# Under Control: Compositional Correctness of Closure Conversion with State (Technical Appendix)

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## Contents

### Prelude

- Sections 1 through 10 present verified compositional closure conversion for an ML-like language to a target language *without* **call/cc**.
- Section 11 shows how to extend the development in the prior sections to obtain a verified closure conversion result from ML to a target *with* **call/cc**.

# 1 Source Language: M

$\tau ::= \alpha \mid \mathsf{unit} \mid \mathsf{int} \mid \forall[\overline{\alpha}].(\overline{\tau}) \to \tau \mid \exists\alpha.\tau \mid \mu\alpha.\tau \mid \mathsf{ref}\,\tau \mid \langle\overline{\tau}\rangle$

$\mathsf{p} ::= + \mid - \mid *$

$\mathsf{v} ::= \mathsf{x} \mid () \mid \mathsf{n} \mid \lambda[\overline{\alpha}](\overline{\mathsf{x}:\tau}).\mathsf{e} \mid \mathsf{pack}\,\langle\tau,\mathsf{v}\rangle\,\mathsf{as}\,\exists\alpha.\tau \mid \mathsf{fold}_{\mu\alpha.\tau}\,\mathsf{v} \mid \ell \mid \langle\overline{\mathsf{v}}\rangle$

$\mathsf{e} ::= \mathsf{v} \mid \mathsf{v}\ \mathsf{p}\ \mathsf{v} \mid \mathsf{if0}\,\mathsf{v}\ \mathsf{e}\ \mathsf{e} \mid \mathsf{v}\,[\overline{\tau}]\,\overline{\mathsf{v}} \mid \mathsf{unpack}\,\langle\alpha,\mathsf{x}\rangle = \mathsf{v}\ \mathsf{in}\ \mathsf{e} \mid \mathsf{new}\,\mathsf{v} \mid \mathsf{v} := \mathsf{v} \mid !\mathsf{v} \mid \mathsf{unfold}\,\mathsf{v} \mid \pi_{\mathsf{i}}(\mathsf{v})$

$\quad\mid \mathsf{let}\,\mathsf{x} = \mathsf{e}\ \mathsf{in}\ \mathsf{e}$

$\mathsf{E} ::= [\cdot] \mid \mathsf{let}\,\mathsf{x} = \mathsf{E}\ \mathsf{in}\ \mathsf{e}$

$\mathsf{H} ::= \cdot \mid \mathsf{H}, \ell \mapsto \mathsf{v}$

$\Psi ::= \cdot \mid \Psi, \ell : \tau$

$\Delta ::= \cdot \mid \Delta, \alpha$

$\Gamma ::= \cdot \mid \Gamma, \mathsf{x} : \tau$

## 1.1 Well-Formed Types $\boxed{\Delta \vdash \tau}$

$$\frac{\alpha \in \Delta}{\Delta \vdash \alpha} \qquad \frac{}{\Delta \vdash \mathsf{unit}} \qquad \frac{}{\Delta \vdash \mathsf{int}} \qquad \frac{\overline{\Delta, \overline{\alpha} \vdash \tau} \qquad \Delta, \overline{\alpha} \vdash \tau'}{\Delta \vdash \forall[\overline{\alpha}].(\overline{\tau}) \to \tau'} \qquad \frac{\Delta, \alpha \vdash \tau}{\Delta \vdash \exists\alpha.\tau} \qquad \frac{\Delta, \alpha \vdash \tau}{\Delta \vdash \mu\alpha.\tau} \qquad \frac{\Delta \vdash \tau}{\Delta \vdash \mathsf{ref}\,\tau}$$

$$\frac{\Delta \vdash \tau_1 \quad \cdots \quad \Delta \vdash \tau_n}{\Delta \vdash \langle \tau_1, \ldots, \tau_n \rangle}$$

## 1.2 Well-Formed Heap Types $\boxed{\vdash \Psi}$

$$\frac{\cdot \vdash \tau_1 \quad \cdots \quad \cdot \vdash \tau_n}{\vdash \ell_1 : \tau_1, \ldots, \ell_n : \tau_n}$$

## 1.3 Well-Formed Type Environment $\boxed{\Delta \vdash \Gamma}$

$$\frac{}{\Delta \vdash \cdot} \qquad \frac{\Delta \vdash \Gamma \qquad \Delta \vdash \tau}{\Delta \vdash \Gamma, \mathsf{x} : \tau}$$

## 1.4 Well-Typed Heap Fragments $\boxed{\Psi \vdash \mathsf{H} : \Psi'}$

$$\frac{\mathrm{dom}(\Psi) \cap \mathrm{dom}(\Psi') = \emptyset \qquad \vdash \Psi' \qquad (\Psi, \Psi'); \cdot; \cdot \vdash \mathsf{v}_1 : \Psi'(\ell_1), \ldots, (\Psi, \Psi'); \cdot; \cdot \vdash \mathsf{v}_n : \Psi'(\ell_n)}{\vdash \{\ell_1 \mapsto \mathsf{v}_1, \ldots, \ell_n \mapsto \mathsf{v}_n\} : \Psi'}$$

## 1.5 Well-Typed Terms $\boxed{\Psi;\Delta;\Gamma \vdash e : \tau}$

$$\frac{x{:}\tau \in \Gamma}{\Psi;\Delta;\Gamma \vdash x : \tau} \qquad \frac{}{\Psi;\Delta;\Gamma \vdash () : \mathsf{unit}} \qquad \frac{}{\Psi;\Delta;\Gamma \vdash n : \mathsf{int}} \qquad \frac{\Psi;\Delta;\Gamma \vdash v_1 : \mathsf{int} \qquad \Psi;\Delta;\Gamma \vdash v_2 : \mathsf{int}}{\Psi;\Delta;\Gamma \vdash v_1\ p\ v_2 : \mathsf{int}}$$

$$\frac{\Psi;\Delta;\Gamma \vdash v : \mathsf{int} \qquad \Psi;\Delta;\Gamma \vdash e_1 : \tau \qquad \Psi;\Delta;\Gamma \vdash e_2 : \tau}{\Psi;\Delta;\Gamma \vdash \mathsf{if0}\, v\ e_1\ e_2 : \tau} \qquad \frac{\Psi;\Delta,\overline{\alpha};\Gamma,\overline{x{:}\tau} \vdash e : \tau'}{\Psi;\Delta;\Gamma \vdash \lambda[\overline{\alpha}](\overline{x{:}\tau}).e : \forall[\overline{\alpha}].(\overline{\tau}) \to \tau'}$$

$$\frac{\Psi;\Delta;\Gamma \vdash v_0 : \forall[\overline{\alpha}].(\overline{\tau}) \to \tau' \qquad \Delta \vdash \overline{\tau_0} \qquad \Psi;\Delta;\Gamma \vdash \overline{v} : \overline{\tau[\overline{\tau_0/\alpha}]}}{\Psi;\Delta;\Gamma \vdash v_0\ [\overline{\tau_0}]\ \overline{v} : \tau'[\overline{\tau_0/\alpha}]} \qquad \frac{\Psi;\Delta;\Gamma \vdash v : \tau[\tau'/\alpha]}{\Psi;\Delta;\Gamma \vdash \mathsf{pack}\,\langle \tau', v\rangle\ \mathsf{as}\ \exists\alpha.\tau : \exists\alpha.\tau}$$

$$\frac{\Psi;\Delta;\Gamma \vdash v : \exists\alpha.\tau \qquad \Delta \vdash \tau' \qquad \Psi;\Delta,\alpha;\Gamma,x{:}\tau \vdash e : \tau'}{\Psi;\Delta;\Gamma \vdash \mathsf{unpack}\,\langle \alpha, x\rangle = v\ \mathsf{in}\ e : \tau'} \qquad \frac{\Psi;\Delta;\Gamma \vdash v : \tau[\mu\alpha.\tau/\alpha]}{\Psi;\Delta;\Gamma \vdash \mathsf{fold}_{\mu\alpha.\tau}\, v : \mu\alpha.\tau}$$

$$\frac{\Psi;\Delta;\Gamma \vdash v : \mu\alpha.\tau}{\Psi;\Delta;\Gamma \vdash \mathsf{unfold}\, v : \tau[\mu\alpha.\tau/\alpha]} \qquad \frac{\Psi(\ell) = \tau}{\Psi;\Delta;\Gamma \vdash \ell : \mathsf{ref}\ \tau} \qquad \frac{\Psi;\Delta;\Gamma \vdash v : \tau}{\Psi;\Delta;\Gamma \vdash \mathsf{new}\, v : \mathsf{ref}\ \tau}$$

$$\frac{\Psi;\Delta;\Gamma \vdash v_1 : \mathsf{ref}\ \tau \qquad \Psi;\Delta;\Gamma \vdash v_2 : \tau}{\Psi;\Delta;\Gamma \vdash v_1 := v_2 : \mathsf{unit}} \qquad \frac{\Psi;\Delta;\Gamma \vdash v : \mathsf{ref}\ \tau}{\Psi;\Delta;\Gamma \vdash\ !v : \tau} \qquad \frac{\Psi;\Delta;\Gamma \vdash v_1 : \tau_1 \qquad \cdots \qquad \Psi;\Delta;\Gamma \vdash v_n : \tau_n}{\Psi;\Delta;\Gamma \vdash \langle v_1, \ldots, v_n\rangle : \langle \tau_1, \ldots, \tau_n\rangle}$$

$$\frac{\Psi;\Delta;\Gamma \vdash v : \langle \tau_1, \ldots, \tau_n\rangle}{\Psi;\Delta;\Gamma \vdash \pi_i(v) : \tau_i} \qquad \frac{\Psi;\Delta;\Gamma \vdash e_1 : \tau_1 \qquad \Psi;\Delta;\Gamma,x{:}\tau_1 \vdash e_2 : \tau_2}{\Psi;\Delta;\Gamma \vdash \mathsf{let}\, x = e_1\ \mathsf{in}\ e_2 : \tau_2}$$

## 1.6 Reduction Relation $\boxed{\langle H \mid e\rangle \longmapsto \langle H' \mid e'\rangle}$

$$\begin{array}{lll}
\langle H \mid E[n_1\ p\ n_2]\rangle & \longmapsto \langle H \mid E[\mathsf{prim}(p, n_1, n_2)]\rangle & \\
\langle H \mid E[\mathsf{if0}\, 0\ e_1\ e_2]\rangle & \longmapsto \langle H \mid E[e_1]\rangle & \\
\langle H \mid E[\mathsf{if0}\, n\ e_1\ e_2]\rangle & \longmapsto \langle H \mid E[e_2]\rangle & n \neq 0 \\
\langle H \mid E[(\lambda[\overline{\alpha}](\overline{x{:}\tau}).e)\ [\overline{\tau'}]\ \overline{v}]\rangle & \longmapsto \langle H \mid E[e[\overline{\tau'}/\overline{\alpha}][\overline{v}/\overline{x}]]\rangle & \\
\langle H \mid E[\mathsf{unpack}\,\langle \alpha, x\rangle = (\mathsf{pack}\,\langle \tau', v\rangle\ \mathsf{as}\ \exists\alpha.\tau)\ \mathsf{in}\ e]\rangle & \longmapsto \langle H \mid E[e[\tau'/\alpha][v/x]]\rangle & \\
\langle H \mid E[\mathsf{unfold}\,(\mathsf{fold}_{\mu\alpha.\tau}\, v)]\rangle & \longmapsto \langle H \mid E[v]\rangle & \\
\langle H \mid E[\mathsf{new}\, v]\rangle & \longmapsto \langle H[\ell \mapsto v] \mid E[\ell]\rangle & \ell \notin H \\
\langle H \mid E[\ell := v]\rangle & \longmapsto \langle H[\ell \mapsto v] \mid E[()]\rangle & \ell \in H \\
\langle H \mid E[!\ell]\rangle & \longmapsto \langle H \mid E[v]\rangle & H(\ell) = v \\
\langle H \mid E[\pi_i(\langle v_1, \ldots, v_n\rangle)]\rangle & \longmapsto \langle H \mid E[v_i]\rangle & \\
\langle H \mid E[\mathsf{let}\, x = v\ \mathsf{in}\ e]\rangle & \longmapsto \langle H \mid E[e[v/x]]\rangle &
\end{array}$$

## 2 Closure-Conversion Target Language: C

$\tau ::= \alpha \mid \mathsf{unit} \mid \mathsf{int} \mid \forall[\overline{\alpha}].(\overline{\tau}) \to \tau \mid \exists\alpha.\tau \mid \mu\alpha.\tau \mid \mathsf{ref}\,\tau \mid \langle\overline{\tau}\rangle$

