell_2 \mapsto \overline{w}]; [j \mapsto K[\ell_2]]\} \otimes \Sigma_0$.
18. $\Sigma \rightrightarrows \Sigma'$.
19. Write $\eta'' = (\emptyset, \{\emptyset; [j \mapsto K[\ell_2]]\})$.
20. Pick $n \notin \mathrm{dom}(W'.\omega)$.
21. Write $P = \exists \overline{x}, \overline{y}.\ \bigwedge \overline{x \preceq^{\mathcal{V}} y : \tau} \wedge (\ell_1 \mapsto_{\mathrm{I}} (\overline{x}) * \ell_2 \mapsto_{\mathrm{S}} (\overline{y}))$.
22. Write $\iota = ((\{1\}, \emptyset, \emptyset, \lambda\_.\emptyset), [\![\lambda\_.P]\!]^{\rho}_{W'k}, 1, \emptyset)$.
23. Write $W'' = (W'.k, W.\omega \uplus [n \mapsto \iota])$.
24. $|W''| \models^{\rho} \ell_1 \preceq^{\mathcal{V}} \ell_2 : \textbf{ref}(\overline{\tau})$.
25. $\triangleright W'' \overset{\text{guar}}{\sqsupseteq} W'$.
26. $\triangleright |W''| \overset{\text{rely}}{\sqsupseteq} \triangleright |W'|$.

27. $\triangleright|W''| \overset{\text{rely}}{\sqsupseteq} U'$.
28. $h \uplus [\ell_1 \mapsto \overline{v}], \Sigma' : W'', \eta'' \otimes \eta_f$.
29. $h \uplus [\ell_1 \mapsto \overline{v}], \Sigma' : \triangleright W'', \eta'' \otimes \eta_f$ by Lem. 12 .
30. $\triangleright W'', \eta'' \models^\rho [i \mapsto \ell_1]@i \left\langle x.\ Q \right\rangle$.

□

3.3.2 Fork

Lemma 19.

$$\frac{\Delta; \Gamma \models e_1 \preceq e_2 : \mathbf{1}}{\Delta; \Gamma \models \mathsf{fork}\ e_1 \preceq \mathsf{fork}\ e_2 : \mathbf{1}}$$

Proof.

1. Let $U, \rho : \Delta \to \mathrm{VRel}, \gamma_1, \gamma_2 : \mathrm{dom}(\Gamma) \to \mathrm{Val}$.
2. Suppose $\forall x \in \mathrm{dom}(\Gamma).\ U \models^\rho \gamma_1(x) \preceq^{\mathcal{V}} \gamma_2(x) : \Gamma(x)$.
3. Write $e_1' = e_1[\gamma_1/\Gamma], e_2' = e_2[\gamma_2/\Gamma]$.
4. Let K, i, j.
5. Write $\eta = (\emptyset, \{\emptyset; [j \mapsto K[\mathsf{fork}\ e_2']]\})$.
6. Write $Q = \exists x_2.\ x_1 \preceq^{\mathcal{V}} x_2 : \mathbf{1} \wedge j \rightarrowtail_{\mathrm{S}} K[x_2]$.
7. Suffices to show $U, \eta \models^\rho [i \mapsto \mathsf{fork}\ e_1']@i \left\langle x_1.\ Q \right\rangle$. by Lem. 17.
8. Let $W \overset{\text{rely}}{\sqsupseteq} U, \eta_f \# \eta$.
9. Suppose $W.k > 0$ and $h, \Sigma : W, \eta \otimes \eta_f$.
10. $h; [i \mapsto \mathsf{fork}\ e_1'] \to h; [i \mapsto ()] \uplus [i' \mapsto e_1']$.
11. Pick j' with $\forall h'; T' \in \Sigma.\ j' \notin \mathrm{dom}(T')$.
12. Write $\Sigma = \{\emptyset; [j \mapsto K[\mathsf{fork}\ e_2']]\} \otimes \Sigma_0$.
13. Write $\Sigma' = \{\emptyset; [j \mapsto K[()]] \uplus [j' \mapsto e_2']\} \otimes \Sigma_0$.
14. $\Sigma \rightrightarrows \Sigma'$.
15. Write $\eta' = (\emptyset, \{\emptyset; [j \mapsto K[()]]\})$.
16. Write $\eta'' = (\emptyset, \{\emptyset; [j' \mapsto e_2']\})$.
17. $h, \Sigma' : W, \eta' \otimes \eta'' \otimes \eta_f$.
18. $h, \Sigma' : \triangleright W, \eta' \otimes \eta'' \otimes \eta_f$ by Lem. 12.
19. $\triangleright W, \eta' \models^\rho [i \mapsto ()]@i \left\langle x_1.\ Q \right\rangle$.
20. $|\triangleright W|, \eta'' \models^\rho [i' \mapsto e_1']@i' \left\langle x_1.\ \mathsf{tt} \right\rangle$ by assumption and Lem. 17. .
21. $\triangleright W, \eta' \otimes \eta'' \models^\rho [i \mapsto ()] \uplus [i' \mapsto e_1']@i \left\langle x_1.\ Q \right\rangle$ by Lem. 15.

□

3.3.3 Function Application and Abstraction

Lemma 20. For $U.k \neq 0$ we have $U \models^\rho \textbf{rec}\ f(x).e_1 \preceq^{\mathcal{V}} \textbf{rec}\ f(x).e_2 : \tau_1 \to \tau_2$ equivalent to

$$\begin{aligned}
&\forall w_1, w_2.\ \forall U' \overset{\text{rely}}{\sqsupseteq} \triangleright U.\\
&\quad \left[U' \models^\rho w_1 \preceq^{\mathcal{V}} w_2 : \tau_1\right]\\
&\quad\quad \Longrightarrow\\
&\quad \left[\forall K, j, i.\ U', (\emptyset, \{\emptyset; [j \mapsto K[e_2[\textbf{rec}\ f(x).e_2/f, w_2/x]]]\}) \models^\rho\right.\\
&\quad\quad \left.[i \mapsto e_1[\textbf{rec}\ f(x).e_1/f, w_1/x]]@i\ \left\langle x_1.\ \exists x_2.\ x_1 \preceq^{\mathcal{V}} x_2 : \tau_2 \wedge j \rightarrowtail_{\mathrm{S}} K[x_2]\right\rangle\right].
\end{aligned}$$

Lemma 21.

$$\frac{\Delta; \Gamma \models e_1 \preceq e_2 : \tau_1 \to \tau_2 \qquad \Delta; \Gamma \models f_1 \preceq f_2 : \tau_1}{\Delta; \Gamma \models e_1\ f_1 \preceq e_2\ f_2 : \tau_2}$$

Proof.

1. Let $U, \rho : \Delta \to \mathrm{VRel}, \gamma_1, \gamma_2 : \mathrm{dom}(\Gamma) \to \mathrm{Val}$.
2. Suppose $\forall x \in \mathrm{dom}(\Gamma).\ U \models^\rho \gamma_1(x) \preceq^{\mathcal{V}} \gamma_2(x) : \Gamma(x)$.
3. Write $e_1' = e_1[\gamma_1/\Gamma], e_2' = e_2[\gamma_2/\Gamma]$.
4. Write $f_1' = f_1[\gamma_1/\Gamma], f_2' = f_2[\gamma_2/\Gamma]$.
5. Let K, i, j.
6. Write $\eta = (\emptyset, \{\emptyset; [j \mapsto K[e_2'\ f_2']]\})$.
7. Write $Q = \exists x_2.\ x_1 \preceq^{\mathcal{V}} x_2 : \tau_2 \wedge j \rightarrowtail_{\mathrm{S}} K[x_2]$.
8. Suffices to show $U, \eta \models^\rho [i \mapsto e_1'\ f_1']@i\ \langle x_1.\ Q\rangle$.
   by Lem. 17.
9. Write $Q' = \exists x_2'.\ x_1' \preceq^{\mathcal{V}} x_2' : \tau_1 \to \tau_2 \wedge j \rightarrowtail_{\mathrm{S}} K[x_2'\ f_2']$.
10. $U, \eta \models^\rho [i \mapsto e_1']@i\ \langle x_1'.\ Q'\rangle$. by assumption and Lem. 17.
11. $U, \eta \models^\rho [i \mapsto e_1']@i\ \langle x_1'.\ Q' \wedge \cdot \overset{\text{rely}}{\sqsupseteq} U\rangle$ by Cor. 2.
12. Let $v_1' \in \mathrm{Val}$.
13. Let W', η' with $W', \eta' \models^\rho Q'[v_1'/x_1'] \wedge \cdot \overset{\text{rely}}{\sqsupseteq} U$.
14. Suffices to show $W', \eta' \models^\rho [i \mapsto v_1'\ f_1']@i\ \langle x_1.\ Q\rangle$
    by Lem. 16.
15. Suffices to show $|W'|, \eta' \models^\rho [i \mapsto v_1'\ f_1']@i\ \langle x_1.\ Q\rangle$
    by Cor. 1.
16. Suppose $W'.k > 0$ WLOG.
17. Pick v_2' with
    $|W'| \models^\rho v_1' \preceq^{\mathcal{V}} v_2' : \tau_1 \to \tau_2$,
    $\eta' = (\emptyset, \{\emptyset; [j \mapsto K[v_2'\ f_2']]\})$,
    $|W'| \overset{\text{rely}}{\sqsupseteq} U$.
18. Write $Q'' = \exists x_2''.\ x_1'' \preceq^{\mathcal{V}} x_2'' : \tau_1 \wedge j \rightarrowtail_{\mathrm{S}} K[v_2'\ x_2'']$.
19. $|W'|, \eta' \models^\rho [i \mapsto f_1']@i\ \langle x_1''.\ Q''\rangle$. by assumption and Lem. 17.
20. $|W'|, \eta' \models^\rho [i \mapsto f_1']@i\ \langle x_1''.\ Q'' \wedge \cdot \overset{\text{rely}}{\sqsupseteq} |W'|\rangle$
    by Cor. 2.
21. Let $v_1'' \in \mathrm{Val}$.