$\mathsf{p} ::= + \mid - \mid *$

$\mathsf{v} ::= \mathsf{x} \mid () \mid \mathsf{n} \mid \lambda[\overline{\alpha}](\overline{\mathsf{x}:\tau}).\mathsf{e} \mid \mathsf{pack}\,\langle\tau,\mathsf{v}\rangle\,\mathsf{as}\,\exists\alpha.\tau \mid \mathsf{fold}_{\mu\alpha.\tau}\,\mathsf{v} \mid \ell \mid \langle\overline{\mathsf{v}}\rangle \mid \mathsf{v}\,[\tau]$

$\mathsf{e} ::= \mathsf{v} \mid \mathsf{v}\;\mathsf{p}\;\mathsf{v} \mid \mathsf{if0}\;\mathsf{v}\;\mathsf{e}\;\mathsf{e} \mid \mathsf{v}\,[]\,\overline{\mathsf{v}} \mid \mathsf{unpack}\,\langle\alpha,\mathsf{x}\rangle = \mathsf{v}\;\mathsf{in}\;\mathsf{e} \mid \mathsf{unfold}\,\mathsf{v} \mid \mathsf{new}\,\mathsf{v} \mid \mathsf{v} := \mathsf{v} \mid !\mathsf{v} \mid \pi_i(\mathsf{v})$
$\quad \mid \mathsf{let}\,\mathsf{x} = \mathsf{e}\;\mathsf{in}\;\mathsf{e}$

$\mathsf{E} ::= [\cdot] \mid \mathsf{let}\,\mathsf{x} = \mathsf{E}\;\mathsf{in}\;\mathsf{e}$

$\mathsf{H} ::= \cdot \mid \mathsf{H}, \ell \mapsto \mathsf{v}$

$\Psi ::= \cdot \mid \Psi, \ell : \tau$

$\Delta ::= \cdot \mid \Delta, \alpha$

$\Gamma ::= \cdot \mid \Gamma, \mathsf{x} : \tau$

### 2.1 Well-Formed Types $\boxed{\Delta \vdash \tau}$

$$\frac{\alpha \in \Delta}{\Delta \vdash \alpha} \qquad \frac{}{\Delta \vdash \mathsf{unit}} \qquad \frac{}{\Delta \vdash \mathsf{int}} \qquad \frac{\overline{\Delta, \overline{\alpha} \vdash \tau} \qquad \Delta, \overline{\alpha} \vdash \tau'}{\Delta \vdash \forall[\overline{\alpha}].(\overline{\tau}) \to \tau'} \qquad \frac{\Delta, \alpha \vdash \tau}{\Delta \vdash \exists\alpha.\tau} \qquad \frac{\Delta, \alpha \vdash \tau}{\Delta \vdash \mu\alpha.\tau} \qquad \frac{\Delta \vdash \tau}{\Delta \vdash \mathsf{ref}\,\tau}$$

$$\frac{\Delta \vdash \tau_1 \qquad \cdots \qquad \Delta \vdash \tau_n}{\Delta \vdash \langle\tau_1, \ldots, \tau_n\rangle}$$

### 2.2 Well-Formed Heap Types $\boxed{\vdash \Psi}$

$$\frac{\cdot \vdash \tau_1 \qquad \cdots \qquad \cdot \vdash \tau_n}{\vdash \ell_1 : \tau_1, \ldots, \ell_n : \tau_n}$$

### 2.3 Well-Formed Type Environment $\boxed{\Delta \vdash \Gamma}$

$$\frac{}{\Delta \vdash \cdot} \qquad\qquad \frac{\Delta \vdash \Gamma \qquad \Delta \vdash \tau}{\Delta \vdash \Gamma, \mathsf{x} : \tau}$$

### 2.4 Well-Typed Heap Fragments $\boxed{\Psi \vdash \mathsf{H} : \Psi'}$

$$\frac{\mathrm{dom}(\Psi) \cap \mathrm{dom}(\Psi') = \emptyset \qquad \vdash \Psi' \qquad (\Psi, \Psi'); \cdot; \cdot \vdash \mathsf{v}_1 : \Psi'(\ell_1), \ldots, (\Psi, \Psi'); \cdot; \cdot \vdash \mathsf{v}_n : \Psi'(\ell_n)}{\vdash \{\ell_1 \mapsto \mathsf{v}_1, \ldots, \ell_n \mapsto \mathsf{v}_n\} : \Psi'}$$

## 2.5 Well-Typed Terms $\boxed{\Psi;\Delta;\Gamma \vdash e : \tau}$

$$\frac{x : \tau \in \Gamma}{\Psi;\Delta;\Gamma \vdash x : \tau} \qquad \frac{}{\Psi;\Delta;\Gamma \vdash () : \mathsf{unit}} \qquad \frac{}{\Psi;\Delta;\Gamma \vdash n : \mathsf{int}} \qquad \frac{\Psi;\Delta;\Gamma \vdash v_1 : \mathsf{int} \qquad \Psi;\Delta;\Gamma \vdash v_2 : \mathsf{int}}{\Psi;\Delta;\Gamma \vdash v_1 \ p \ v_2 : \mathsf{int}}$$

$$\frac{\Psi;\Delta;\Gamma \vdash v : \mathsf{int} \qquad \Psi;\Delta;\Gamma \vdash e_1 : \tau \qquad \Psi;\Delta;\Gamma \vdash e_2 : \tau}{\Psi;\Delta;\Gamma \vdash \mathsf{if0}\ v\ e_1\ e_2 : \tau} \qquad \frac{\Psi;\overline{\alpha};\overline{x : \tau} \vdash e : \tau'}{\Psi;\Delta;\Gamma \vdash \lambda[\overline{\alpha}](\overline{x : \tau}).e : \forall[\overline{\alpha}].(\overline{\tau}) \to \tau'}$$

$$\frac{\Psi;\Delta;\Gamma \vdash v : \forall[].(\overline{\tau}) \to \tau' \qquad \Psi;\Delta;\Gamma \vdash \overline{v} : \overline{\tau}}{\Psi;\Delta;\Gamma \vdash v\ []\ \overline{v} : \tau'} \qquad \frac{\Psi;\Delta;\Gamma \vdash v : \forall[\beta,\overline{\alpha}].(\overline{\tau}) \to \tau' \qquad \Delta \vdash \tau_0}{\Psi;\Delta;\Gamma \vdash v[\tau_0] : \forall[\overline{\alpha}].(\overline{\tau[\tau_0/\beta]}) \to \tau'[\tau_0/\beta]}$$

$$\frac{\Psi;\Delta;\Gamma \vdash v : \tau[\tau'/\alpha] \qquad \Delta \vdash \tau'}{\Psi;\Delta;\Gamma \vdash \mathsf{pack}\ \langle \tau', v\rangle\ \mathsf{as}\ \exists\alpha.\tau : \exists\alpha.\tau} \qquad \frac{\Psi;\Delta;\Gamma \vdash v : \exists\alpha.\tau \qquad \Delta \vdash \tau' \qquad \Psi;\Delta,\alpha;\Gamma,x : \tau \vdash e : \tau'}{\Psi;\Delta;\Gamma \vdash \mathsf{unpack}\ \langle \alpha, x\rangle = v\ \mathsf{in}\ e : \tau'}$$

$$\frac{\Psi;\Delta;\Gamma \vdash v : \tau[\mu\alpha.\tau/\alpha]}{\Psi;\Delta;\Gamma \vdash \mathsf{fold}_{\mu\alpha.\tau}\ v : \mu\alpha.\tau} \qquad \frac{\Psi;\Delta;\Gamma \vdash v : \mu\alpha.\tau}{\Psi;\Delta;\Gamma \vdash \mathsf{unfold}\ v : \tau[\mu\alpha.\tau/\alpha]} \qquad \frac{\Psi(\ell) = \tau}{\Psi;\Delta;\Gamma \vdash \ell : \mathsf{ref}\ \tau}$$

$$\frac{\Psi;\Delta;\Gamma \vdash v : \tau}{\Psi;\Delta;\Gamma \vdash \mathsf{new}\ v : \mathsf{ref}\ \tau} \qquad \frac{\Psi;\Delta;\Gamma \vdash v_1 : \mathsf{ref}\ \tau \qquad \Psi;\Delta;\Gamma \vdash v_2 : \tau}{\Psi;\Delta;\Gamma \vdash v_1 := v_2 : \mathsf{ref}\ \tau} \qquad \frac{\Psi;\Delta;\Gamma \vdash v : \mathsf{ref}\ \tau}{\Psi;\Delta;\Gamma \vdash !v : \tau}$$

$$\frac{\Psi;\Delta;\Gamma \vdash v_1 : \tau_1 \qquad \cdots \qquad \Psi;\Delta;\Gamma \vdash v_n : \tau_n}{\Psi;\Delta;\Gamma \vdash \langle v_1, \ldots, v_n\rangle : \langle \tau_1, \ldots, \tau_n\rangle} \qquad \frac{\Psi;\Delta;\Gamma \vdash v : \langle \tau_1, \ldots, \tau_n\rangle}{\Psi;\Delta;\Gamma \vdash \pi_i(v) : \tau_i}$$

$$\frac{\Psi;\Delta;\Gamma \vdash e_1 : \tau_1 \qquad \Psi;\Delta;\Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Psi;\Delta;\Gamma \vdash \mathsf{let}\ x = e_1\ \mathsf{in}\ e_2 : \tau_2}$$

## 2.6 Reduction Relation $\boxed{\langle H \mid e\rangle \longmapsto \langle H' \mid e'\rangle}$

| | | |
|---|---|---|
| $\langle H \mid E[n_1\ p\ n_2]\rangle$ | $\longmapsto \langle H \mid E[\delta(p, n_1, n_2)]\rangle$ | |
| $\langle H \mid E[\mathsf{if0}\ 0\ e_1\ e_2]\rangle$ | $\longmapsto \langle H \mid E[e_1]\rangle$ | |
| $\langle H \mid E[\mathsf{if0}\ n\ e_1\ e_2]\rangle$ | $\longmapsto \langle H \mid E[e_2]\rangle$ | $n \neq 0$ |
| $\langle H \mid E[(\lambda[\overline{\alpha}](\overline{x : \tau}).e)\ [\overline{\tau'}]\ \overline{v}]\rangle$ | $\longmapsto \langle H \mid E[e[\overline{\tau'/\alpha}][\overline{v/x}]]\rangle$ | |
| $\langle H \mid E[\mathsf{unpack}\ \langle \alpha, x\rangle = (\mathsf{pack}\ \langle \tau', v\rangle\ \mathsf{as}\ \exists\alpha.\tau)\ \mathsf{in}\ e]\rangle$ | $\longmapsto \langle H \mid E[e[\tau'/\alpha][v/x]]\rangle$ | |
| $\langle H \mid E[\mathsf{unfold}\ (\mathsf{fold}_{\mu\alpha.\tau}\ v)]\rangle$ | $\longmapsto \langle H \mid E[v]\rangle$ | |
| $\langle H \mid E[\mathsf{new}\ v]\rangle$ | $\longmapsto \langle H[\ell \mapsto v] \mid E[\ell]\rangle$ | $\ell \notin H$ |
| $\langle H \mid E[\ell := v]\rangle$ | $\longmapsto \langle H[\ell \mapsto v] \mid E[\ell]\rangle$ | $\ell \in H$ |
| $\langle H \mid E[!\ell]\rangle$ | $\longmapsto \langle H \mid E[v]\rangle$ | $H(\ell) = v$ |
| $\langle H \mid E[\pi_i(\langle v_1, \ldots, v_n\rangle)]\rangle$ | $\longmapsto \langle H \mid E[v_i]\rangle$ | |
| $\langle H \mid E[\mathsf{let}\ x = v\ \mathsf{in}\ e]\rangle$ | $\longmapsto \langle H \mid E[e[v/x]]\rangle$ | |

## 2.7 M to C Translation

### 2.7.1 Compiler Type Translation

$$\alpha^{\mathcal{C}} = \boldsymbol{\alpha} \qquad \forall[\overline{\alpha}].(\overline{\tau}) \to \tau'^{\mathcal{C}} = \exists\boldsymbol{\beta}.\langle(\forall[\overline{\boldsymbol{\alpha}}].(\boldsymbol{\beta}, \overline{\tau^{\mathcal{C}}}) \to \tau'^{\mathcal{C}}), \boldsymbol{\beta}\rangle$$

$$\mathsf{unit}^{\mathcal{C}} = \mathbf{unit} \qquad \exists\alpha.\tau^{\mathcal{C}} = \exists\boldsymbol{\alpha}.\tau^{\mathcal{C}}$$

$$\mathsf{ref}\,\tau^{\mathcal{C}} = \mathbf{ref}\,\tau^{\mathcal{C}} \qquad \mu\alpha.\tau^{\mathcal{C}} = \boldsymbol{\mu\alpha}.\tau^{\mathcal{C}}$$

$$\langle\tau_1, \ldots, \tau_n\rangle^{\mathcal{C}} = \langle\tau_1{}^{\mathcal{C}}, \ldots, \tau_n{}^{\mathcal{C}}\rangle \qquad \mathsf{int}^{\mathcal{C}} = \mathbf{int}$$