22. Let W'', η'' with $W'', \eta'' \models^\rho Q''[v_1''/x_1''] \wedge \cdot \overset{\text{rely}}{\sqsupseteq} |W'|$.
23. Suffices to show $W'', \eta'' \models^\rho [i \mapsto v_1'\ v_1'']@i \left\langle x_1.\ Q \right\rangle$ by Lem. 16.
24. Suffices to show $|W''|, \eta'' \models^\rho [i \mapsto v_1'\ v_1'']@i \left\langle x_1.\ Q \right\rangle$ by Cor. 1.
25. Suppose $W''.k > 0$ WLOG.
26. Pick v_2'' with
    $|W''| \models^\rho v_1'' \preceq^{\mathcal{V}} v_2'' : \tau_1$,
    $\eta'' = (\emptyset, \{\emptyset; [j \mapsto K[v_2'\ v_2'']]\})$,
    $|W''| \overset{\text{rely}}{\sqsupseteq} |W'|$.
27. Let $W''' \overset{\text{rely}}{\sqsupseteq} |W''|, \eta_f \# \eta''$.
28. Suppose $W'''.k > 0$ and $h, \Sigma : W''', \eta'' \otimes \eta_f$.
29. Write $v_1' = \textbf{rec}\ f(x).g_1'$ and $v_2' = \textbf{rec}\ f(x).g_2'$.
30. $h; [i \mapsto v_1'\ v_1''] \to h; [i \mapsto g_1'[v_1'/f, v_1''/x]]$.
31. Write $\Sigma = \{\emptyset; [j \mapsto K[v_2'\ v_2'']]\} \otimes \Sigma_0$.
32. Write $\Sigma' = \{\emptyset; [j \mapsto K[g_2'[v_2'/f, v_2''/x]]]\} \otimes \Sigma_0$.
33. $\Sigma \rightrightarrows \Sigma'$.
34. Write $\eta''' = (\emptyset, \{\emptyset; [j \mapsto K[g_2'[v_2'/f, v_2''/x]]]\})$.
35. $h, \Sigma : \triangleright W''', \eta'' \otimes \eta_f$ by Lem. 12.
36. $h, \Sigma' : \triangleright W''', \eta''' \otimes \eta_f$.
37. Suffices to show $\triangleright W''', \eta''' \models^\rho [i \mapsto g_1'[v_1'/f, v_1''/x]]@i \left\langle x_1.\ Q \right\rangle$.
38. Suffices to show $|\triangleright W'''|, \eta''' \models^\rho [i \mapsto g_1'[v_1'/f, v_1''/x]]@i \left\langle x_1.\ Q \right\rangle$ by Cor. 1.
39. Suppose $W'''.k > 0$ WLOG.
40. $|W'''| \models^\rho v_1' \preceq^{\mathcal{V}} v_2' : \tau_1 \to \tau_2$.
41. $|\triangleright W'''| = \triangleright |W'''|$.
42. $|\triangleright W'''| \models^\rho v_1'' \preceq^{\mathcal{V}} v_2'' : \tau_1$.
43. $|\triangleright W'''|, \eta''' \models^\rho [i \mapsto g_1'[v_1'/f, v_1''/x]]@i \left\langle x_1.\ Q \right\rangle$ by Lem. 20.

□

Lemma 22.

$$\frac{\Delta; \Gamma, f : \tau_1 \to \tau_2, x : \tau_1 \models e_1 \preceq e_2 : \tau_2}{\Delta; \Gamma \models \textbf{rec}\ f(x).e_1 \preceq \textbf{rec}\ f(x).e_2 : \tau_1 \to \tau_2}$$

Proof.

1. Let $U, \rho : \Delta \to \mathrm{VRel}, \gamma_1, \gamma_2 : \mathrm{dom}(\Gamma) \to \mathrm{Val}$.
2. Suppose $\forall x \in \mathrm{dom}(\Gamma).\ U \models^\rho \gamma_1(x) \preceq^{\mathcal{V}} \gamma_2(x) : \Gamma(x)$.
3. Write $e_1' = e_1[\gamma_1/\Gamma], e_2' = e_2[\gamma_2/\Gamma]$.
4. Let K, i, j.
5. Write $\eta = (\emptyset, \{\emptyset; [j \mapsto K[\textbf{rec}\ f(x).e_2']]\})$.
6. Write $Q = \exists x_2.\ x_1 \preceq^{\mathcal{V}} x_2 : \tau_1 \to \tau_2 \wedge j \rightarrowtail_{\mathrm{S}} K[x_2]$.

7. Suffices to show $U, \eta \models^\rho [i \mapsto \textbf{rec}\ f(x).e_1'] @ i \left\langle x_1.\ Q \right\rangle$.
by Lem. 17.
8. Suffices to show $U, \eta \models^\rho Q[\textbf{rec}\ f(x).e_1'/x_1]$.
9. Suffices to show $U \models^\rho \textbf{rec}\ f(x).e_1' \preceq^{\mathcal{V}} \textbf{rec}\ f(x).e_2' : \tau_1 \to \tau_2$.
10. Suffices to show $\forall U' \overset{\text{rely}}{\sqsupseteq} U.\ U' \models^\rho \textbf{rec}\ f(x).e_1' \preceq^{\mathcal{V}} \textbf{rec}\ f(x).e_2' : \tau_1 \to \tau_2$.
11. Proceed by induction on $U'.k$.
12. Suppose $U'.k > 0$.
13. Let $w_1, w_2, U'' \overset{\text{rely}}{\sqsupseteq} \triangleright U'$ with $U'' \models^\rho w_1 \preceq^{\mathcal{V}} w_2 : \tau_1$.
14. Let K, j, i.
15. Write $\eta' = (\emptyset, \{\emptyset; [j \mapsto K[e_2'[\textbf{rec}\ f(x).e_2'/f, w_2/x]]]\})$.
16. Write $Q' = \exists x_2.\ x_1 \preceq^{\mathcal{V}} x_2 : \tau_2 \wedge j \rightarrowtail_{\mathrm{S}} K[x_2]$.
17. Suffices to show $U'', \eta' \models^\rho [i \mapsto e_1'[\textbf{rec}\ f(x).e_1'/f, w_1/x]] @ i \left\langle x_1.\ Q' \right\rangle$
by Lem. 20.
18. $\forall x \in \mathrm{dom}(\Gamma).\ U'' \models^\rho \gamma_1(x) \preceq^{\mathcal{V}} \gamma_2(x) : \Gamma(x)$.
19. $U'' \models^\rho \textbf{rec}\ f(x).e_1' \preceq^{\mathcal{V}} \textbf{rec}\ f(x).e_2' : \tau_1 \to \tau_2$
by induction hypothesis.
20. $U'', \eta' \models^\rho [i \mapsto e_1'[\textbf{rec}\ f(x).e_1'/f, w_1/x]] @ i \left\langle x_1.\ Q' \right\rangle$
by assumption and Lem. 17.

□

3.3.4 CAS

Lemma 23. For $U.k \neq 0$ we have that $U \models^\rho \ell_1 \preceq^{\mathcal{V}} \ell_2 : \textbf{ref}(\overline{\tau})$ implies the existence of an $i \in \mathrm{dom}(U.\omega)$ such that we have

$$\mathcal{I}[\![U.\omega(i)]\!]_U = \left\{([\ell_1 \mapsto \overline{v_1}], \{[\ell_2 \mapsto \overline{v_2}]; \emptyset\}) \mid \bigwedge \overline{\triangleright U \models^\rho v_1 \preceq^{\mathcal{V}} v_2 : \tau}\right\}.$$

Lemma 24. Assume that we have $U \models^\rho v_1 \preceq^{\mathcal{V}} v_2 : \sigma$ and $U \models^\rho w_1 \preceq^{\mathcal{V}} w_2 : \sigma$. If $U.k \neq 0$ and there are η, h and Σ such that $h, \Sigma : U, \eta$ holds, then we have that

$$v_1 = w_1 \iff v_2 = w_2.$$

Lemma 25.

$$\frac{\tau_n = \sigma \qquad \begin{array}{c}\Delta; \Gamma \models e_1 \preceq e_2 : \textbf{ref}(\overline{\tau}) \\ \Delta; \Gamma \models f_1 \preceq f_2 : \sigma \end{array} \qquad \Delta; \Gamma \models g_1 \preceq g_2 : \sigma}{\Delta; \Gamma \models \mathsf{CAS}(e_1[n], f_1, g_1) \preceq \mathsf{CAS}(e_2[n], f_2, g_2) : \mathbf{B}}$$

Proof.

1. Let $U, \rho : \Delta \to \mathrm{VRel}, \gamma_1, \gamma_2 : \mathrm{dom}(\Gamma) \to \mathrm{Val}$.
2. Suppose $\forall x \in \mathrm{dom}(\Gamma).\ U \models^\rho \gamma_1(x) \preceq^{\mathcal{V}} \gamma_2(x) : \Gamma(x)$.
3. Write $e_1' = e_1[\gamma_1/\Gamma], e_2' = e_2[\gamma_2/\Gamma]$.
4. Write $f_1' = f_1[\gamma_1/\Gamma], f_2' = f_2[\gamma_2/\Gamma]$.
5. Write $g_1' = g_1[\gamma_1/\Gamma], g_2' = g_2[\gamma_2/\Gamma]$.
6. Let K, i, j.
7. Write $\eta = (\emptyset, \{\emptyset; [j \mapsto K[\mathsf{CAS}(e_2'[n], f_2', g_2')]]\})$.
8. Write $Q = \exists x_2.\ x_1 \preceq^{\mathcal{V}} x_2 : \mathbf{B} \wedge j \rightarrowtail_{\mathrm{S}} K[x_2]$.