### 2.7.2 Compiler Term Translation $\boxed{\Psi; \Delta; \Gamma \vdash \mathsf{e} : \tau \rightsquigarrow \mathbf{e}}$

(where $\Psi; \Delta; \Gamma \vdash \mathsf{e} : \tau$ and $\Psi^{\mathcal{C}}; \Delta^{\mathcal{C}}; \Gamma^{\mathcal{C}} \vdash \mathbf{e} : \tau^{\mathcal{C}}$)

$$\frac{\mathsf{x} : \tau \in \Gamma}{\Psi; \Delta; \Gamma \vdash \mathsf{x} : \tau \rightsquigarrow \mathbf{x}} \qquad \frac{}{\Psi; \Delta; \Gamma \vdash () : \mathsf{unit} \rightsquigarrow ()} \qquad \frac{}{\Psi; \Delta; \Gamma \vdash \mathsf{n} : \mathsf{int} \rightsquigarrow \mathbf{n}}$$

$$\frac{\Psi; \Delta; \Gamma \vdash \mathsf{v}_1 : \mathsf{int} \rightsquigarrow \mathbf{v_1} \qquad \Psi; \Delta; \Gamma \vdash \mathsf{v}_2 : \mathsf{int} \rightsquigarrow \mathbf{v_2}}{\Psi; \Delta; \Gamma \vdash \mathsf{v}_1\ \mathsf{p}\ \mathsf{v}_2 : \mathsf{int} \rightsquigarrow \mathbf{v_1\ p\ v_2}}$$

$$\frac{\Psi; \Delta; \Gamma \vdash \mathsf{v}_1 : \mathsf{int} \rightsquigarrow \mathbf{v_1} \qquad \Psi; \Delta; \Gamma \vdash \mathsf{e}_2 : \tau \rightsquigarrow \mathbf{e_2} \qquad \Psi; \Delta; \Gamma \vdash \mathsf{e}_3 : \tau \rightsquigarrow \mathbf{e_3}}{\Psi; \Delta; \Gamma \vdash \mathsf{if0}\,\mathsf{v}_1\ \mathsf{e}_2\ \mathsf{e}_3 : \tau \rightsquigarrow \mathbf{if0\,v_1\ e_2\ e_3}}$$

$$\frac{\begin{array}{c}\Psi; \Delta, \overline{\alpha}; \Gamma, \overline{\mathsf{x} : \tau} \vdash \mathsf{e} : \tau' \rightsquigarrow \mathbf{e} \qquad \mathsf{y}_1, \ldots, \mathsf{y}_m = \mathrm{fv}(\lambda[\overline{\alpha}](\overline{\mathsf{x} : \tau}).\mathsf{e}) \qquad \beta_1, \ldots, \beta_k = \mathrm{ftv}(\lambda[\overline{\alpha}](\overline{\mathsf{x} : \tau}).\mathsf{e}) \\ \boldsymbol{\tau_{env}} = \langle(\Gamma(\mathsf{y}_1))^{\mathcal{C}}, \ldots, (\Gamma(\mathsf{y}_m))^{\mathcal{C}}\rangle \qquad \mathbf{v} = \boldsymbol{\lambda}[\overline{\boldsymbol{\beta}}, \overline{\boldsymbol{\alpha}}](\mathbf{z} : \boldsymbol{\tau_{env}}, \overline{\mathbf{x} : \tau^{\mathcal{C}}}).\mathbf{let\ y_1} = \boldsymbol{\pi_1}(\mathbf{z})\ \mathbf{in}\ \ldots\mathbf{let\ y_m} = \boldsymbol{\pi_m}(\mathbf{z})\ \mathbf{in\ e}\end{array}}{\Psi; \Delta; \Gamma \vdash \lambda[\overline{\alpha}](\overline{\mathsf{x} : \tau}).\mathsf{e} : \forall[\overline{\alpha}].(\overline{\tau}) \to \tau' \rightsquigarrow \mathbf{pack}\,\langle\boldsymbol{\tau_{env}}, \langle\mathbf{v}[\overline{\boldsymbol{\beta}}], \langle\overline{\mathbf{y}}\rangle\rangle\rangle\ \mathbf{as}\ \exists\boldsymbol{\alpha'}.\langle(\forall[\overline{\boldsymbol{\alpha}}].(\boldsymbol{\alpha'}, \overline{\tau^{\mathcal{C}}}) \to \tau'^{\mathcal{C}}), \boldsymbol{\alpha'}\rangle}$$

$$\frac{\Psi; \Delta; \Gamma \vdash \mathsf{v} : \tau[\tau'/\alpha] \rightsquigarrow \mathbf{v}}{\Psi; \Delta; \Gamma \vdash \mathsf{pack}\,\langle\tau', \mathsf{v}\rangle\ \mathsf{as}\,\exists\alpha.\tau : \exists\alpha.\tau \rightsquigarrow \mathbf{pack}\,\langle\tau'^{\mathcal{C}}, \mathbf{v}\rangle\ \mathbf{as}\,\exists\alpha.\tau^{\mathcal{C}}}$$

$$\frac{\Psi; \Delta; \Gamma \vdash \mathsf{v} : \exists\alpha.\tau \rightsquigarrow \mathbf{v} \qquad \Psi; \Delta, \alpha; \Gamma, \mathsf{x} : \tau \vdash \mathsf{e} : \tau' \rightsquigarrow \mathbf{e}}{\Psi; \Delta; \Gamma \vdash \mathsf{unpack}\,\langle\alpha, \mathsf{x}\rangle = \mathsf{v}\ \mathsf{in}\ \mathsf{e} : \tau' \rightsquigarrow \mathbf{unpack}\,\langle\boldsymbol{\alpha}, \mathbf{x}\rangle = \mathbf{v\ in\ e}} \qquad \frac{\Psi; \Delta; \Gamma \vdash \mathsf{v} : \tau[\mu\alpha.\tau/\alpha] \rightsquigarrow \mathbf{v}}{\Psi; \Delta; \Gamma \vdash \mathsf{fold}_{\mu\alpha.\tau}\,\mathsf{v} : \mu\alpha.\tau \rightsquigarrow \mathbf{fold}_{\mu\alpha.\tau^{\mathcal{C}}}\,\mathbf{v}}$$

$$\frac{\Psi(\ell) = \tau}{\Psi; \Delta; \Gamma \vdash \ell : \mathsf{ref}\,\tau \rightsquigarrow \boldsymbol{\ell}} \qquad \frac{\Psi; \Delta; \Gamma \vdash \mathsf{v}_1 : \tau_1 \rightsquigarrow \mathbf{v_1} \qquad \ldots \qquad \Psi; \Delta; \Gamma \vdash \mathsf{v}_n : \tau_n \rightsquigarrow \mathbf{v_n}}{\Psi; \Delta; \Gamma \vdash \langle\mathsf{v}_1, \ldots, \mathsf{v}_n\rangle : \langle\tau_1, \ldots, \tau_n\rangle \rightsquigarrow \langle\mathbf{v_1}, \ldots, \mathbf{v_n}\rangle}$$

$$\frac{\Psi; \Delta; \Gamma \vdash \mathsf{e}_1 : \tau_1 \rightsquigarrow \mathbf{e_1} \qquad \Psi; \Delta; \Gamma, \mathsf{x} : \tau_1 \vdash \mathsf{e}_2 : \tau_2 \rightsquigarrow \mathbf{e_2}}{\Psi; \Delta; \Gamma \vdash \mathsf{let}\,\mathsf{x} = \mathsf{e}_1\ \mathsf{in}\ \mathsf{e}_2 : \tau_2 \rightsquigarrow \mathbf{let\ x} = \mathbf{e_1\ in\ e_2}}$$

$$\frac{\Psi; \Delta; \Gamma \vdash \mathsf{v}_0 : \forall[\overline{\alpha}].(\overline{\tau_1}) \to \tau_2 \rightsquigarrow \mathbf{v_0} \qquad \Delta \vdash \overline{\tau} \qquad \Psi; \Delta; \Gamma \vdash \overline{\mathsf{v}} : \overline{\tau_1[\overline{\tau}/\overline{\alpha}]} \rightsquigarrow \overline{\mathbf{v}}}{\Psi; \Delta; \Gamma \vdash \mathsf{v}_0\ [\overline{\tau}]\ \overline{\mathsf{v}} : \tau_2[\overline{\tau}/\overline{\alpha}] \rightsquigarrow \mathbf{unpack}\,\langle\boldsymbol{\beta}, \mathbf{z}\rangle = \mathbf{v_0\ in\ let}\,(\mathbf{f}, \mathbf{y}) = (\boldsymbol{\pi_1}(\mathbf{z}), \boldsymbol{\pi_2}(\mathbf{z}))\ \mathbf{in\ f}\ [\overline{\tau^{\mathcal{C}}}]\ (\mathbf{y}, \overline{\mathbf{v}})}$$

$$\frac{\Psi; \Delta; \Gamma \vdash \mathsf{v} : \mu\alpha.\tau \rightsquigarrow \mathbf{v}}{\Psi; \Delta; \Gamma \vdash \mathsf{unfold}\,\mathsf{v} : \tau[\mu\alpha.\tau/\alpha] \rightsquigarrow \mathbf{unfold\,v}} \qquad \frac{\Psi; \Delta; \Gamma \vdash \mathsf{v} : \tau \rightsquigarrow \mathbf{v}}{\Psi; \Delta; \Gamma \vdash \mathsf{new}\,\mathsf{v} : \mathsf{ref}\,\tau \rightsquigarrow \mathbf{new\,v}}$$

$$\frac{\Psi; \Delta; \Gamma \vdash \mathsf{v}_1 : \mathsf{ref}\,\tau \rightsquigarrow \mathbf{v_1} \qquad \Psi; \Delta; \Gamma \vdash \mathsf{v}_2 : \tau \rightsquigarrow \mathbf{v_2}}{\Psi; \Delta; \Gamma \vdash \mathsf{v}_1 := \mathsf{v}_2 : \mathsf{unit} \rightsquigarrow \mathbf{v_1} := \mathbf{v_2}} \qquad \frac{\Psi; \Delta; \Gamma \vdash \mathsf{v} : \mathsf{ref}\,\tau \rightsquigarrow \mathbf{v}}{\Psi; \Delta; \Gamma \vdash !\mathsf{v} : \tau \rightsquigarrow \mathbf{!v}} \qquad \frac{\Psi; \Delta; \Gamma \vdash \mathsf{v} : \langle\tau_1, \ldots, \tau_n\rangle \rightsquigarrow \mathbf{v}}{\Psi; \Delta; \Gamma \vdash \pi_i(\mathsf{v}) : \tau_i \rightsquigarrow \boldsymbol{\pi_i}(\mathbf{v})}$$

# 3 Closure conversion: M to C

## 3.1 Compiler Type Translation

$$
\begin{aligned}
\alpha^{\mathcal{C}} &= \boldsymbol{\alpha} & \forall[\overline{\alpha}].(\overline{\tau}) \to \tau'^{\mathcal{C}} &= \exists \boldsymbol{\beta}.\langle(\forall[\overline{\boldsymbol{\alpha}}].(\boldsymbol{\beta}, \overline{\tau^{\mathcal{C}}}) \to \tau'^{\mathcal{C}}), \boldsymbol{\beta}\rangle \\
\mathsf{unit}^{\mathcal{C}} &= \mathbf{unit} & \exists \alpha.\tau^{\mathcal{C}} &= \exists \boldsymbol{\alpha}.\tau^{\mathcal{C}} \\
\mathsf{ref}\ \tau^{\mathcal{C}} &= \mathbf{ref}\ \tau^{\mathcal{C}} & \mu\alpha.\tau^{\mathcal{C}} &= \mu\boldsymbol{\alpha}.\tau^{\mathcal{C}} \\
\langle \tau_1, \ldots, \tau_n \rangle^{\mathcal{C}} &= \langle {\tau_1}^{\mathcal{C}}, \ldots, {\tau_n}^{\mathcal{C}} \rangle & \mathsf{int}^{\mathcal{C}} &= \mathbf{int}
\end{aligned}
$$

## 3.2 Compiler Term Translation $\boxed{\Psi; \Delta; \Gamma \vdash \mathsf{e} : \tau \rightsquigarrow \mathbf{e}}$

(where $\Psi; \Delta; \Gamma \vdash \mathsf{e} : \tau$ and $\Psi^{\mathcal{C}}; \Delta^{\mathcal{C}}; \Gamma^{\mathcal{C}} \vdash \mathbf{e} : \tau^{\mathcal{C}}$)

$$
\frac{\mathsf{x} : \tau \in \Gamma}{\Psi; \Delta; \Gamma \vdash \mathsf{x} : \tau \rightsquigarrow \mathbf{x}}
\qquad
\frac{}{\Psi; \Delta; \Gamma \vdash () : \mathsf{unit} \rightsquigarrow \mathbf{()}}
\qquad
\frac{}{\Psi; \Delta; \Gamma \vdash \mathsf{n} : \mathsf{int} \rightsquigarrow \mathbf{n}}
$$

$$
\frac{\Psi; \Delta; \Gamma \vdash \mathsf{v}_1 : \mathsf{int} \rightsquigarrow \mathbf{v_1} \qquad \Psi; \Delta; \Gamma \vdash \mathsf{v}_2 : \mathsf{int} \rightsquigarrow \mathbf{v_2}}{\Psi; \Delta; \Gamma \vdash \mathsf{v}_1\ \mathsf{p}\ \mathsf{v}_2 : \mathsf{int} \rightsquigarrow \mathbf{v_1\ p\ v_2}}
$$

$$
\frac{\Psi; \Delta; \Gamma \vdash \mathsf{v}_1 : \mathsf{int} \rightsquigarrow \mathbf{v_1} \qquad \Psi; \Delta; \Gamma \vdash \mathsf{e}_2 : \tau \rightsquigarrow \mathbf{e_2} \qquad \Psi; \Delta; \Gamma \vdash \mathsf{e}_3 : \tau \rightsquigarrow \mathbf{e_3}}{\Psi; \Delta; \Gamma \vdash \mathsf{if0}\ \mathsf{v}_1\ \mathsf{e}_2\ \mathsf{e}_3 : \tau \rightsquigarrow \mathbf{if0\ v_1\ e_2\ e_3}}
$$

$$
\frac{\begin{array}{c}\Psi; \Delta, \overline{\alpha}; \Gamma, \overline{\mathsf{x} : \tau} \vdash \mathsf{e} : \tau' \rightsquigarrow \mathbf{e} \qquad \mathsf{y}_1, \ldots, \mathsf{y}_m = \mathrm{fv}(\lambda[\overline{\alpha}](\overline{\mathsf{x} : \tau}).\mathsf{e}) \qquad \beta_1, \ldots, \beta_k = \mathrm{ftv}(\lambda[\overline{\alpha}](\overline{\mathsf{x} : \tau}).\mathsf{e}) \\ \boldsymbol{\tau_{env}} = \langle (\Gamma(\mathsf{y}_1))^{\mathcal{C}}, \ldots, (\Gamma(\mathsf{y}_m))^{\mathcal{C}} \rangle \qquad \mathbf{v} = \boldsymbol{\lambda}[\overline{\boldsymbol{\beta}}, \overline{\boldsymbol{\alpha}}](\mathbf{z} : \boldsymbol{\tau_{env}}, \overline{\mathbf{x} : \tau^{\mathcal{C}}}).\mathbf{let\ y_1 = \pi_1(z)\ in\ \ldots let\ y_m = \pi_m(z)\ in\ e}\end{array}}{\Psi; \Delta; \Gamma \vdash \lambda[\overline{\alpha}](\overline{\mathsf{x} : \tau}).\mathsf{e} : \forall[\overline{\alpha}].(\overline{\tau}) \to \tau' \rightsquigarrow \mathbf{pack}\ \langle \boldsymbol{\tau_{env}}, \langle \mathbf{v}[\overline{\boldsymbol{\beta}}], \langle \overline{\mathbf{y}} \rangle \rangle \rangle\ \mathbf{as}\ \exists \boldsymbol{\alpha}'.\langle (\forall[\overline{\boldsymbol{\alpha}}].(\boldsymbol{\alpha}', \overline{\tau^{\mathcal{C}}}) \to \tau'^{\mathcal{C}}), \boldsymbol{\alpha}' \rangle}
$$

$$
\frac{\Psi; \Delta; \Gamma \vdash \mathsf{v} : \tau[\tau'/\alpha] \rightsquigarrow \mathbf{v}}{\Psi; \Delta; \Gamma \vdash \mathsf{pack}\ \langle \tau', \mathsf{v} \rangle\ \mathsf{as}\ \exists \alpha.\tau : \exists \alpha.\tau \rightsquigarrow \mathbf{pack}\ \langle \tau'^{\mathcal{C}}, \mathbf{v} \rangle\ \mathbf{as}\ \exists \boldsymbol{\alpha}.\tau^{\mathcal{C}}}
$$

$$
\frac{\Psi; \Delta; \Gamma \vdash \mathsf{v} : \exists \alpha.\tau \rightsquigarrow \mathbf{v} \qquad \Psi; \Delta, \alpha; \Gamma, \mathsf{x} : \tau \vdash \mathsf{e} : \tau' \rightsquigarrow \mathbf{e}}{\Psi; \Delta; \Gamma \vdash \mathsf{unpack}\ \langle \alpha, \mathsf{x} \rangle = \mathsf{v}\ \mathsf{in}\ \mathsf{e} : \tau' \rightsquigarrow \mathbf{unpack}\ \langle \boldsymbol{\alpha}, \mathbf{x} \rangle = \mathbf{v\ in\ e}}
\qquad
\frac{\Psi; \Delta; \Gamma \vdash \mathsf{v} : \tau[\mu\alpha.\tau/\alpha] \rightsquigarrow \mathbf{v}}{\Psi; \Delta; \Gamma \vdash \mathsf{fold}_{\mu\alpha.\tau}\ \mathsf{v} : \mu\alpha.\tau \rightsquigarrow \mathbf{fold}_{\mu\boldsymbol{\alpha}.\tau^{\mathcal{C}}}\ \mathbf{v}}
$$

$$
\frac{\Psi(\ell) = \tau}{\Psi; \Delta; \Gamma \vdash \ell : \mathsf{ref}\ \tau \rightsquigarrow \boldsymbol{\ell}}
\qquad
\frac{\Psi; \Delta; \Gamma \vdash \mathsf{v}_1 : \tau_1 \rightsquigarrow \mathbf{v_1} \qquad \ldots \qquad \Psi; \Delta; \Gamma \vdash \mathsf{v}_n : \tau_n \rightsquigarrow \mathbf{v_n}}{\Psi; \Delta; \Gamma \vdash \langle \mathsf{v}_1, \ldots, \mathsf{v}_n \rangle : \langle \tau_1, \ldots, \tau_n \rangle \rightsquigarrow \langle \mathbf{v_1}, \ldots, \mathbf{v_n} \rangle}
$$

$$
\frac{\Psi; \Delta; \Gamma \vdash \mathsf{e}_1 : \tau_1 \rightsquigarrow \mathbf{e_1} \qquad \Psi; \Delta; \Gamma, \mathsf{x} : \tau_1 \vdash \mathsf{e}_2 : \tau_2 \rightsquigarrow \mathbf{e_2}}{\Psi; \Delta; \Gamma \vdash \mathsf{let}\ \mathsf{x} = \mathsf{e}_1\ \mathsf{in}\ \mathsf{e}_2 : \tau_2 \rightsquigarrow \mathbf{let\ x = e_1\ in\ e_2}}
$$

$$
\frac{\Psi; \Delta; \Gamma \vdash \mathsf{v}_0 : \forall[\overline{\alpha}].(\overline{\tau_1}) \to \tau_2 \rightsquigarrow \mathbf{v_0} \qquad \Delta \vdash \overline{\tau} \qquad \Psi; \Delta; \Gamma \vdash \overline{\mathsf{v}} : \overline{\tau_1 \overline{[\tau/\alpha]}} \rightsquigarrow \overline{\mathbf{v}}}{\Psi; \Delta; \Gamma \vdash \mathsf{v}_0\ [\overline{\tau}]\ \overline{\mathsf{v}} : \tau_2 \overline{[\tau/\alpha]} \rightsquigarrow \mathbf{unpack}\ \langle \boldsymbol{\beta}, \mathbf{z} \rangle = \mathbf{v_0\ in\ let\ (f, y) = (\pi_1(z), \pi_2(z))\ in\ f}\ [\overline{\tau^{\mathcal{C}}}]\ (\mathbf{y}, \overline{\mathbf{v}})}
$$

$$
\frac{\Psi; \Delta; \Gamma \vdash \mathsf{v} : \mu\alpha.\tau \rightsquigarrow \mathbf{v}}{\Psi; \Delta; \Gamma \vdash \mathsf{unfold}\ \mathsf{v} : \tau[\mu\alpha.\tau/\alpha] \rightsquigarrow \mathbf{unfold\ v}}
\qquad
\frac{\Psi; \Delta; \Gamma \vdash \mathsf{v} : \tau \rightsquigarrow \mathbf{v}}{\Psi; \Delta; \Gamma \vdash \mathsf{new}\ \mathsf{v} : \mathsf{ref}\ \tau \rightsquigarrow \mathbf{new\ v}}
$$

$$
\frac{\Psi; \Delta; \Gamma \vdash \mathsf{v}_1 : \mathsf{ref}\ \tau \rightsquigarrow \mathbf{v_1} \qquad \Psi; \Delta; \Gamma \vdash \mathsf{v}_2 : \tau \rightsquigarrow \mathbf{v_2}}{\Psi; \Delta; \Gamma \vdash \mathsf{v}_1 := \mathsf{v}_2 : \mathsf{unit} \rightsquigarrow \mathbf{v_1 := v_2}}
\qquad
\frac{\Psi; \Delta; \Gamma \vdash \mathsf{v} : \mathsf{ref}\ \tau \rightsquigarrow \mathbf{v}}{\Psi; \Delta; \Gamma \vdash !\mathsf{v} : \tau \rightsquigarrow \mathbf{!v}}
\qquad
\frac{\Psi; \Delta; \Gamma \vdash \mathsf{v} : \langle \tau_1, \ldots, \tau_n \rangle \rightsquigarrow \mathbf{v}}{\Psi; \Delta; \Gamma \vdash \pi_i(\mathsf{v}) : \tau_i \rightsquigarrow \boldsymbol{\pi_i}(\mathbf{v})}
$$

## 4 Multi-Language Semantics: M + C

Here we define the grammar of the multi-language between M and C.

$$\tau ::= \cdots \mid \mathsf{L}\langle\boldsymbol{\tau}\rangle \qquad \tau ::= \tau \mid \boldsymbol{\tau}$$
$$\mathsf{e} ::= \cdots \mid {}^{\tau}\mathcal{MC}\,\mathbf{e} \qquad e ::= \mathsf{e} \mid \mathbf{e}$$
$$\mathsf{v} ::= \cdots \mid {}^{\mathsf{ref}\,\tau}\mathcal{MC}\,\boldsymbol{\ell} \mid {}^{\mathsf{L}\langle\boldsymbol{\tau}\rangle}\mathcal{MC}\,\mathbf{v} \qquad v ::= \mathsf{v} \mid \mathbf{v}$$
$$\mathsf{E} ::= \cdots \mid {}^{\tau}\mathcal{MC}\,\mathbf{E} \qquad E ::= \mathsf{E} \mid \mathbf{E}$$
$$\boldsymbol{\tau} ::= \cdots \mid \lceil\alpha\rceil \qquad H ::= (\mathsf{H}, \mathbf{H})$$
$$\mathbf{e} ::= \cdots \mid \mathcal{CM}^{\tau}\,\mathsf{e} \qquad \Psi ::= (\Psi, \boldsymbol{\Psi})$$
$$\mathbf{v} ::= \cdots \mid \mathcal{CM}^{\mathsf{ref}\,\tau}\,\ell \qquad \Delta ::= \cdot \mid \Delta, \alpha \mid \Delta, \boldsymbol{\alpha}$$
$$\mathbf{E} ::= \cdots \mid \mathcal{CM}^{\tau}\,\mathsf{E} \qquad \Gamma ::= \cdot \mid \Gamma, \mathsf{x}\!:\!\tau \mid \Gamma, \mathbf{x}\!:\!\boldsymbol{\tau}$$

Interoperability between M and C should be guided by the closure-conversion type translation. However, notice that the multi-language specification has binding forms from both M and C languages. Using the closure-conversion type translation leads to problems with binding.