9. Suffices to show $U, \eta \models^\rho [i \mapsto \mathsf{CAS}(e_1'[n], f_1', g_1')]@i \left\langle x_1.\ Q \right\rangle$.
by Lem. 17.
10. Write $Q' = \exists x_2'.\ x_1' \preceq^{\mathcal{V}} x_2' : \mathbf{ref}(\overline{\tau}) \wedge j \rightarrowtail_{\mathrm{S}} K[\mathsf{CAS}(x_2'[n], f_2', g_2')]$.
11. $U, \eta \models^\rho [i \mapsto e_1']@i \left\langle x_1'.\ Q' \right\rangle$. by assumption and Lem. 17.
12. $U, \eta \models^\rho [i \mapsto e_1']@i \left\langle x_1'.\ Q' \wedge \cdot \overset{\text{rely}}{\sqsupseteq} U \right\rangle$. by Cor. 2.
13. Let $v_1' \in \mathrm{Val}$.
14. Let W', η' with $W', \eta' \models^\rho Q'[v_1'/x_1'] \wedge \cdot \overset{\text{rely}}{\sqsupseteq} U$.
15. Suffices to show $W', \eta' \models^\rho [i \mapsto \mathsf{CAS}(v_1'[n], f_1', g_1')]@i \left\langle x_1.\ Q \right\rangle$
by Lem. 16.
16. Suffices to show $|W'|, \eta' \models^\rho [i \mapsto \mathsf{CAS}(v_1'[n], f_1', g_1')]@i \left\langle x_1.\ Q \right\rangle$
by Cor. 1.
17. Suppose $W'.k > 0$ WLOG.
18. Pick v_2' with
$|W'| \models^\rho v_1' \preceq^{\mathcal{V}} v_2' : \mathbf{ref}(\overline{\tau})$,
$\eta' = (\emptyset, \{\emptyset; [j \mapsto K[\mathsf{CAS}(v_2'[n], f_2', g_2')]]\})$,
$|W'| \overset{\text{rely}}{\sqsupseteq} U$.
19. Write $Q'' = \exists x_2''.\ x_1'' \preceq^{\mathcal{V}} x_2'' : \sigma \wedge j \rightarrowtail_{\mathrm{S}} K[\mathsf{CAS}(v_2'[n], x_2'', g_2')]$.
20. $|W'|, \eta' \models^\rho [i \mapsto f_1']@i \left\langle x_1''.\ Q'' \right\rangle$ by assumption and Lem. 17.
21. $|W'|, \eta' \models^\rho [i \mapsto f_1']@i \left\langle x_1''.\ Q'' \wedge \cdot \overset{\text{rely}}{\sqsupseteq} |W'| \right\rangle$
by Cor. 2.
22. Let $v_1'' \in \mathrm{Val}$.
23. Let W'', η'' with $W'', \eta'' \models^\rho Q''[v_1''/x_1''] \wedge \cdot \overset{\text{rely}}{\sqsupseteq} |W'|$.
24. Suffices to show $W'', \eta'' \models^\rho [i \mapsto \mathsf{CAS}(v_1'[n], v_1'', g_1')]@i \left\langle x_1.\ Q \right\rangle$
by Lem. 16.
25. Suffices to show $|W''|, \eta'' \models^\rho [i \mapsto \mathsf{CAS}(v_1'[n], v_1'', g_1')]@i \left\langle x_1.\ Q \right\rangle$
by Cor. 1.
26. Suppose $W''.k > 0$ WLOG.
27. Pick v_2'' with
$|W''| \models^\rho v_1'' \preceq^{\mathcal{V}} v_2'' : \sigma$,
$\eta'' = (\emptyset, \{\emptyset; [j \mapsto K[\mathsf{CAS}(v_2'[n], v_2'', g_2')]]\})$,
$|W''| \overset{\text{rely}}{\sqsupseteq} |W'|$.
28. Write $Q''' = \exists x_2'''.\ x_1''' \preceq^{\mathcal{V}} x_2''' : \sigma \wedge j \rightarrowtail_{\mathrm{S}} K[\mathsf{CAS}(v_2'[n], v_2'', x_2''')]$.
29. $|W''|, \eta' \models^\rho [i \mapsto g_1']@i \left\langle x_1'''.\ Q''' \right\rangle$ by assumption and Lem. 17.
30. $|W''|, \eta' \models^\rho [i \mapsto g_1']@i \left\langle x_1'''.\ Q'' \wedge \cdot \overset{\text{rely}}{\sqsupseteq} |W'| \right\rangle$
by Cor. 2.
31. Let $v_1''' \in \mathrm{Val}$.
32. Let W''', η''' with $W''', \eta''' \models^\rho Q'''[v_1'''/x_1'''] \wedge \cdot \overset{\text{rely}}{\sqsupseteq} |W''|$.
33. Suffices to show $W''', \eta''' \models^\rho [i \mapsto \mathsf{CAS}(v_1'[n], v_1'', v_1''')]@i \left\langle x_1.\ Q \right\rangle$
by Lem. 16.
34. Suffices to show $|W'''|, \eta''' \models^\rho [i \mapsto \mathsf{CAS}(v_1'[n], v_1'', v''')]@i \left\langle x_1.\ Q \right\rangle$
by Cor. 1.

35. Suppose $W'''.k > 0$ WLOG.
36. Pick v_2''' with
    $|W'''| \models^\rho v_1''' \preceq^{\mathcal{V}} v_2''' : \sigma$,
    $\eta''' = (\emptyset, \{\emptyset; [j \mapsto K[\mathsf{CAS}(v_2'[n], v_2'', v_2''')]]\})$,
    $|W'''| \overset{\text{rely}}{\sqsupseteq} |W''|$.
37. Let $W'''' \overset{\text{rely}}{\sqsupseteq} W''', \eta_f \# \eta'''$.
38. Suppose $W''''.k > 0$ and $h, \Sigma : W'''', \eta''' \otimes \eta_f$.
39. Suppose $h; [i \mapsto \mathsf{CAS}(v_1'[n], v_1'', v''')] \to h'; T'$.
40. Suppose $W''''.k > 1$ WLOG.
41. Write $v_1' = \ell_1$ and $v_2' = \ell_2$.
42. Pick $\overline{v}, \overline{w}, h_0, \Sigma_0$ with
    $\bigwedge \overline{\triangleright |W''''| \models^\rho v \preceq^{\mathcal{V}} w : \tau}$,
    $h = [\ell_1 \mapsto \overline{v}] \uplus h_0$,
    $\Sigma = \{[\ell_2 \mapsto \overline{w}]; \emptyset\} \otimes \{\emptyset; [j \mapsto K[\mathsf{CAS}(v_2'[n], v_2'', v_2''')]]\} \otimes \Sigma_0$
    by Lem. 23.
43. $\triangleright |W''''| \models^\rho v_n \preceq^{\mathcal{V}} w_n : \sigma$.
44. $\triangleright |W''''| \models^\rho v_1'' \preceq^{\mathcal{V}} v_2'' : \sigma$.
45. $h, \Sigma : \triangleright |W''''|, \eta''' \otimes \eta_f$ by Lem. 12.
46. $v_n = v_1'' \Leftrightarrow w_n = v_2''$ by Lem. 24.

Case $\boxed{v_n = v_1'' \wedge w_n = v_2''}$

47. Write $v_n^\dagger = v_1'''$ and $v_m^\dagger = v_m, m \neq n$.
48. $h' = h[\ell_1 \mapsto \overline{v^\dagger}]$.
49. $T' = [i \mapsto \mathbf{true}]$.
50. Write $w_n^\dagger = v_2'''$ and $w_m^\dagger = w_m, m \neq n$.
51. Write $\Sigma' = \{[\ell_2 \mapsto \overline{w^\dagger}]; \emptyset\} \otimes \{\emptyset; [j \mapsto K[\mathbf{true}]]\} \otimes \Sigma_0$.
52. $\Sigma \rightrightarrows \Sigma'$.
53. Write $\eta'''' = (\emptyset, \{\emptyset; [j \mapsto K[\mathbf{true}]]\})$.
54. $\bigwedge \overline{\triangleright |W''''| \models^\rho v^\dagger \preceq^{\mathcal{V}} w^\dagger : \tau}$.
55. $h', \Sigma' : W'''', \eta'''' \otimes \eta_f$.
56. $h', \Sigma' : \triangleright W'''', \eta'''' \otimes \eta_f$ by Lem. 12.
57. $\triangleright W'''', \eta'''' \models^\rho [i \mapsto \mathbf{true}]@i \, \langle x_1.\, Q \rangle$.

Case $\boxed{v_n \neq v_1'' \wedge w_n \neq v_2''}$

58. $h' = h$.
59. $T' = [i \mapsto \mathbf{false}]$.
60. Write $\Sigma' = \{[\ell_2 \mapsto \overline{w}]; \emptyset\} \otimes \{\emptyset; [j \mapsto K[\mathbf{false}]]\} \otimes \Sigma_0$.
61. $\Sigma \rightrightarrows \Sigma'$.
62. Write $\eta'''' = (\emptyset, \{\emptyset; [j \mapsto K[\mathbf{false}]]\})$.
63. $h, \Sigma' : W'''', \eta'''' \otimes \eta_f$.

64. $h, \Sigma' : \triangleright W'''', \eta'''' \otimes \eta_f$ by Lem. 12 .

65. $\triangleright W'''', \eta'''' \models^\rho [i \mapsto \mathsf{false}]@i \left\langle x_1.\ Q \right\rangle$.

□

3.4 May-refinement

Theorem 1 (May-refinement)**.** Suppose $\cdot;\cdot \models e_1 \preceq e_2 : \mathbf{N}$ holds. Let h_1, h_2, i, j and n be arbitrary. If we have

$$\exists h_1', T_1.\ h_1; [i \mapsto e_1] \to^* h_1'; [i \mapsto n] \uplus T_1$$

then we also have

$$\exists h_2', T_2.\ h_2; [j \mapsto e_2] \to^* h_2'; [j \mapsto n] \uplus T_2.$$

Proof. Let M be the number of steps in the assumed reduction. Write

$$h_1; [i \mapsto e_1] = h_1^0; T_1^0 \to h_1^1; T_1^1 \to \cdots \to h_1^M; T_1^M = h_1'; [i \mapsto n] \uplus T_1.$$

We proceed to prove by induction the claim that for all $0 \le m \le M$ there are W_m, $\eta_m \# \eta$, and Σ_m with the following properties, where $\eta = (h_1, \{h_2; \emptyset\})$ and $\Sigma = \{h_2; [j \mapsto e_2]\}$:

- $W_m, \eta_m \models^\rho T_1^m@i \left\langle x_1.\ \exists x_2.\ x_1 \preceq^{\mathcal{V}} x_2 : \mathbf{N} \wedge j \rightarrowtail_{\mathrm{S}} x_2 \right\rangle$.
- $h_1^m, \Sigma_m : W_m, \eta_m \otimes \eta$.
- $W_m.k = 1 + M - m$.
- $\Sigma \rightrightarrows \Sigma_m$.