For instance, take the following example from [2]. Consider the type $\forall[\alpha].(\alpha) \to \alpha$. Since $\alpha^{\mathcal{C}} = \boldsymbol{\alpha}$, the translation of this type is

$$(\forall[\alpha].(\alpha) \to \alpha)^{\mathcal{C}} = \exists\boldsymbol{\beta}.\langle(\forall[\boldsymbol{\alpha}].(\boldsymbol{\beta}, \boldsymbol{\alpha}) \to \boldsymbol{\alpha}), \boldsymbol{\beta}\rangle.$$

If we naïvely try to translate a C value of type $(\forall[\alpha].(\alpha) \to \alpha)^{\mathcal{C}}$ to a M value of type $\forall[\alpha].(\alpha) \to \alpha$, we get the following:

$$^{\forall[\alpha].(\alpha)\to\alpha}\mathbf{MC}(\mathbf{v}) = \lambda[\alpha](\mathsf{x}\!:\alpha).{}^{\alpha}\mathcal{MC}\;(\mathbf{unpack}\;\langle\boldsymbol{\beta}, \mathbf{z}\rangle = \mathbf{v}\;\mathbf{in}\;\mathbf{let}\;(\mathbf{f}, \mathbf{y}) = (\boldsymbol{\pi_1}(\mathbf{z}), \boldsymbol{\pi_2}(\mathbf{z}))\;\mathbf{in}\;\mathbf{f}\;[\overline{\alpha^{\mathcal{C}}}]\;(\mathbf{y}, \overline{\mathcal{CM}^{\alpha}\,\mathsf{x}}))$$

Note that we have not expanded $\alpha^{\mathcal{C}}$ in the application produced by this translation. It would expand to a C type variable $\boldsymbol{\alpha}$, but we cannot allow this because that $\boldsymbol{\alpha}$ would be unbound! What we really want is that when $\alpha$ is instantiated with a concrete type $\tau$, the positions inside language C where that type is needed receive $\tau^{\mathcal{C}}$ instead.

The solution (from Perconti and Ahmed [2]) is to come up with an "interoperability" type translation—which Perconti and Ahmed refer to as the "operational" type translation—that is essentially the same as the compiler type translation except that it delays translating unbound type variables and includes a rule for lump types, discussed below. Suspended type variables get translated once a concrete type is substituted for them. Any lemmas that operate over the multi-language must use the interoperability type translation, since this is the type translation that directs the interoperation of the multi-language. (That said, this interoperability type translation serves us well when we state the correctness of closure conversion 10.21, a statement that uses the compiler type and term translation but then manually turns free C type variables into suspended M type variables.)

What are the lumps in the interoperability type translation all about? Again, let us refer to the explanation in [2]. Consider translating values of type $\forall[\alpha].(\alpha) \to \alpha$ from M into C. Once again, the existing machinery is not quite sufficient. Here is a naïve attempt:

$$\mathbf{CM}^{\forall[\alpha].(\alpha)\to\alpha}(\mathsf{v}) = \mathbf{pack}\;\langle\mathbf{unit}, \langle\mathbf{v}, ()\rangle\rangle\;\mathbf{as}\;(\forall[\alpha].(\alpha) \to \alpha)^{\langle\mathcal{C}\rangle}$$
$$\text{where } \mathbf{v} = \boldsymbol{\lambda}[\boldsymbol{\alpha}](\mathbf{z}\!:\mathbf{unit}, \mathbf{x}\!:\boldsymbol{\alpha}).\mathcal{CM}^{\alpha}\;(\mathsf{v}\;[\alpha]\;{}^{\alpha}\mathcal{MC}\,\mathbf{x}).$$

This time, we have translated the binder for $\alpha$ into a C binder for $\boldsymbol{\alpha}$, but we are left with free occurrences of $\alpha$ in the result! This is not a suitable translation, as we must produce a closed value. Note that the boundary terms in the body of $\mathbf{v}$ expect to be annotated with a type that translates to $\boldsymbol{\alpha}$.

To fix this problem, we introduce a *lump type* $\mathsf{L}\langle\boldsymbol{\tau}\rangle$ that allows us to pass C values to M terms as opaque lumps. The introduction form for the lump type is the boundary term ${}^{\mathsf{L}\langle\boldsymbol{\tau}\rangle}\mathcal{MC}\,\mathbf{e}$, and the elimination form is $\mathcal{CM}^{\mathsf{L}\langle\boldsymbol{\tau}\rangle}\,\mathsf{e}$. A pair of opposite boundaries at lump type cancel, to yield the underlying C value. We extend the boundary type translation by defining $\mathsf{L}\langle\boldsymbol{\tau}\rangle^{\langle\mathcal{C}\rangle} = \boldsymbol{\tau}$.

Now the three free occurrences of $\alpha$ in $\mathbf{v}$ can be replaced with $\mathsf{L}\langle\boldsymbol{\alpha}\rangle$, yielding a well-typed translation.

### 4.1 Specification Type Translation

$$\alpha^{\langle \mathcal{C} \rangle} = \lceil \alpha \rceil \qquad \mathsf{L}\langle \boldsymbol{\tau} \rangle^{\langle \mathcal{C} \rangle} = \boldsymbol{\tau}$$

$$\mathsf{unit}^{\langle \mathcal{C} \rangle} = \mathbf{unit} \qquad \mathsf{int}^{\langle \mathcal{C} \rangle} = \mathbf{int}$$

$$\mathsf{ref}\,\tau^{\langle \mathcal{C} \rangle} = \mathbf{ref}\,\tau^{\langle \mathcal{C} \rangle} \qquad \mu\alpha.\tau^{\langle \mathcal{C} \rangle} = \boldsymbol{\mu\alpha.}(\tau^{\langle \mathcal{C} \rangle}[\boldsymbol{\alpha}/\lceil \alpha \rceil])$$

$$\langle \tau_1, \dots, \tau_n \rangle^{\langle \mathcal{C} \rangle} = \langle \tau_1{}^{\langle \mathcal{C} \rangle}, \dots, \tau_n{}^{\langle \mathcal{C} \rangle} \rangle \qquad \exists\alpha.\tau^{\langle \mathcal{C} \rangle} = \boldsymbol{\exists\alpha.}\tau^{\langle \mathcal{C} \rangle}[\boldsymbol{\alpha}/\lceil \alpha \rceil]$$

$$\forall[\overline{\alpha}].(\overline{\tau}) \rightarrow \tau'^{\langle \mathcal{C} \rangle} = \boldsymbol{\exists\beta.}\langle \left( \forall[\overline{\boldsymbol{\alpha}}].(\boldsymbol{\beta}, \overline{\tau^{\langle \mathcal{C} \rangle}\,\overline{[\boldsymbol{\alpha}/\lceil \alpha \rceil]}}) \rightarrow \tau'^{\langle \mathcal{C} \rangle}\,\overline{[\boldsymbol{\alpha}/\lceil \alpha \rceil]} \right), \boldsymbol{\beta} \rangle$$

### 4.2 Type Substitution

$$\lceil \alpha \rceil [\tau/\alpha] = \tau^{\langle \mathcal{C} \rangle}$$

### 4.3 Well-formed Types $\boxed{\Delta \vdash \tau}$

Adapt the rules for $\Delta \vdash \tau$ and $\boldsymbol{\Delta} \vdash \boldsymbol{\tau}$ by changing the environments to $\Delta$ (multi-language environment instead of a single-language environment), and add the following rules:

$$\frac{\Delta \vdash \boldsymbol{\tau}}{\Delta \vdash \mathsf{L}\langle \boldsymbol{\tau} \rangle} \qquad \frac{\alpha \in \Delta}{\Delta \vdash \lceil \alpha \rceil}$$

### 4.4 Well-Typed Store $\boxed{\vdash H : \Psi}$

$$\frac{(\cdot, \boldsymbol{\Psi}) \vdash \mathsf{H} : \Psi \qquad (\Psi, \boldsymbol{\cdot}) \vdash \mathbf{H} : \boldsymbol{\Psi}}{\vdash (\mathsf{H}, \mathbf{H}) : (\Psi, \boldsymbol{\Psi})}$$

### 4.5 Well-Typed Terms $\boxed{\Psi; \Delta; \Gamma \vdash e : \tau}$

Adapt the corresponding judgments for M and C by changing all the environments to the appropriate multi-language environment, and add the following rules:

$$\frac{\Psi; \Delta; \Gamma \vdash \mathbf{e} : \tau^{\langle \mathcal{C} \rangle}}{\Psi; \Delta; \Gamma \vdash {}^{\tau}\mathcal{MC}\,\mathbf{e} : \tau} \qquad \frac{\Psi; \Delta; \Gamma \vdash \mathsf{e} : \tau}{\Psi; \Delta; \Gamma \vdash \mathcal{CM}^{\tau}\,\mathsf{e} : \tau^{\langle \mathcal{C} \rangle}}$$

### 4.6 Value Translation

Below are the definitions for the meta functions that perform translations of value forms between languages. These are invoked by the operational semantics. We assume well-typedness of values, this ensures certain levels of well-formedness, such as the $\boldsymbol{\mu\alpha.\tau} = \mu\alpha.\tau^{\langle \mathcal{C} \rangle}$ in ${}^{\mu\alpha.\tau}\mathbf{MC}(\mathbf{fold}_{\boldsymbol{\mu\alpha.\tau}}\,\mathbf{v})$ rule. Rules that cancel adjacent boundaries around locations require the side condition $\mathsf{ref}\,\tau^{\langle \mathcal{C} \rangle} = \mathsf{ref}\,\tau'^{\langle \mathcal{C} \rangle}$ because in the boundary cancellation lemma we might wish to cancel boundaries where $\tau = \hat{\tau}$ and $\tau' = \mathsf{L}\langle \hat{\tau}^{\langle \mathcal{C} \rangle} \rangle$.

For readability, we skip some let bindings below in the definitions of $\mathbf{CM}^{\forall[\overline{\alpha}].(\overline{\tau}) \rightarrow \tau'}(\mathsf{v})$ and ${}^{\forall[\overline{\alpha}].(\overline{\tau}) \rightarrow \tau'}\mathbf{MC}(\mathbf{v})$. Technically, in the former, when defining $\mathbf{v}$ we should let-bind the terms ${}^{\tau\overline{[\mathsf{L}\langle \boldsymbol{\alpha} \rangle/\alpha]}}\mathcal{MC}\,\mathbf{x}$, and in the latter, when defining $\mathbf{e}$ we should let-bind the terms $\mathcal{CM}^{\tau}\,\mathsf{x}$.

$$
\begin{aligned}
&\mathbf{CM}^{\mathsf{unit}}(()) &&= \mathbf{()}\\
&\mathbf{CM}^{\mathsf{int}}(\mathsf{n}) &&= \mathbf{n}\\
&\mathbf{CM}^{\forall[\overline{\alpha}].(\overline{\tau})\to\tau'}(\mathsf{v}) &&= \mathbf{pack}\ \langle\mathbf{unit}, \langle\mathbf{v}, ()\rangle\rangle\ \mathbf{as}\ (\forall[\overline{\alpha}].(\overline{\tau})\to\tau')^{\langle\mathcal{C}\rangle}\\
&\qquad \text{where } \mathbf{v} = \lambda[\overline{\alpha}](\mathbf{z}:\mathbf{unit}, \overline{\mathbf{x}:\tau^{\langle\mathcal{C}\rangle}[\alpha/\lceil\alpha\rceil]}).\mathcal{CM}^{\tau'[\overline{\mathsf{L}\langle\alpha\rangle/\alpha}]}\ (\mathsf{v}\ [\overline{\mathsf{L}\langle\alpha\rangle}]\ \overline{{}^{\tau[\overline{\mathsf{L}\langle\alpha\rangle/\alpha}]}\mathcal{MC}\ \mathbf{x}})\\
&\mathbf{CM}^{\exists\alpha.\tau}(\mathsf{pack}\ \langle\tau', \mathsf{v}\rangle\ \mathsf{as}\ \exists\alpha.\tau) &&= \mathbf{pack}\ \langle\tau'^{\langle\mathcal{C}\rangle}, \mathbf{v}\rangle\ \mathbf{as}\ \exists\alpha.\tau^{\langle\mathcal{C}\rangle} \quad \text{where } \mathbf{CM}^{\tau[\tau'/\alpha]}(\mathsf{v}) = \mathbf{v}\\
&\mathbf{CM}^{\mu\alpha.\tau}(\mathsf{fold}_{\mu\alpha.\tau}\ \mathsf{v}) &&= \mathbf{fold}_{\mu\alpha.\tau^{\langle\mathcal{C}\rangle}}\ \mathbf{v} \qquad \text{where } \mathbf{CM}^{\tau[\mu\alpha.\tau/\alpha]}(\mathsf{v}) = \mathbf{v}\\
&\mathbf{CM}^{\mathsf{ref}\ \tau}(\ell) &&= \mathcal{CM}^{\mathsf{ref}\ \tau}\ \ell\\
&\mathbf{CM}^{\mathsf{ref}\ \tau}({}^{\mathsf{ref}\ \tau'}\mathcal{MC}\ \mathbf{v}) &&= \mathbf{v} \qquad \text{where } \mathsf{ref}\ \tau^{\langle\mathcal{C}\rangle} = \mathsf{ref}\ \tau'^{\langle\mathcal{C}\rangle}\\
&\mathbf{CM}^{\langle\tau_1,\dots,\tau_n\rangle}(\langle\mathsf{v}_1,\dots,\mathsf{v}_n\rangle) &&= \langle\mathbf{v_1},\dots,\mathbf{v_n}\rangle \qquad \text{where } \mathbf{CM}^{\tau_i}(\mathsf{v})_i = \mathbf{v_i}\\
&\mathbf{CM}^{\mathsf{L}\langle\tau\rangle}({}^{\mathsf{L}\langle\tau\rangle}\mathcal{MC}\ \mathbf{v}) &&= \mathbf{v}
\end{aligned}
$$

$$
\begin{aligned}
&{}^{\mathsf{unit}}\mathbf{MC}(\mathbf{()}) &&= ()\\
&{}^{\mathsf{int}}\mathbf{MC}(\mathbf{n}) &&= \mathsf{n}\\
&{}^{\forall[\overline{\alpha}].(\overline{\tau})\to\tau'}\mathbf{MC}(\mathbf{v}) &&= \lambda[\overline{\alpha}](\overline{\mathsf{x}:\tau}).{}^{\tau'}\mathcal{MC}\ \mathbf{e}\\
&\quad \text{where } \mathbf{e} = (\mathbf{unpack}\ \langle\beta, \mathbf{z}\rangle = \mathbf{v}\ \mathbf{in}\ \mathbf{let}\ (\mathbf{f}, \mathbf{y}) = (\pi_1(\mathbf{z}), \pi_2(\mathbf{z}))\ \mathbf{in}\ \mathbf{f}\ [\overline{\lceil\alpha\rceil}]\ (\mathbf{y}, \overline{\mathcal{CM}^{\tau}\ \mathsf{x}})\\
&{}^{\exists\alpha.\tau}\mathbf{MC}(\mathbf{pack}\ \langle\tau', \mathbf{v}\rangle\ \mathbf{as}\ \exists\alpha.\tau^{\langle\mathcal{C}\rangle}) &&= \mathsf{pack}\ \langle\mathsf{L}\langle\tau'\rangle, \mathsf{v}\rangle\ \mathsf{as}\ \exists\alpha.\tau \quad \text{where } {}^{\tau[\mathsf{L}\langle\tau'\rangle/\alpha]}\mathbf{MC}(\mathbf{v}) = \mathsf{v}\\
&{}^{\mu\alpha.\tau}\mathbf{MC}(\mathbf{fold}_{\mu\alpha.\tau}\ \mathbf{v}) &&= \mathsf{fold}_{\mu\alpha.\tau}\ \mathsf{v} \qquad \text{where } {}^{\tau[\mu\alpha.\tau/\alpha]}\mathbf{MC}(\mathbf{v}) = \mathsf{v}\\
&{}^{\mathsf{ref}\ \tau}\mathbf{MC}(\boldsymbol{\ell}) &&= {}^{\mathsf{ref}\ \tau}\mathcal{MC}\ \boldsymbol{\ell}\\
&{}^{\mathsf{ref}\ \tau}\mathbf{MC}(\mathcal{CM}^{\mathsf{ref}\ \tau'}\ \mathsf{v}) &&= \mathsf{v} \qquad \text{where } \mathsf{ref}\ \tau^{\langle\mathcal{C}\rangle} = \mathsf{ref}\ \tau'^{\langle\mathcal{C}\rangle}\\
&{}^{\langle\tau_1,\dots,\tau_n\rangle}\mathbf{MC}(\langle\mathbf{v_1},\dots,\mathbf{v_n}\rangle) &&= \langle\mathsf{v}_1,\dots,\mathsf{v}_n\rangle \qquad \text{where } {}^{\tau_i}\mathbf{MC}(\mathbf{v})_i = \mathsf{v_i}\\
&{}^{\mathsf{L}\langle\tau\rangle}\mathbf{MC}(\mathbf{v}) &&= {}^{\mathsf{L}\langle\tau\rangle}\mathcal{MC}\ \mathbf{v}
\end{aligned}
$$

## 4.7 Reduction Relation $\boxed{\langle H \mid e\rangle \longmapsto \langle H' \mid e'\rangle}$

Lift the M and C reduction rules to the new configuration—with heaps ($\mathsf{H}$, $\mathbf{H}$)—replacing evaluation contexts $\mathsf{E}$ and $\mathbf{E}$ with $E$.

Also add the following rules for boundary forms:

$$
\frac{\mathbf{CM}^{\tau}(\mathsf{v}) = \mathbf{v} \qquad \mathcal{CM}^{\tau}\ \mathsf{v} \text{ not a value}}{\langle H \mid E[\mathcal{CM}^{\tau}\ \mathsf{v}]\rangle \longmapsto \langle H' \mid E[\mathbf{v}]\rangle}
\qquad\qquad
\frac{{}^{\tau}\mathbf{MC}(\mathbf{v}) = \mathsf{v} \qquad {}^{\tau}\mathcal{MC}\ \mathbf{v} \text{ not a value}}{\langle H \mid E[{}^{\tau}\mathcal{MC}\ \mathbf{v}]\rangle \longmapsto \langle H' \mid E[\mathsf{v}]\rangle}
$$

$$
\begin{aligned}
&\langle H \mid E[\mathbf{!}(\mathcal{CM}^{\mathsf{ref}\ \tau}\ \mathsf{v})]\rangle &&\longmapsto \langle H \mid E[\mathcal{CM}^{\tau}\ !\mathsf{v}]\rangle && \mathcal{CM}^{\mathsf{ref}\ \tau}\ \mathsf{v} \text{ a value}\\
&\langle H \mid E[!({}^{\mathsf{ref}\ \tau}\mathcal{MC}\ \mathbf{v})]\rangle &&\longmapsto \langle H \mid E[{}^{\tau}\mathcal{MC}\ \mathbf{!v}]\rangle && {}^{\mathsf{ref}\ \tau}\mathcal{MC}\ \mathbf{v} \text{ a value}\\
&\langle H \mid E[(\mathcal{CM}^{\mathsf{ref}\ \tau}\ \mathsf{v})\ \mathbf{:=}\ \mathbf{v}]\rangle &&\longmapsto \langle H \mid E[\mathcal{CM}^{\mathsf{unit}}\ (\mathsf{v} := {}^{\tau}\mathcal{MC}\ \mathbf{v})]\rangle && \mathcal{CM}^{\mathsf{ref}\ \tau}\ \mathsf{v} \text{ a value}\\
&\langle H \mid E[({}^{\mathsf{ref}\ \tau}\mathcal{MC}\ \mathbf{v}) := \mathsf{v}]\rangle &&\longmapsto \langle H \mid E[{}^{\mathsf{unit}}\mathcal{MC}\ (\mathbf{v}\ \mathbf{:=}\ \mathcal{CM}^{\tau}\ \mathsf{v})]\rangle && {}^{\mathsf{ref}\ \tau}\mathcal{MC}\ \mathbf{v} \text{ a value}
\end{aligned}
$$

# 5 Contexts and Contextual Equivalence

$$
\begin{array}{rcl}
\mathsf{C}^{\mathsf{v}} & ::= & [\cdot]^{\mathsf{v}} \mid \lambda[\overline{\alpha}](\overline{\mathsf{x}:\tau}).\mathsf{C} \mid \mathsf{pack}\,\langle\tau,\mathsf{C}^{\mathsf{v}}\rangle\,\mathsf{as}\,\exists\alpha.\tau \mid \mathsf{fold}_{\mu\alpha.\tau}\,\mathsf{C}^{\mathsf{v}} \mid \langle\overline{\mathsf{v}},\mathsf{C}^{\mathsf{v}},\overline{\mathsf{v}}\rangle \\
\mathsf{C} & ::= & [\cdot] \mid \mathsf{C}^{\mathsf{v}} \mid \mathsf{C}^{\mathsf{v}}\ \mathsf{p}\ \mathsf{v} \mid \mathsf{v}\ \mathsf{p}\ \mathsf{C}^{\mathsf{v}} \mid \mathsf{if0}\,\mathsf{C}^{\mathsf{v}}\ \mathsf{e}\ \mathsf{e} \mid \mathsf{if0}\,\mathsf{v}\ \mathsf{C}\ \mathsf{e} \mid \mathsf{if0}\,\mathsf{v}\ \mathsf{e}\ \mathsf{C} \mid \mathsf{C}^{\mathsf{v}}\,[\overline{\tau}]\,\overline{\mathsf{v}} \mid \mathsf{v}\,[\overline{\tau}]\,\overline{\mathsf{v}}\ \mathsf{C}^{\mathsf{v}}\ \overline{\mathsf{v}} \\
& & \mid \mathsf{unpack}\,\langle\alpha,\mathsf{x}\rangle = \mathsf{C}^{\mathsf{v}}\ \mathsf{in}\ \mathsf{e} \mid \mathsf{unpack}\,\langle\alpha,\mathsf{x}\rangle = \mathsf{v}\ \mathsf{in}\ \mathsf{C} \mid \mathsf{unfold}\,\mathsf{C}^{\mathsf{v}} \mid \mathsf{new}\,\mathsf{C}^{\mathsf{v}} \mid \mathsf{C}^{\mathsf{v}} := \mathsf{v} \mid \mathsf{v} := \mathsf{C}^{\mathsf{v}} \mid !\mathsf{C}^{\mathsf{v}} \mid \pi_{\mathsf{i}}(\mathsf{C}^{\mathsf{v}}) \\
& & \mid \mathsf{let}\,\mathsf{x} = \mathsf{C}\ \mathsf{in}\ \mathsf{e} \mid \mathsf{let}\,\mathsf{x} = \mathsf{e}\ \mathsf{in}\ \mathsf{C} \mid {}^{\tau}\mathcal{MC}\,\mathbf{C} \\
\mathbf{C}^{\mathbf{v}} & ::= & [\cdot]^{\mathbf{v}} \mid \boldsymbol{\lambda}[\overline{\alpha}](\overline{\mathbf{x}:\tau}).\mathbf{C} \mid \mathbf{pack}\,\langle\tau,\mathbf{C}^{\mathbf{v}}\rangle\,\mathbf{as}\,\exists\alpha.\tau \mid \mathbf{fold}_{\mu\alpha.\tau}\,\mathbf{C}^{\mathbf{v}} \mid \langle\overline{\mathbf{v}},\mathbf{C}^{\mathbf{v}},\overline{\mathbf{v}}\rangle \mid \mathbf{C}^{\mathbf{v}}[\tau] \\
\mathbf{C} & ::= & [\cdot] \mid \mathbf{C}^{\mathbf{v}} \mid \mathbf{C}^{\mathbf{v}}\ \mathbf{p}\ \mathbf{v} \mid \mathbf{v}\ \mathbf{p}\ \mathbf{C}^{\mathbf{v}} \mid \mathbf{if0}\,\mathbf{C}^{\mathbf{v}}\ \mathbf{e}\ \mathbf{e} \mid \mathbf{if0}\,\mathbf{v}\ \mathbf{C}\ \mathbf{e} \mid \mathbf{if0}\,\mathbf{v}\ \mathbf{e}\ \mathbf{C} \mid \mathbf{C}^{\mathbf{v}}\,[]\,\overline{\mathbf{v}} \mid \mathbf{v}\,[]\,\overline{\mathbf{v}}\ \mathbf{C}^{\mathbf{v}}\ \overline{\mathbf{v}} \\
& & \mid \mathbf{unpack}\,\langle\alpha,\mathbf{x}\rangle = \mathbf{C}^{\mathbf{v}}\ \mathbf{in}\ \mathbf{e} \mid \mathbf{unpack}\,\langle\alpha,\mathbf{x}\rangle = \mathbf{v}\ \mathbf{in}\ \mathbf{C} \mid \mathbf{unfold}\,\mathbf{C}^{\mathbf{v}} \mid \mathbf{new}\,\mathbf{C}^{\mathbf{v}} \mid \mathbf{C}^{\mathbf{v}} := \mathbf{v} \mid \mathbf{v} := \mathbf{C}^{\mathbf{v}} \mid !\mathbf{C}^{\mathbf{v}} \\
& & \mid \pi_{\mathbf{i}}(\mathbf{C}^{\mathbf{v}}) \mid \mathbf{let}\,\mathbf{x} = \mathbf{C}\ \mathbf{in}\ \mathbf{e} \mid \mathbf{let}\,\mathbf{x} = \mathbf{e}\ \mathbf{in}\ \mathbf{C} \mid \mathcal{CM}^{\tau}\,\mathsf{C} \\
C & ::= & \mathsf{C} \mid \mathbf{C}
\end{array}
$$