Let us initially consider the base case $m = 0$. We choose $W_0 = (1 + M, \emptyset)$, $\eta_0 = (\emptyset, \{\emptyset; [j \mapsto e_2]\})$ and $\Sigma_0 = \{h_2; [j \mapsto e_2]\}$. The different properties are easily verified; the only nontrivial is the first and that follows from the initial assumption $\cdot;\cdot \models e_1 \preceq e_2 : \mathbf{N}$. The induction step comes down to unrolling the definition of threadpool triples; we omit the details.

Instantiating our claim at $m = M$ now gives us W_M, $\eta_M \# \eta$, and Σ_M such that:

- $W_M, \eta_M \models^\rho [i \mapsto n] \uplus T_1@i \left\langle x_1.\ \exists x_2.\ x_1 \preceq^{\mathcal{V}} x_2 : \mathbf{N} \wedge j \rightarrowtail_{\mathrm{S}} x_2 \right\rangle$.
- $h_1', \Sigma_M : W_M, \eta_M \otimes \eta$.
- $W_M = 1$.
- $\Sigma \rightrightarrows \Sigma_M$.

A final unrolling of the definition of threadpool triples and a few calculations gives us Σ' with $\Sigma \rightrightarrows \Sigma'$ and $\Sigma' = \{\emptyset; [j \mapsto n]\} \otimes \Sigma_0$. All that remains is to pick an element from the nonempty set Σ'. □

4 The atomic triple

$$h; e \text{ atomic} \triangleq \forall h', e'.\ h; e \hookrightarrow h'; e' \implies e' \text{ a value} \quad \wedge \quad h; e \hookrightarrow -$$

$$\begin{array}{l}\rho\colon (\!|P|\!)\ e\ (\!|x.\ Q|\!) \triangleq \forall W, \eta \models^\rho P.\ \forall \eta_F \# \eta.\ W.k > 0 \text{ implies} \\ \quad (\eta \otimes \eta_F).h; e \text{ atomic and if } (\eta \otimes \eta_F).h; e \hookrightarrow h'; v \text{ then } \exists \eta' \# \eta_F. \\ \quad\quad h' = (\eta' \otimes \eta_F).h, \quad (\eta \otimes \eta_F).\Sigma \rightrightarrows (\eta' \otimes \eta_F).\Sigma, \quad \triangleright W, \eta' \models^\rho Q[v/x]\end{array}$$

$$\frac{\begin{array}{c}P, Q \text{ token-pure} \\ \forall \iota \overset{\text{rely}}{\sqsupseteq} \iota_0.\ \exists \iota' \overset{\text{guar}}{\sqsupseteq} \iota.\ \exists Q.\ \ \rho : (\iota' \wedge Q \Rightarrow R) \ \wedge\ \rho\colon (\!|\triangleright \iota. J(\iota.s) * P|\!)\ e\ (\!|x.\ \triangleright \iota'. J(\iota'.s) * Q|\!)\end{array}}{\rho\colon \langle \iota_0 \wedge P \rangle\ e\ \langle x.\ R \rangle}$$

Lemma 26 (Island focus)**.** The island-focusing inference rule is sound.

Proof.

1. Fix i and $W_0, \eta \models^\rho \iota_0 \wedge P$
2. Suffices to show $W_0, \eta \models^\rho [i \mapsto e]@i\ \langle x.\ R \rangle$
3. Fix $W \overset{\text{rely}}{\sqsupseteq} W_0$ and $\eta_F \# \eta$
4. Suppose $W.k > 0$ and $h, \Sigma : W, \eta \otimes \eta_F$
5. $W, \eta \models^\rho \iota_0$
6. $\exists \iota \overset{\text{rely}}{\sqsupseteq} \iota_0, \omega_F, j.$
   $W.\omega = \omega_F \uplus [j \mapsto [\![\iota]\!]^\rho_{W.k}]$
7. $\exists \eta_\iota, \eta'_F.$ — by semantics of world satisfaction
   $(h, \Sigma) = \eta_\iota \otimes \eta \otimes \eta_F \otimes \eta'_F$
   $\eta_\iota \in \mathcal{I}[\![[\![\iota]\!]^\rho_{W.k}]\!]_W, \quad \eta'_F \in \mathcal{I}[\![\omega_F]\!]_W$
8. $\triangleright |W|, \eta_\iota \models \iota. J(\iota.s)$
9. $\exists \iota' \overset{\text{guar}}{\sqsupseteq} \iota,\ Q.$ — by assumption
   $\rho : (\iota' \wedge Q \Rightarrow R)$
   $\rho\colon (\!|\triangleright \iota. J(\iota.s) * P|\!)\ e\ (\!|x.\ \triangleright \iota'. J(\iota'.s) * Q|\!)$
10. Let $\widehat{\eta} = \eta \otimes \eta_\iota$
11. $|W|, \widehat{\eta} \models^\rho \triangleright \iota. J(\iota.s) * P$ — by (1, 8), token-purity of P
12. Let $\widehat{\eta}_F = \eta_F \otimes \eta'_F$
13. $h = (\widehat{\eta} \otimes \widehat{\eta}_F).h$
14. $h; e$ atomic — by (9)
15. Suppose $h; [i \mapsto e] \to h'; T$
16. Suppose $W.k > 1$ WLOG
17. $\exists v.\ h; e \hookrightarrow h'; v,\ T = [i \mapsto v]$ — by inversion, (14)
18. $\exists \widehat{\eta}'.\ h' = (\widehat{\eta}' \otimes \widehat{\eta}_F).h,$ — by (9)
    $(\widehat{\eta} \otimes \widehat{\eta}_F).\Sigma \rightrightarrows (\widehat{\eta}' \otimes \widehat{\eta}_F).\Sigma,$
    $\triangleright |W|, \widehat{\eta}' \models^\rho \triangleright \iota'. J(\iota'.s) * Q[v/x]$
19. Let $W' = (W.k - 1, \lfloor \omega_F \rfloor_{W.k-1} \uplus [j \mapsto [\![\iota']\!]^\rho_{W.k-1}])$
20. $W' \overset{\text{guar}}{\sqsupseteq} W$
21. $\triangleright |W'| \overset{\text{rely}}{\sqsupseteq} |W'| \overset{\text{rely}}{\sqsupseteq} |\triangleright W| = \triangleright |W|$

22. $\exists \eta'_\iota, \eta'.\ \widehat{\eta}' = \eta' \otimes \eta'_\iota,$ $\triangleright |W'|, \eta'_\iota \models \iota'.J(\iota'.s),$ $W', \eta' \models^\rho Q[v/x]$ — by semantics of assertions, token-purity of Q

23. Write $\Sigma' = (\widehat{\eta}' \otimes \widehat{\eta}_F).\Sigma$

24. $\Sigma \rightrightarrows \Sigma'$

25. $h', \Sigma' : W', \eta' \otimes \eta_F$ — by (7, 22)

26. $W', \eta' \models^\rho Q[v/x] \wedge \iota'$

27. $W', \eta' \models^\rho R[v/x]$ — by (9)

28. $W', \eta' \models^\rho [i \mapsto v]@i \left\langle x.\ R \right\rangle$

□

5 Hoare-style reasoning

The following inference rules are sound:

$$\frac{\langle P\rangle\ e\ \langle x.\ P'\rangle \qquad \langle P'\rangle\ e'\ \langle y.\ P''\rangle}{\langle P\rangle\ \mathbf{let}\ x = e\ \mathbf{in}\ e'\ \langle y.\ P''\rangle} \qquad \frac{P \Rightarrow P' \qquad \langle P'\rangle\ e\ \langle x.\ Q'\rangle \qquad Q' \Rightarrow Q}{\langle P\rangle\ e\ \langle x.\ Q\rangle}$$

$$\frac{\triangleright P \Rightarrow P}{P} \qquad \frac{\langle P\rangle\ e\ \langle x.\ Q\rangle}{\langle P * R\rangle\ e\ \langle x.\ Q * R\rangle} \qquad \frac{P \Rightarrow \langle Q\rangle\ e\ \langle x.\ R\rangle}{\langle P \wedge Q\rangle\ e\ \langle x.\ R\rangle}$$

$$\frac{\langle P_1\rangle\ e\ \langle x.\ Q_1\rangle \qquad \langle P_2\rangle\ e\ \langle x.\ Q_2\rangle}{\langle P_1 \vee P_2\rangle\ e\ \langle x.\ Q_1 \vee Q_2\rangle} \qquad \langle \mathsf{emp}\rangle\ \mathsf{cons}(x, y)\ \langle z.\ z \mapsto_{\mathrm{I}} (x, y)\rangle$$

$$\langle \mathsf{emp}\rangle\ v\ \langle x.\ x = v \wedge \mathsf{emp}\rangle \qquad \frac{(\!|P|\!)\ e\ (\!|x.\ \diamond Q|\!)}{(\!|P|\!)\ e\ (\!|x.\ Q|\!)}$$

Note: the rule of conjunction

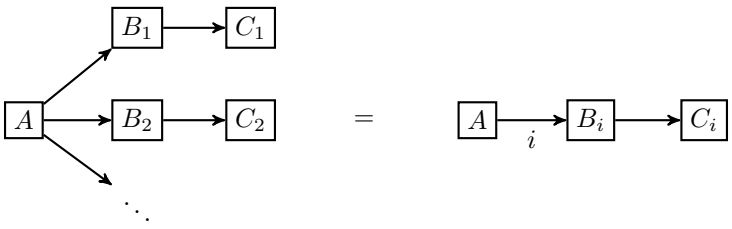
$$\frac{\langle P_1\rangle\ e\ \langle x.\ Q_1\rangle \qquad \langle P_2\rangle\ e\ \langle x.\ Q_2\rangle}{\langle P_1 \wedge P_2\rangle\ e\ \langle x.\ Q_1 \wedge Q_2\rangle}$$

is *probably unsound* in our logic (as it is with many concurrent separation logics), because, in the termination branch of the triple, the splitting of resources between the main thread and remaining threads can be chosen arbitrarily.

An assertion P is *token-pure* if $W, \eta \models^{\rho} P$ iff $|W|, \eta \models^{\rho} P$.

6 Examples

We use a compact notation to draw structured branches:



6.1 Counters

```
counter: 1 → { inc : 1 → N }
counter ≜ λ(). let c = new 0 in {
  inc = rec try().
    let x = get c in
    if CAS(c, x, x + 1) then x else try()
}
```

The ref island suffices.

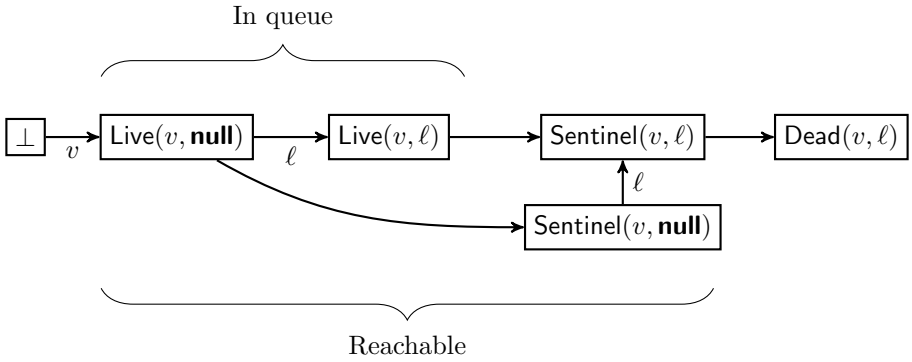
6.2 Michael-Scott queue

```
MSQ: ∀α.1 → { enq : α → 1, deq : 1 → ref?(α) }
MSQ ≜ Λ. λ(). let head = new cons(null, null) in {
  enq = λx. let n = cons(some(new x), null) in
            let rec try(c). case c[2] of
                null ⇒ if CAS(c[2], null, n) then () else try(c)
              | some(c′) ⇒ try(c′)
            in try(getVal(head[1])),
  deq = rec try().
    let c = head[1], n = getVal(c) in case n[2] of
        null ⇒ null        (* queue is empty *)
      | some(n′) ⇒ if CAS(head[1], c, n′) then n′[1] else try()
}
```

```
CGQ: ∀α.1 → { enq : α → 1, deq : 1 → ref?(α) }
CGQ ≜ Λ. λ(). let head = new null, lock = new false in {
  deq = λ(). sync(lock) { case head[1] of }
    null ⇒ null  |  some(n) ⇒ head[1] := n[2]; some(new n[1]),
  enq = λx. sync(lock) {  }
            case head[1] of
                null ⇒ head[1] := cons(x, null)
              | some(c) ⇒
                let rec try(c). case c[2] of
                    null ⇒ c[2] := cons(x, null)
                  | some(c′) ⇒ try(c′)
                in try(c)
}
```

Each instance of the MSQueue will have a fixed location for the head reference, and similarly on the specification side. We fix these as head_I and head_S respectively. In addition, the spec has a lock, lock, of type **ref**(**B**).

For every location ℓ, we have an instance of the following transition system, which tracks the life story of a node at that location:



This leads us to our state space—S_0 is per-location, while S gives the state space for the whole island:

$$\begin{aligned} S_0 &\triangleq \{\bot\} \\ &\cup \{\mathsf{Live}(v, v') \mid v, v' \in \mathrm{Val}\} \\ &\cup \{\mathsf{Sentinel}(v, v') \mid v, v' \in \mathrm{Val}\} \\ &\cup \{\mathsf{Dead}(v, \ell) \mid v \in \mathrm{Val}, \ell \in \mathrm{Loc}\} \\ S &\triangleq \mathrm{Loc} \xrightarrow{\text{fin}} S_0 \end{aligned}$$

The notation $\overset{\text{fin}}{\rightharpoonup}$ here means that a state only maps finitely-many locations to a non-$\bot$ state—so $\bot$ is a useful pun.

We define $\rightsquigarrow_0$ according to the transition system drawn above, and lift this pointwise to $\rightsquigarrow$. There are no tokens in this example, so all that's left is the interpretation. We use a pattern-matching notation on states s of the space S; when nothing matches, the interpretation is ff.

$$\begin{array}{rcl} I(s) & \triangleq & \mathsf{head}_{\mathrm{I}} \mapsto_{\mathrm{I}} \ell_0 * \ell_0 \mapsto_{\mathrm{I}} (v_0, v_{\mathrm{I}}) * \mathsf{lock} \mapsto_{\mathrm{S}} \textbf{false} * \exists v_{\mathrm{S}}.\ \mathsf{head}_{\mathrm{S}} \mapsto_{\mathrm{S}} v_{\mathrm{S}} * \mathsf{link}(v_{\mathrm{I}}, v_{\mathrm{S}}, s_L) \\ & * & \ast_{\ell \in \mathrm{dom}(s_D)} \exists v, \ell'. s_D(\ell) = \mathsf{Dead}(v, \ell') * s(\ell') \neq \bot * \ell \mapsto_{\mathrm{I}} (v, \ell') \\ \multicolumn{3}{r}{\text{when } s = [\ell_0 \mapsto \mathsf{Sentinel}(v_0, v_{\mathrm{I}})] \uplus s_L \uplus s_D} \end{array}$$

$$\begin{array}{rcl} \mathsf{link}(\textbf{null}, \textbf{null}, \emptyset) & \triangleq & \mathsf{emp} \\ \mathsf{link}(\ell_{\mathrm{I}}, \ell_{\mathrm{S}}, [\ell_{\mathrm{I}} \mapsto \mathsf{Live}(v_{\mathrm{I}}, v'_{\mathrm{I}})] \uplus s) & \triangleq & \exists \ell, v_{\mathrm{S}}, v'_{\mathrm{S}}.\ \ell \mapsto_{\mathrm{I}} v_{\mathrm{I}} * v_{\mathrm{I}} \preceq^{\mathcal{V}} v_{\mathrm{S}} : \alpha \\ & * & \ell_{\mathrm{I}} \mapsto_{\mathrm{I}} (\ell, v'_{\mathrm{I}}) * \ell_{\mathrm{S}} \mapsto_{\mathrm{I}} (v_{\mathrm{S}}, v'_{\mathrm{S}}) * \mathsf{link}(v'_{\mathrm{I}}, v'_{\mathrm{S}}, s) \end{array}$$

Let Θ be the transition system above. We use the shorthand

$$x \propto s_0 \quad \triangleq \quad (\theta, I, [x \mapsto s_0], \emptyset)$$

for $s_0 \in S_0$.

6.2.1 Proof outline for enq

Stable assumptions: $(\theta, I, \emptyset, \emptyset)$ and $x_{\mathrm{I}} \preceq^{\mathcal{V}} x_{\mathrm{S}} : \alpha$.
Let $P \triangleq j \rightarrowtail_{\mathrm{S}} K[\mathsf{enq}_{\mathrm{S}}(x_{\mathrm{S}})]$.

$\langle P \rangle$
  let $n = \mathsf{cons}(\textbf{some}(\textbf{new}\ x_{\mathrm{I}}), \textbf{null})$ **in**
$\langle P * \exists \ell.\ n \mapsto_{\mathrm{I}} (\ell, \textbf{null}) * \ell \mapsto_{\mathrm{I}} x_{\mathrm{I}} \rangle$
  let rec $\mathsf{try}(c)$.
  $\langle P * c \propto \mathsf{Live}(-,-) * \exists \ell.\ n \mapsto_{\mathrm{I}} (\ell, \textbf{null}) * \ell \mapsto_{\mathrm{I}} x_{\mathrm{I}} \rangle$
    case $c[2]$ **of**
      null $\Rightarrow \langle P * c \propto \mathsf{Live}(-,-) * \exists \ell.\ n \mapsto_{\mathrm{I}} (\ell, \textbf{null}) * \ell \mapsto_{\mathrm{I}} x_{\mathrm{I}} \rangle$
        if $\mathsf{CAS}(c[2], \textbf{null}, n)$
        then $\langle n \propto \mathsf{Live}(x_{\mathrm{I}}, \textbf{null}) * j \rightarrowtail_{\mathrm{S}} K[()] \rangle$
          $()$
        else $\langle P * c \propto \mathsf{Live}(-,-) * \exists \ell.\ n \mapsto_{\mathrm{I}} (\ell, \textbf{null}) * \ell \mapsto_{\mathrm{I}} x_{\mathrm{I}} \rangle$
          $\mathsf{try}(c)$
      | **some**$(c') \Rightarrow \langle P * c' \propto \mathsf{Live}(-,-) * \exists \ell.\ n \mapsto_{\mathrm{I}} (\ell, \textbf{null}) * \ell \mapsto_{\mathrm{I}} x_{\mathrm{I}} \rangle$
        $\mathsf{try}(c')$
  $\langle \mathsf{ret}.\ \mathsf{ret} = () \wedge j \rightarrowtail_{\mathrm{S}} K[()] \rangle$
  in $\mathsf{try}(\mathsf{getVal}(\mathsf{head}[1]))$
$\langle \mathsf{ret}.\ \mathsf{ret} = () \wedge j \rightarrowtail_{\mathrm{S}} K[()] \rangle$

At the atomic triple level, the reasoning depends on a case analysis of rely-future states, which determine in particular whether the CAS succeeds. Because Dead nodes must have non-**null** next pointers, the CAS will never succeed on a Dead node.