## 5.1 Well-Typed Context $\boxed{\vdash C : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau')}$

$$
\frac{\Psi \subseteq \Psi' \qquad \Delta \subseteq \Delta' \qquad \Gamma \subseteq \Gamma'}{\vdash [\cdot]^{\mathsf{v}} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau)}
\qquad
\frac{\vdash \mathsf{C} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';(\Delta',\overline{\alpha});(\Gamma',\overline{\mathsf{x}:\tau}) \vdash \tau')}{\vdash \lambda[\overline{\alpha}](\overline{\mathsf{x}:\tau}).\mathsf{C} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \forall[\overline{\alpha}].(\overline{\tau}) \to \tau')}
$$

$$
\frac{\vdash \mathsf{C}^{\mathsf{v}} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau[\tau'/\alpha])}{\vdash \mathsf{pack}\,\langle\tau',\mathsf{C}^{\mathsf{v}}\rangle\,\mathsf{as}\,\exists\alpha.\tau : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \exists\alpha.\tau)}
$$

$$
\frac{\vdash \mathsf{C}^{\mathsf{v}} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau[\mu\alpha.\tau/\alpha])}{\vdash \mathsf{fold}_{\mu\alpha.\tau}\,\mathsf{C}^{\mathsf{v}} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \mu\alpha.\tau)}
$$

$$
\frac{\begin{array}{c}\Psi';\Delta';\Gamma' \vdash \mathsf{v}_1 : \tau_1 \qquad \cdots \qquad \Psi';\Delta';\Gamma' \vdash \mathsf{v}_{\mathsf{i}} : \tau_{\mathsf{i}} \\ \vdash \mathsf{C}^{\mathsf{v}} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau_{\mathsf{i}+1}) \qquad \Psi';\Delta';\Gamma' \vdash \mathsf{v}_{\mathsf{i}+2} : \tau_{\mathsf{i}+2} \qquad \cdots \qquad \Psi';\Delta';\Gamma' \vdash \mathsf{v}_{\mathsf{n}} : \tau_{\mathsf{n}}\end{array}}{\vdash \langle\mathsf{v}_1,\ldots,\mathsf{v}_{\mathsf{i}},\mathsf{C}^{\mathsf{v}},\mathsf{v}_{\mathsf{i}+2},\ldots,\mathsf{v}_{\mathsf{n}}\rangle : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \langle\tau_1,\ldots,\tau_{\mathsf{n}}\rangle)}
$$

$$
\frac{\Psi \subseteq \Psi' \qquad \Delta \subseteq \Delta' \qquad \Gamma \subseteq \Gamma'}{\vdash [\cdot] : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau)}
\qquad
\frac{\vdash \mathsf{C}^{\mathsf{v}} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \mathsf{int}) \qquad \Psi';\Delta';\Gamma' \vdash \mathsf{v} : \mathsf{int}}{\vdash \mathsf{C}^{\mathsf{v}}\ \mathsf{p}\ \mathsf{v} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \mathsf{int})}
$$

$$
\frac{\Psi';\Delta';\Gamma' \vdash \mathsf{v} : \mathsf{int} \qquad \vdash \mathsf{C}^{\mathsf{v}} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \mathsf{int})}{\vdash \mathsf{v}\ \mathsf{p}\ \mathsf{C}^{\mathsf{v}} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \mathsf{int})}
$$

$$
\frac{\vdash \mathsf{C}^{\mathsf{v}} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \mathsf{int}) \qquad \Psi';\Delta';\Gamma' \vdash \mathsf{e}_1 : \tau \qquad \Psi';\Delta';\Gamma' \vdash \mathsf{e}_2 : \tau}{\vdash \mathsf{if0}\,\mathsf{C}^{\mathsf{v}}\ \mathsf{e}_1\ \mathsf{e}_2 : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau)}
$$

$$
\frac{\Psi';\Delta';\Gamma' \vdash \mathsf{v} : \mathsf{int} \qquad \vdash \mathsf{C} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau) \qquad \Psi';\Delta';\Gamma' \vdash \mathsf{e}_2 : \tau}{\vdash \mathsf{if0}\,\mathsf{v}\ \mathsf{C}\ \mathsf{e}_2 : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau)}
$$

$$
\frac{\Psi';\Delta';\Gamma' \vdash \mathsf{v} : \mathsf{int} \qquad \Psi';\Delta';\Gamma' \vdash \mathsf{e}_1 : \tau \qquad \vdash \mathsf{C} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau)}{\vdash \mathsf{if0}\,\mathsf{v}\ \mathsf{e}_1\ \mathsf{C} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau)}
$$

$$\frac{\vdash \mathsf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \forall[\overline{\alpha}].(\tau_1,\ldots,\tau_n) \to \tau') \quad \Delta \vdash \overline{\tau} \quad \Psi';\Delta';\Gamma' \vdash \mathsf{v_1} : \tau_1\overline{[\tau/\alpha]} \quad \cdots \quad \Psi';\Delta';\Gamma' \vdash \mathsf{v_n} : \tau_n\overline{[\tau/\alpha]}}{\vdash \mathsf{C^v}\ [\overline{\tau}]\ \mathsf{v_1}\cdots\mathsf{v_n} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau'\overline{[\tau/\alpha]})}$$

$$\frac{\begin{array}{c}\Psi';\Delta';\Gamma' \vdash \mathsf{v} : \forall[\overline{\alpha}].(\tau_1,\ldots,\tau_n) \to \tau' \qquad \Delta \vdash \overline{\tau} \\ \Psi';\Delta';\Gamma' \vdash \mathsf{v_1} : \tau_1\overline{[\tau/\alpha]} \quad \cdots \quad \Psi';\Delta';\Gamma' \vdash \mathsf{v_i} : \tau_i\overline{[\tau/\alpha]} \quad \vdash \mathsf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau_{i+1}\overline{[\tau/\alpha]}) \\ \Psi';\Delta';\Gamma' \vdash \mathsf{v_{i+2}} : \tau_{i+2}\overline{[\tau/\alpha]} \quad \cdots \quad \Psi';\Delta';\Gamma' \vdash \mathsf{v_n} : \tau_n\overline{[\tau/\alpha]}\end{array}}{\vdash \mathsf{v}\ [\overline{\tau}]\ \mathsf{v_1}\cdots\mathsf{v_i}\ \mathsf{C^v}\ \mathsf{v_{i+2}}\cdots\mathsf{v_n} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau'\overline{[\tau/\alpha]})}$$

$$\frac{\vdash \mathsf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \mu\alpha.\tau)}{\vdash \mathsf{unfold}\ \mathsf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau[\mu\alpha.\tau/\alpha])}$$

$$\frac{\vdash \mathsf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \exists\alpha.\tau) \qquad \Psi';\Delta',\alpha;\Gamma',\mathsf{x}:\tau \vdash \mathsf{e} : \tau'}{\vdash \mathsf{unpack}\ \langle\alpha,\mathsf{x}\rangle = \mathsf{C^v}\ \mathsf{in}\ \mathsf{e} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \exists\alpha.\tau)}$$

$$\frac{\Psi';\Delta';\Gamma' \vdash \mathsf{v} : \exists\alpha.\tau \qquad \vdash \mathsf{C} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta',\alpha;\Gamma',\mathsf{x}:\tau \vdash \tau')}{\vdash \mathsf{unpack}\ \langle\alpha,\mathsf{x}\rangle = \mathsf{v}\ \mathsf{in}\ \mathsf{C} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau')}$$

$$\frac{\vdash \mathsf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau')}{\vdash \mathsf{new}\ \mathsf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \mathsf{ref}\ \tau')}$$

$$\frac{\vdash \mathsf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \mathsf{ref}\ \tau') \qquad \Psi';\Delta';\Gamma' \vdash \mathsf{v_2} : \tau'}{\vdash \mathsf{C^v} := \mathsf{v_2} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \mathsf{ref}\ \tau')}$$

$$\frac{\vdash \mathsf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau') \qquad \Psi';\Delta';\Gamma' \vdash \mathsf{v_1} : \mathsf{ref}\ \tau'}{\vdash \mathsf{v_1} := \mathsf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \mathsf{ref}\ \tau')}$$

$$\frac{\vdash \mathsf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \mathsf{ref}\ \tau')}{\vdash\ !\mathsf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau')} \qquad \frac{\vdash \mathsf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \langle\tau_1,\ldots,\tau_n\rangle)}{\vdash \pi_i(\mathsf{C^v}) : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau_i)}$$

$$\frac{\vdash \mathsf{C} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau_1) \qquad \Psi';\Delta';\Gamma',\mathsf{x}:\tau_1 \vdash \mathsf{e_2} : \tau_2}{\vdash \mathsf{let}\,\mathsf{x} = \mathsf{C}\ \mathsf{in}\ \mathsf{e_2} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau_2)}$$

$$\frac{\vdash \mathsf{C} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma',\mathsf{x}:\tau_1 \vdash \tau_2) \qquad \Psi';\Delta';\Gamma' \vdash \mathsf{e_1} : \tau_1}{\vdash \mathsf{let}\,\mathsf{x} = \mathsf{e_1}\ \mathsf{in}\ \mathsf{C} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau_2)}$$

$$\frac{\vdash \mathbf{C} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau^{\langle\mathcal{C}\rangle})}{\vdash {}^{\tau}\mathcal{MC}\ \mathbf{C} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi';\Delta';\Gamma' \vdash \tau)}$$

$$\frac{\Psi \subseteq \Psi' \qquad \Delta \subseteq \Delta' \qquad \Gamma \subseteq \Gamma'}{\vdash [\cdot]^{\mathsf{v}} : (\Psi; \Delta; \Gamma \vdash \boldsymbol{\tau}) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \boldsymbol{\tau})} \qquad \frac{\vdash \mathsf{C} : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; (\overline{\alpha}); (\overline{\mathsf{x} : \boldsymbol{\tau}}) \vdash \boldsymbol{\tau}')}{\vdash \boldsymbol{\lambda}[\overline{\alpha}](\overline{\mathsf{x} : \boldsymbol{\tau}}).\mathsf{C} : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \forall[\overline{\alpha}].(\overline{\boldsymbol{\tau}}) \rightarrow \boldsymbol{\tau}')}$$

$$\frac{\vdash \mathsf{C}^{\mathsf{v}} : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \boldsymbol{\tau}[\boldsymbol{\tau}'/\alpha])}{\vdash \mathsf{pack}\ \langle \boldsymbol{\tau}', \mathsf{C}^{\mathsf{v}} \rangle\ \mathsf{as}\ \exists \alpha.\boldsymbol{\tau} : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \exists \alpha.\boldsymbol{\tau})}$$

$$\frac{\vdash \mathsf{C}^{\mathsf{v}} : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \boldsymbol{\tau}[\mu\alpha.\boldsymbol{\tau}/\alpha])}{\vdash \mathsf{fold}_{\mu\alpha.\boldsymbol{\tau}}\ \mathsf{C}^{\mathsf{v}} : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \mu\alpha.\boldsymbol{\tau})}$$

$$\frac{\begin{array}{c}\Psi'; \Delta'; \Gamma' \vdash \mathsf{v}_1 : \boldsymbol{\tau}_1 \qquad \cdots \qquad \Psi'; \Delta'; \Gamma' \vdash \mathsf{v}_i : \boldsymbol{\tau}_i \\ \vdash \mathsf{C}^{\mathsf{v}} : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \boldsymbol{\tau}_{i+1}) \qquad \Psi'; \Delta'; \Gamma' \vdash \mathsf{v}_{i+2} : \boldsymbol{\tau}_{i+2} \qquad \cdots \qquad \Psi'; \Delta'; \Gamma' \vdash \mathsf{v}_n : \boldsymbol{\tau}_n\end{array}}{\vdash \langle \mathsf{v}_1, \ldots, \mathsf{v}_i, \mathsf{C}^{\mathsf{v}}, \mathsf{v}_{i+2}, \ldots, \mathsf{v}_n \rangle : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \langle \boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n \rangle)}$$