6.2.2 Proof outline for deq

Stable assumptions: $(\theta, I, \emptyset, \emptyset)$.
Let $P \triangleq j \rightarrowtail_{\mathrm{S}} K[\mathsf{deq}_{\mathrm{S}}()]$.

$\langle P \rangle$
  rec try().
  $\langle P \rangle$
    let $c = \mathsf{head}[1]$ **in**
  $\langle c \propto \mathsf{Sentinel}(-,-) \rangle$
    let $n = \mathsf{getVal}(c)$ **in**
  $\langle n \propto \mathsf{Sentinel}(-,-) \rangle$
    case $n[2]$ **of**
      null $\Rightarrow \langle j \rightarrowtail_{\mathrm{S}} K[\mathbf{null}] \rangle$
        null
      | **some**$(n') \Rightarrow \langle P * n \propto \mathsf{Sentinel}(-, n') \rangle$
        if $\mathsf{CAS}(\mathsf{head}[1], c, n')$
        then $\left\langle \begin{array}{l} n \propto \mathsf{Dead}(-, n') * \exists \ell_{\mathrm{I}}, v_{\mathrm{I}}, \ell_{\mathrm{S}}, v_{\mathrm{S}}.\ v_{\mathrm{I}} \preceq^{\mathcal{V}} v_{\mathrm{S}} : \alpha * \\ n' \propto \mathsf{Sentinel}(\ell_{\mathrm{I}}, -) * \ell_{\mathrm{I}} \mapsto_{\mathrm{I}} v_{\mathrm{I}} * \\ j \rightarrowtail_{\mathrm{S}} K[\ell_{\mathrm{S}}] * \ell_{\mathrm{S}} \mapsto_{\mathrm{S}} v_{\mathrm{S}} \end{array} \right\rangle$
          $n'[1]$
        else $\langle P \rangle$
          try()
  $\langle \mathsf{ret}_{\mathrm{I}}.\ \exists \mathsf{ret}_{\mathrm{S}}.\ \mathsf{ret}_{\mathrm{I}} \preceq^{\mathcal{V}} \mathsf{ret}_{\mathrm{S}} : \mathbf{ref}_?(\alpha) \wedge j \rightarrowtail_{\mathrm{S}} K[\mathsf{ret}_{\mathrm{S}}] \rangle$
$\langle \mathsf{ret}_{\mathrm{I}}.\ \exists \mathsf{ret}_{\mathrm{S}}.\ \mathsf{ret}_{\mathrm{I}} \preceq^{\mathcal{V}} \mathsf{ret}_{\mathrm{S}} : \mathbf{ref}_?(\alpha) \wedge j \rightarrowtail_{\mathrm{S}} K[\mathsf{ret}_{\mathrm{S}}] \rangle$

6.3 Late/early choice

$$
\begin{array}{rcl}
\mathsf{rand} & \triangleq & \lambda().\ \textbf{let}\ y = \textbf{new false in}\ (\textbf{fork}\ y\ :=\ \textbf{true}); y[1] \\
\mathsf{lateChoice} & \triangleq & \lambda x.\ x\ :=\ 0; \mathsf{rand}() \\
\mathsf{earlyChoice} & \triangleq & \lambda x.\ \textbf{let}\ r = \mathsf{rand}()\ \textbf{in}\ x\ :=\ 0; r
\end{array}
$$

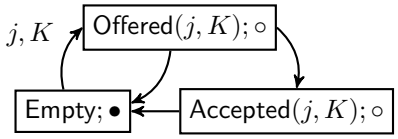
No internal protocol.
We use the *atomic Hoare triple* to show the details of the proof outline.
We have $\langle \mathsf{emp} \rangle\ \mathsf{rand}()\ \langle \mathsf{ret}.\ \mathsf{ret} = \textbf{true} \lor \mathsf{ret} = \textbf{false} \rangle$.

$\langle x_\mathrm{I} \preceq^{\mathcal{V}} x_\mathrm{S} : \textbf{ref}(\mathbf{N}) \land j \rightarrowtail_\mathrm{S} K[\mathsf{earlyChoice}(x_\mathrm{S})] \rangle$
$\quad (\!| \triangleright (\exists y_\mathrm{I}, y_\mathrm{S}.\ y_\mathrm{I} \preceq^{\mathcal{V}} y_\mathrm{S} : \mathbf{N}\ *\ x_\mathrm{I} \mapsto_\mathrm{I} y_\mathrm{I}\ *\ x_\mathrm{S} \mapsto_\mathrm{S} y_\mathrm{S})\ *\ j \rightarrowtail_\mathrm{S} K[\mathsf{earlyChoice}(x_\mathrm{S})] |\!)$
$\quad (\!| x_\mathrm{I} \mapsto_\mathrm{I} -\ *\ x_\mathrm{S} \mapsto_\mathrm{S} -\ *\ j \rightarrowtail_\mathrm{S} K[\mathsf{earlyChoice}(x_\mathrm{S})] |\!)$
$\qquad x_\mathrm{I}\ :=\ 0$
$\quad (\!| x_\mathrm{I} \mapsto_\mathrm{I} 0\ *\ x_\mathrm{S} \mapsto_\mathrm{S} -\ *\ j \rightarrowtail_\mathrm{S} K[\mathsf{earlyChoice}(x_\mathrm{S})] |\!)$
$\quad (\!| x_\mathrm{I} \mapsto_\mathrm{I} 0\ *\ x_\mathrm{S} \mapsto_\mathrm{S} 0\ *\ \mathsf{speculate}\,(j \rightarrowtail_\mathrm{S} K[\textbf{true}] \lor j \rightarrowtail_\mathrm{S} K[\textbf{false}]) |\!)$
$\langle x_\mathrm{I} \preceq^{\mathcal{V}} x_\mathrm{S} : \textbf{ref}(\mathbf{N}) \land \mathsf{speculate}\,(j \rightarrowtail_\mathrm{S} K[\textbf{true}] \lor j \rightarrowtail_\mathrm{S} K[\textbf{false}]) \rangle$
$\langle \mathsf{speculate}\,(j \rightarrowtail_\mathrm{S} K[\textbf{true}] \lor j \rightarrowtail_\mathrm{S} K[\textbf{false}]) \rangle$
$\quad \mathsf{rand}()$
$\langle \mathsf{ret}.\ (\mathsf{ret} = \textbf{true} \lor \mathsf{ret} = \textbf{false}) * \mathsf{speculate}\,(j \rightarrowtail_\mathrm{S} K[\textbf{true}] \lor j \rightarrowtail_\mathrm{S} K[\textbf{false}]) \rangle$
$\langle \mathsf{ret}.\ j \rightarrowtail_\mathrm{S} K[\mathsf{ret}] \rangle$

6.4 Red/blue flags

We use the *atomic Hoare triple* to show the details of the proof outline.

$\mathsf{redFlag} \triangleq \lambda().\ \textbf{let}\ \mathit{flag} = \textbf{new}\ \textbf{true},\ \mathit{chan} = \textbf{new}\ 0\ \textbf{in}\ \{$
$\quad \mathsf{flip} = \textbf{rec}\ \mathsf{try}().$
$\quad\quad \textbf{if}\ \mathsf{CAS}(\mathit{chan}, 1, 2)\ \textbf{then}\ ()\ \textbf{else}$
$\quad\quad \textbf{if}\ \mathsf{CAS}(\mathit{flag}, \textbf{true}, \textbf{false})\ \textbf{then}\ ()\ \textbf{else}$
$\quad\quad \textbf{if}\ \mathsf{CAS}(\mathit{flag}, \textbf{false}, \textbf{true})\ \textbf{then}\ ()\ \textbf{else}$
$\quad\quad \textbf{if}\ \mathsf{CAS}(\mathit{chan}, 0, 1)\ \textbf{then}$
$\quad\quad\quad \textbf{if}\ \mathsf{CAS}(\mathit{chan}, 1, 0)\ \textbf{then}\ \mathsf{try}()\ \textbf{else}\ \mathit{chan} := 0$
$\quad\quad \textbf{else}\ \mathsf{try}(),$
$\quad \mathsf{read} = \lambda().\ \mathit{flag}[1]$
$\}$
$\mathsf{blueFlag} \triangleq \lambda().\ \textbf{let}\ \mathit{flag} = \textbf{new}\ \textbf{true}, \mathsf{lock} = \textbf{new}\ \textbf{false}\ \textbf{in}\ \{$
$\quad \mathsf{flip} = \lambda().\ \mathsf{sync(lock)}\ \{\ \mathit{flag} := \textbf{not}\ \mathit{flag}[1]\ \},$
$\quad \mathsf{read} = \lambda().\ \mathsf{sync(lock)}\ \{\ \mathit{flag}[1]\ \}$
$\}$



We use the shorthand $\exists x : \mathbf{B}.P$ for $\exists x.\ (x = \textbf{true} \lor x = \textbf{false}) \land P$.

$$\begin{aligned}
Q &\triangleq \exists x : \mathbf{B}.\ \mathit{flag}_{\mathrm{I}} \mapsto_{\mathrm{I}} x * \mathit{flag}_{\mathrm{S}} \mapsto_{\mathrm{S}} x * \mathsf{lock} \mapsto_{\mathrm{S}} \textbf{false} \\
I(s) &\triangleq Q * I_0(s) \\
I_0(\mathsf{Empty}) &\triangleq \mathit{chan} \mapsto_{\mathrm{I}} 0 \\
I_0(\mathsf{Offered}(j, K)) &\triangleq \mathit{chan} \mapsto_{\mathrm{I}} 1 * j \rightarrowtail_{\mathrm{S}} K[\mathsf{flip}_{\mathrm{S}}()] \\
I_0(\mathsf{Accepted}(j, K)) &\triangleq \mathit{chan} \mapsto_{\mathrm{I}} 2 * j \rightarrowtail_{\mathrm{S}} K[()]
\end{aligned}$$

An interesting aspect of this example: it does not matter which order we perform the *top-level* CASes in. Each CAS either succeeds and completes the spec, or fails and leaves our knowledge unchanged. Thus, we give triples only for the CASes themselves; they can be plugged together, within if expressions, in any order.

Let θ be the transition system above.

Let $P = (\theta, I, \mathsf{Empty}, \emptyset) \land j \rightarrowtail_{\mathrm{S}} K[\mathsf{flip}_{\mathrm{S}}()]$.

For each top-level CAS e, we want to prove the triple

$$\langle P \rangle\ \ e\ \ \langle \mathsf{ret}.\ (\mathsf{ret} = \textbf{true} * j \rightarrowtail_{\mathrm{S}} K[()]) \lor (\mathsf{ret} = \textbf{false} * P) \rangle$$

We do this by proving atomic triples, and we need give the case where the CAS will succeed, since otherwise P trivially continues to hold (since a failed CAS has no effect).

For the first CAS, the abstract state tells us whether the CAS will succeed; we consider only the case where it does. Note that we use the diamond modality (defined in Section 2 of the appendix) to 'take specification steps in the postcondition':

$(\!|\triangleright I(\mathsf{Offered}(j', K')) * j \rightarrowtail_{\mathrm{S}} K[\mathsf{flip}_{\mathrm{S}}()]|\!)$
$(\!|Q * \mathit{chan} \mapsto_{\mathrm{I}} 1 * j' \rightarrowtail_{\mathrm{S}} K'[\mathsf{flip}_{\mathrm{S}}()] * j \rightarrowtail_{\mathrm{S}} K[\mathsf{flip}_{\mathrm{S}}()]|\!)$
$\quad \mathsf{CAS}(\mathit{chan}, 1, 2)$
$(\!|\mathsf{ret}.\ \mathsf{ret} = \textbf{true} * Q * \mathit{chan} \mapsto_{\mathrm{I}} 2 * j' \rightarrowtail_{\mathrm{S}} K'[\mathsf{flip}_{\mathrm{S}}()] * j \rightarrowtail_{\mathrm{S}} K[\mathsf{flip}_{\mathrm{S}}()]|\!)$
$(\!|\mathsf{ret}.\ \mathsf{ret} = \textbf{true} * \exists x : \mathbf{B}.\ \mathit{flag}_{\mathrm{I}} \mapsto_{\mathrm{I}} x * \mathit{flag}_{\mathrm{S}} \mapsto_{\mathrm{S}} x \quad * \mathit{chan} \mapsto_{\mathrm{I}} 2 * j' \rightarrowtail_{\mathrm{S}} K'[\mathsf{flip}_{\mathrm{S}}()] * j \rightarrowtail_{\mathrm{S}} K[\mathsf{flip}_{\mathrm{S}}()]|\!)$
$(\!|\mathsf{ret}.\ \mathsf{ret} = \textbf{true} * \exists x : \mathbf{B}.\ \mathit{flag}_{\mathrm{I}} \mapsto_{\mathrm{I}} x * \mathit{flag}_{\mathrm{S}} \mapsto_{\mathrm{S}} \neg x * \mathit{chan} \mapsto_{\mathrm{I}} 2 * j' \rightarrowtail_{\mathrm{S}} K'[()] * j \rightarrowtail_{\mathrm{S}} K[\mathsf{flip}_{\mathrm{S}}()]|\!)$
$(\!|\mathsf{ret}.\ \mathsf{ret} = \textbf{true} * \exists x : \mathbf{B}.\ \mathit{flag}_{\mathrm{I}} \mapsto_{\mathrm{I}} x * \mathit{flag}_{\mathrm{S}} \mapsto_{\mathrm{S}} x \quad * \mathit{chan} \mapsto_{\mathrm{I}} 2 * j' \rightarrowtail_{\mathrm{S}} K'[()] * j \rightarrowtail_{\mathrm{S}} K[()]|\!)$

$(\!|\mathsf{ret.\ ret} = \mathbf{true} * I(\mathsf{Accepted}(j', K')) * j \rightarrowtail_{\mathrm{S}} K[()]|\!)$

We prove the second CAS for any state s:
$(\!|\triangleright I(s) * j \rightarrowtail_{\mathrm{S}} K[\mathsf{flip}_{\mathrm{S}}()]|\!)$
$(\!|\exists x : \mathbf{B}.\ \mathit{flag}_{\mathrm{I}} \mapsto_{\mathrm{I}} x * \mathit{flag}_{\mathrm{S}} \mapsto_{\mathrm{S}} x * I_0(s) * j \rightarrowtail_{\mathrm{S}} K[\mathsf{flip}_{\mathrm{S}}()]|\!)$
  $\mathsf{CAS}(\mathit{flag}, \mathbf{true}, \mathbf{false})$
$(\!|\mathsf{ret.}\ ((\mathsf{ret} = \mathbf{true} * \mathit{flag}_{\mathrm{I}} \mapsto_{\mathrm{I}} \mathbf{false} * \mathit{flag}_{\mathrm{S}} \mapsto_{\mathrm{S}} \mathbf{true}) \vee (\mathsf{ret} = \mathbf{false} * \mathit{flag}_{\mathrm{I}} \mapsto_{\mathrm{I}} \mathbf{false} * \mathit{flag}_{\mathrm{S}} \mapsto_{\mathrm{S}} \mathbf{false}))$
  $* I_0(s) * j \rightarrowtail_{\mathrm{S}} K[\mathsf{flip}_{\mathrm{S}}()]|\!)$
$(\!|\mathsf{ret.}\ (\mathsf{ret} = \mathbf{true} * I(s) * j \rightarrowtail_{\mathrm{S}} K[()])$
  $\vee\ (\mathsf{ret} = \mathbf{false} * I(s) * j \rightarrowtail_{\mathrm{S}} K[\mathsf{flip}_{\mathrm{S}}()])|\!)$

The proof for $\mathsf{CAS}(\mathit{flag}, \mathbf{false}, \mathbf{true})$ is symmetric.

That leaves only the final top-level CAS, in which we make an offer:
  $\langle P \rangle$
  $\langle (\theta, I, \mathsf{Empty}, \emptyset) \wedge j \rightarrowtail_{\mathrm{S}} K[\mathsf{flip}_{\mathrm{S}}()] \rangle$
    $\mathsf{CAS}(\mathit{chan}, 0, 1)$
  $\langle \mathsf{ret.}\ (\mathsf{ret} = \mathbf{true} * (\theta, I, \mathsf{Offered}(j, K), \bullet)) \vee (\mathsf{ret} = \mathbf{false} * P) \rangle$

We are now in a state where we own the token.
For the inner CAS, we therefore need to consider only two possible future states:
  $(\!|\triangleright I(\mathsf{Offered}(j, K))|\!)$
  $(\!|Q * \mathit{chan} \mapsto_{\mathrm{I}} 1 * j \rightarrowtail_{\mathrm{S}} K[\mathsf{flip}_{\mathrm{S}}()]|\!)$
    $\mathsf{CAS}(\mathit{chan}, 1, 0)$ $(\!|\mathsf{ret.}\ \mathsf{ret} = \mathbf{true} * Q * \mathit{chan} \mapsto_{\mathrm{I}} 0 * j \rightarrowtail_{\mathrm{S}} K[\mathsf{flip}_{\mathrm{S}}()]|\!)$
  $(\!|\mathsf{ret.}\ \mathsf{ret} = \mathbf{true} * I(\mathsf{Empty}) * j \rightarrowtail_{\mathrm{S}} K[\mathsf{flip}_{\mathrm{S}}()]|\!)$

  $(\!|\triangleright I(\mathsf{Accepted}(j, K))|\!)$
  $(\!|Q * \mathit{chan} \mapsto_{\mathrm{I}} 2 * j \rightarrowtail_{\mathrm{S}} K[()]|\!)$
    $\mathsf{CAS}(\mathit{chan}, 1, 0)$ $(\!|\mathsf{ret.}\ \mathsf{ret} = \mathbf{false} * Q * \mathit{chan} \mapsto_{\mathrm{I}} 2 * j \rightarrowtail_{\mathrm{S}} K[()]|\!)$
  $(\!|\mathsf{ret.}\ \mathsf{ret} = \mathbf{false} * I(\mathsf{Accepted}(j, K))|\!)$

Thus, the overall triple for the inner CAS is:
$$\langle (\theta, I, \mathsf{Offered}(j, K), \bullet) \rangle\ \mathsf{CAS}(\mathit{chan}, 1, 0)\ \left\langle \begin{array}{ll} \mathsf{ret.} & (\mathsf{ret} = \mathbf{true} * j \rightarrowtail_{\mathrm{S}} K[\mathsf{flip}_{\mathrm{S}}()]) \\ \vee & (\mathsf{ret} = \mathbf{false} * (\theta, I, \mathsf{Accepted}(j, K), \bullet)) \end{array} \right\rangle$$

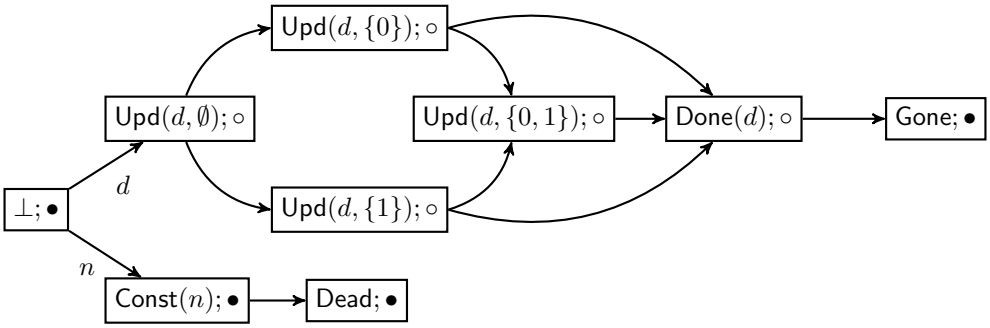
Finally, if the inner CAS fails, there is only one rely-future state: $\mathsf{Accepted}(j, K)$
Thus, we know exactly what the assignment to the channel will see:
  $(\!|\triangleright I(\mathsf{Accepted}(j, K))|\!)$
  $(\!|Q * \mathit{chan} \mapsto_{\mathrm{I}} 2 * j \rightarrowtail_{\mathrm{S}} K[()]|\!)$
    $\mathit{chan} := 0$
  $(\!|Q * \mathit{chan} \mapsto_{\mathrm{I}} 0 * j \rightarrowtail_{\mathrm{S}} K[()]|\!)$
  $(\!|I(\mathsf{Empty}) * j \rightarrowtail_{\mathrm{S}} K[()]|\!)$

6.5 CCAS

$\mathsf{counter}_{\mathrm{S}} \triangleq$
  let $c =$ **new** $0, f =$ **new false**, lock = **new false**
  let get() $=$ sync(lock) { $c[1]$ }
  let setFlag(b) $=$ sync(lock) { $f := b$ }
  let cinc() $=$ **sync**(lock) {
    $c[1] := c[1] +$ **if** $f[1]$ **then** 1 **else** 0 }
  in (get, setFlag, cinc)

$\mathsf{counter}_{\mathrm{I}} \triangleq$
  let $c =$ **new inj**$_1$ $0, f =$ **new false**
  let complete(o, x) $=$
    if $f[1]$ **then** CAS($c, o,$ **inj**$_1$ $(x+1)$)
      else CAS($c, o,$ **inj**$_1$ x)
  let rec get() $=$ **let** $o = c[1]$ **in case** o **of**
    inj$_1$ $x \Rightarrow x$
    | **inj**$_2$ $x \Rightarrow$ complete(o, x); get()
  let setFlag(b) $= f := b$
  let rec cinc() $=$ **let** $o = c[1]$ **in case** o **of**
    inj$_1$ $x \Rightarrow$ **let** $n =$ **inj**$_2$ x **in**
      if CAS(c, o, n) **then** complete(n, x) **else** cinc()
    | **inj**$_2$ $x \Rightarrow$ complete(o, x); cinc()
  in (get, setFlag, cinc)

We have the following "life story" for every "descriptor" (injection into sum) location:



Formally, we have

$$
\begin{array}{rcl}
d & ::= & n, j, K \\
S_0 & \triangleq & \{\bot, \mathsf{Done}(d), \mathsf{Gone}, \mathsf{Const}(n), \mathsf{Dead}\} \cup \{\ \mathsf{Upd}(d, B) \ \mid\ B \subseteq \{0,1\}\ \} \\
S & \triangleq & \left\{\ s \in \mathrm{Loc} \xrightarrow{\mathrm{fin}} S_0 \ \mid\ \exists!\ell.\ (s(\ell) = \mathsf{Const}(-) \vee s(\ell) = \mathsf{Upd}(-))\ \right\} \\
A & \triangleq & \mathrm{Loc}
\end{array}
$$

following the pattern of MSQ. The transition relation on S is the pointwise lifting of that for S_0. Note that the product transition system has one token per location, that is, one token per local transition system. The free tokens F for the product system is the product of the banks for the local systems.

The interpretation is as follows:

$$
\begin{array}{rcl}
d & ::= & n, j, K \qquad\qquad B \subseteq \{0,1\} \qquad\qquad A \triangleq \mathrm{Loc} \\
S_0 & \triangleq & \{\bot, \mathsf{Upd}(d,B), \mathsf{Done}(d), \mathsf{Gone}, \mathsf{Const}(n), \mathsf{Dead}\} \\
S & \triangleq & \left\{ s \in \mathrm{Loc} \xrightarrow{\mathrm{fin}} S_0 \;\middle|\; \exists! \ell.\ s(\ell) \in \{\mathsf{Const}(-), \mathsf{Upd}(-,-)\} \right\} \\
I(s) & \triangleq & \exists b : \mathbf{B}.\ f_{\mathrm{I}} \mapsto_{\mathrm{I}} b * f_{\mathrm{S}} \mapsto_{\mathrm{S}} b * \mathsf{lock} \mapsto_{\mathrm{S}} \mathbf{false} \\
 & & * \begin{cases} \mathsf{linkUpd}(\ell, n, j, K, B) & \exists \ell.\ s(\ell) = \mathsf{Upd}(n, j, K, B) \\ \mathsf{linkConst}(\ell, n) & \exists \ell.\ s(\ell) = \mathsf{Const}(n) \end{cases} \\
 & & * \ast_{s(\ell)=\mathsf{Done}(n,j,K)} \ell \mapsto_{\mathrm{I}} \mathbf{inj}_2\ n * j \rightarrowtail_{\mathrm{S}} K[()] \\
 & & * \ast_{s(\ell)=\mathsf{Gone}} \ell \mapsto_{\mathrm{I}} \mathbf{inj}_2\ - \quad * \quad \ast_{s(\ell)=\mathsf{Dead}} \ell \mapsto_{\mathrm{I}} \mathbf{inj}_1\ - \\
\mathsf{linkConst}(\ell, n) & \triangleq & c_{\mathrm{I}} \mapsto_{\mathrm{I}} \ell * \ell \mapsto_{\mathrm{I}} \mathbf{inj}_1\ n * c_{\mathrm{S}} \mapsto_{\mathrm{S}} n \\
\mathsf{linkUpd}(\ell, n, j, K, B) & \triangleq & c_{\mathrm{I}} \mapsto_{\mathrm{I}} \mathbf{inj}_2\ \ell * \ell \mapsto_{\mathrm{I}} n \\
 & & * \begin{pmatrix} & c_{\mathrm{S}} \mapsto_{\mathrm{S}} n * j \rightarrowtail_{\mathrm{S}} K[\mathsf{cinc}()] & \\ \oplus & c_{\mathrm{S}} \mapsto_{\mathrm{S}} n * j \rightarrowtail_{\mathrm{S}} K[()] & \text{if } 0 \in B \\ \oplus & c_{\mathrm{S}} \mapsto_{\mathrm{S}} (n+1) * j \rightarrowtail_{\mathrm{S}} K[()] & \text{if } 1 \in B \end{pmatrix}
\end{array}
$$

We let θ be the product transition system.

$$
\begin{array}{lll}
\textbf{let } \mathsf{complete}(o,x)\ = & & \langle o \propto \mathsf{Upd}(x,-,-,\emptyset)\rangle \\
\quad \textbf{if } f[1] \textbf{ then} & & \langle o \propto \mathsf{Upd}(x,-,-,\{1\})\rangle \\
\qquad \mathsf{CAS}(c, o, \mathbf{inj}_1\ (x+1)) & & \quad \langle o \propto \mathsf{Done}(x,-,-)\rangle \\
\quad \textbf{else} & & \langle o \propto \mathsf{Upd}(x,-,-,\{0\})\rangle \\
\qquad \mathsf{CAS}(c, o, \mathbf{inj}_1\ x) & & \quad \langle o \propto \mathsf{Done}(x,-,-)\rangle \\
\\
\textbf{let rec } \mathsf{cinc}()\ = & & \langle j \rightarrowtail_{\mathrm{S}} K[\mathsf{cinc}_{\mathrm{S}}()] * (\theta, I, \emptyset, \emptyset)\rangle \\
\quad \textbf{let } o = c[1] \textbf{ in} & & \left\langle j \rightarrowtail_{\mathrm{S}} K[\mathsf{cinc}_{\mathrm{S}}()] * \left(o \propto \mathsf{Const}(-) \vee o \propto \mathsf{Upd}(-,-)\right)\right\rangle \\
\quad \textbf{case } o \textbf{ of} & & \\
\qquad \mathbf{inj}_1\ x \Rightarrow & & \langle j \rightarrowtail_{\mathrm{S}} K[\mathsf{cinc}_{\mathrm{S}}()] * o \propto \mathsf{Const}(x)\rangle \\
\qquad\quad \textbf{let } n = \mathbf{inj}_2\ x \textbf{ in} & & \left\langle j \rightarrowtail_{\mathrm{S}} K[\mathsf{cinc}_{\mathrm{S}}()] * o \propto \mathsf{Const}(x) * n \mapsto \mathbf{inj}_2\ x\right\rangle \\
\qquad\quad \textbf{if } \mathsf{CAS}(c,o,n) \textbf{ then} & & \langle o \propto \mathsf{Dead}(x) \wedge n \propto_\bullet \mathsf{Upd}(x,j,K,\emptyset)\rangle \\
 & \therefore & \quad \langle n \propto_\bullet \mathsf{Upd}(x,j,K,\emptyset)\rangle \\
\qquad\qquad \mathsf{complete}(n,x); & & \quad \langle n \propto_\bullet \mathsf{Done}(x,j,K)\rangle \\
\qquad\qquad () & & \quad \langle \mathsf{ret.\ ret} = () \wedge j \rightarrowtail_{\mathrm{S}} K[()] \wedge n \propto \mathsf{Gone}\rangle \\
 & \therefore & \quad \langle \mathsf{ret.\ ret} = () \wedge j \rightarrowtail_{\mathrm{S}} K[()]\rangle \\
\qquad \textbf{else} & & \langle j \rightarrowtail_{\mathrm{S}} K[\mathsf{cinc}_{\mathrm{S}}()] * (\theta, I, \emptyset, \emptyset)\rangle \\
\qquad\quad \mathsf{cinc}() & & \quad \langle \mathsf{ret.\ ret} = () \wedge j \rightarrowtail_{\mathrm{S}} K[()]\rangle \\
\\
\qquad \mid \mathbf{inj}_2\ x \Rightarrow & & \langle j \rightarrowtail_{\mathrm{S}} K[\mathsf{cinc}_{\mathrm{S}}()] * o \propto \mathsf{Upd}(x,-,-,-)\rangle \\
\qquad\quad \mathsf{complete}(o,x); & & \langle j \rightarrowtail_{\mathrm{S}} K[\mathsf{cinc}_{\mathrm{S}}()] * o \propto \mathsf{Done}(x,-,-)\rangle \\
 & \therefore & \langle j \rightarrowtail_{\mathrm{S}} K[\mathsf{cinc}_{\mathrm{S}}()] * (\theta, I, \emptyset, \emptyset)\rangle \\
\qquad\quad \mathsf{cinc}() & & \langle \mathsf{ret.\ ret} = () \wedge j \rightarrowtail_{\mathrm{S}} K[()]\rangle
\end{array}
$$