$$\frac{\Psi \subseteq \Psi' \qquad \Delta \subseteq \Delta' \qquad \Gamma \subseteq \Gamma'}{\vdash [\cdot] : (\Psi; \Delta; \Gamma \vdash \boldsymbol{\tau}) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \boldsymbol{\tau})} \qquad \frac{\vdash \mathsf{C}^{\mathsf{v}} : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \mathsf{int}) \qquad \Psi'; \Delta'; \Gamma' \vdash \mathsf{v} : \mathsf{int}}{\vdash \mathsf{C}^{\mathsf{v}}\ \mathsf{p}\ \mathsf{v} : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \mathsf{int})}$$

$$\frac{\Psi'; \Delta'; \Gamma' \vdash \mathsf{v} : \mathsf{int} \qquad \vdash \mathsf{C}^{\mathsf{v}} : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \mathsf{int})}{\vdash \mathsf{v}\ \mathsf{p}\ \mathsf{C}^{\mathsf{v}} : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \mathsf{int})}$$

$$\frac{\vdash \mathsf{C}^{\mathsf{v}} : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \mathsf{int}) \qquad \Psi'; \Delta'; \Gamma' \vdash \mathsf{e}_1 : \boldsymbol{\tau} \qquad \Psi'; \Delta'; \Gamma' \vdash \mathsf{e}_2 : \boldsymbol{\tau}}{\vdash \mathsf{if0}\ \mathsf{C}^{\mathsf{v}}\ \mathsf{e}_1\ \mathsf{e}_2 : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \boldsymbol{\tau})}$$

$$\frac{\Psi'; \Delta'; \Gamma' \vdash \mathsf{v} : \mathsf{int} \qquad \vdash \mathsf{C} : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \boldsymbol{\tau}) \qquad \Psi'; \Delta'; \Gamma' \vdash \mathsf{e}_2 : \boldsymbol{\tau}}{\vdash \mathsf{if0}\ \mathsf{v}\ \mathsf{C}\ \mathsf{e}_2 : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \boldsymbol{\tau})}$$

$$\frac{\Psi'; \Delta'; \Gamma' \vdash \mathsf{v} : \mathsf{int} \qquad \Psi'; \Delta'; \Gamma' \vdash \mathsf{e}_1 : \boldsymbol{\tau} \qquad \vdash \mathsf{C} : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \boldsymbol{\tau})}{\vdash \mathsf{if0}\ \mathsf{v}\ \mathsf{e}_1\ \mathsf{C} : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \boldsymbol{\tau})}$$

$$\frac{\vdash \mathbf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \forall[].(\tau_1,\ldots,\tau_n) \to \tau') \quad \Psi';\Delta';\Gamma' \vdash \mathbf{v_1} : \tau_1 \quad \cdots \quad \Psi';\Delta';\Gamma' \vdash \mathbf{v_n} : \tau_n}{\vdash \mathbf{C^v}\ []\ \mathbf{v_1} \cdots \mathbf{v_n} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \tau')}$$

$$\frac{\begin{array}{c}\Psi';\Delta';\Gamma' \vdash \mathbf{v} : \forall[].(\tau_1,\ldots,\tau_n) \to \tau' \quad \Psi';\Delta';\Gamma' \vdash \mathbf{v_1} : \tau_1 \quad \cdots \quad \Psi';\Delta';\Gamma' \vdash \mathbf{v_i} : \tau_i \\ \vdash \mathbf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \tau_{i+1}) \quad \Psi';\Delta';\Gamma' \vdash \mathbf{v_{i+2}} : \tau_{i+2} \quad \cdots \quad \Psi';\Delta';\Gamma' \vdash \mathbf{v_n} : \tau_n\end{array}}{\vdash \mathbf{v}\ []\ \mathbf{v_1} \cdots \mathbf{v_i}\ \mathbf{C^v}\ \mathbf{v_{i+2}} \cdots \mathbf{v_n} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \tau')}$$

$$\frac{\vdash \mathbf{C} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \forall[\beta,\overline{\alpha}].(\overline{\tau}) \to \tau') \quad \Delta \vdash \tau_0}{\vdash \mathbf{C}[\tau_0] : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \forall[\overline{\alpha}].(\overline{\tau[\tau_0/\beta]}) \to \tau'[\tau_0/\beta])}$$

$$\frac{\vdash \mathbf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \exists\alpha.\tau) \quad \Psi';\Delta',\alpha;\Gamma',\mathbf{x}:\tau \vdash \mathbf{e} : \tau'}{\vdash \mathbf{unpack}\ \langle\alpha,\mathbf{x}\rangle = \mathbf{C^v}\ \mathbf{in}\ \mathbf{e} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \exists\alpha.\tau)}$$

$$\frac{\Psi';\Delta';\Gamma' \vdash \mathbf{v} : \exists\alpha.\tau \quad \vdash \mathbf{C} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta',\alpha;\Gamma',\mathbf{x}:\tau \vdash \tau')}{\vdash \mathbf{unpack}\ \langle\alpha,\mathbf{x}\rangle = \mathbf{v}\ \mathbf{in}\ \mathbf{C} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \tau')}$$

$$\frac{\vdash \mathbf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \mu\alpha.\tau)}{\vdash \mathbf{unfold}\ \mathbf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \tau[\mu\alpha.\tau/\alpha])} \qquad \frac{\vdash \mathbf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \tau')}{\vdash \mathbf{new}\ \mathbf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \mathbf{ref}\ \tau')}$$

$$\frac{\vdash \mathbf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \mathbf{ref}\ \tau') \quad \Psi';\Delta';\Gamma' \vdash \mathbf{v_2} : \tau'}{\vdash \mathbf{C^v} := \mathbf{v_2} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \mathbf{ref}\ \tau')}$$

$$\frac{\vdash \mathbf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \tau') \quad \Psi';\Delta';\Gamma' \vdash \mathbf{v_1} : \mathbf{ref}\ \tau'}{\vdash \mathbf{v_1} := \mathbf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \mathbf{ref}\ \tau')}$$

$$\frac{\vdash \mathbf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \mathbf{ref}\ \tau')}{\vdash \mathbf{!C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \tau')} \qquad \frac{\vdash \mathbf{C^v} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \langle\tau_1,\ldots,\tau_n\rangle)}{\vdash \pi_i(\mathbf{C^v}) : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \tau_i)}$$

$$\frac{\vdash \mathbf{C} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \tau_1) \quad \Psi';\Delta';\Gamma',\mathbf{x}:\tau_1 \vdash \mathbf{e_2} : \tau_2}{\vdash \mathbf{let}\ \mathbf{x} = \mathbf{C}\ \mathbf{in}\ \mathbf{e_2} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \tau_2)}$$

$$\frac{\vdash \mathbf{C} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma',\mathbf{x}:\tau_1 \vdash \tau_2) \quad \Psi';\Delta';\Gamma' \vdash \mathbf{e_1} : \tau_1}{\vdash \mathbf{let}\ \mathbf{x} = \mathbf{e_1}\ \mathbf{in}\ \mathbf{C} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \tau_2)}$$

$$\frac{\vdash \mathsf{C} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \tau)}{\vdash \mathcal{CM}^{\tau}\ \mathsf{C} : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \tau^{\langle \mathcal{C} \rangle})}$$

Note: We use $\langle H \mid e \rangle \Downarrow$ as shorthand for $\exists v, H' \,.\, \langle H \mid e \rangle \longmapsto^* \langle H' \mid v \rangle$

### 5.2 Contextual Equivalence

$$\begin{aligned}\Psi; \Delta; \Gamma \vdash e_1 \approx^{ctx}_{M+C} e_2 : \tau \stackrel{\text{def}}{=} {} & \Psi; \Delta; \Gamma \vdash e_1 : \tau \;\wedge\; \Psi; \Delta; \Gamma \vdash e_2 : \tau \;\wedge \\ & \forall C, \Psi', \tau' .\, \vdash C : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \cdot; \cdot \vdash \tau') \;\wedge\; \vdash H : \Psi' \\ & \implies (\langle H \mid C[e_1] \rangle \Downarrow \iff \langle H \mid C[e_2] \rangle \Downarrow)\end{aligned}$$

In the definition above, notice that the hole in the context $C$ may be any one of: $[\cdot]^{\mathsf{v}}$, $[\cdot]$, $[\cdot]^{\mathsf{v}}$, or $[\cdot]$. An implicit requirement in the above definition is that the context $C$ and expressions $e_1$ and $e_2$ are such that the expressions can be legally plugged into the context's hole. In other words, we assume that $C[e_1]$ and $C[e_2]$ are syntactically valid expressions such that $\Psi'; \cdot; \cdot \vdash C[e_i] : \tau'$.

### 5.3 CIU Equivalence

$$\begin{aligned}\Psi; \Delta; \Gamma \vdash e_1 \approx^{ciu}_{M+C} e_2 : \tau \stackrel{\text{def}}{=} {} & \Psi; \Delta; \Gamma \vdash e_1 : \tau \;\wedge\; \Psi; \Delta; \Gamma \vdash e_2 : \tau \;\wedge \\ & \forall \delta, \gamma, E, H, \Psi', \tau' .\, \cdot \vdash \delta : \Delta \;\wedge\; \Psi'; \cdot; \cdot \vdash \gamma : \delta(\Gamma) \;\wedge \\ & \quad \vdash E : (\Psi; \cdot; \cdot \vdash \tau) \rightsquigarrow (\Psi'; \cdot; \cdot \vdash \tau') \;\wedge\; \vdash H : \Psi' \\ & \quad \implies (\langle H \mid E[\delta(\gamma(e_1))] \rangle \Downarrow \iff \langle H \mid E[\delta(\gamma(e_2))] \rangle \Downarrow)\end{aligned}$$

# 6 Logical Relation

With the exception of Perconti and Ahmed [2], existing work on logical relations for multi-language systems is restricted to the setting where only one of the two interoperating languages contains type abstraction (e.g., [?, ?, ?]). We build on the work by Perconti and Ahmed [2], which includes a closure conversion pass between languages similar to the M and C languages here, except they don't include mutable references. Next, we discuss the changes we had to make to their multi-language logical relation order to model mutable references.

Since the multi-language enables the suspension of locations we need to be able to specify relatedness of locations across languages: $\Psi; \Delta; \Gamma \vdash \ell \approx^{log}_{M+C} {}^{\mathsf{ref}\,\tau}\mathcal{MC}\,\boldsymbol{\ell} : \mathsf{ref}\,\tau$. Our first instinct when trying to define relatedness of locations in the logical relation model was to instantly pull out concrete values from heaps given to us by the logical relation and talk about their relatedness. Since these values might be in different languages, we need to convert them into the same language so that we can talk about their relatedness in a traditional binary logical relation. Unfortunately, having the definition of related locations translate into the same language made it impossible to prove the bridge lemma. We won't show here how the proof breaks, but the intuition is that the multi-language delays translating the value that a wrapped location points to until dereference. A semantic model of the multi-language must follow suit, mimicking such delays.

Our solution is to come up with a way of relating values across languages, written $\mathcal{V}[\![\tau, \tau^{\langle \mathcal{C} \rangle}]\!]\rho$ and $\mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}, \tau]\!]\rho$, and say that two locations from different languages are related if the values they point to are related across languages. Making the logical relation more general in this way enables us to prove the bridge lemma.

Technically, we now have a two-dimensional N-by-N matrix of logical relations, where N is the number of languages embedded in our multi-language. On the diagonals we have $\mathcal{V}[\![\tau, \tau]\!]\rho$ and $\mathcal{V}[\![\boldsymbol{\tau}, \boldsymbol{\tau}]\!]\rho$. Off-diagonals include cross-language versions: $\mathcal{V}[\![\tau, \tau^{\langle \mathcal{C} \rangle}]\!]\rho$ and $\mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}, \tau]\!]\rho$. Now, the astute reader may be worried that an approach that requires an N-by-N matrix of logical relations won't scale well. Fortunately, the actual logical relation definition isn't much more involved than a standard (one-dimensional) one because, as we shall see below, we are able to define the off-diagonals in terms of the diagonals.

Our admissibility criteria need to be adapted from [2]. While they have a list of relations ranging from size 1 to size N, depending on the language in question, we need a matrix ranging in size from 1-by-1 to N-by-N. While there are more properties to consider, we formulated the admissibility criteria in a way that ultimately leads to cleaner encodings of the boundary cancellation and bridge lemma requirements of the admissibility criteria.

$$
\begin{array}{lcl}
\mathrm{HeapAtom}_n & \stackrel{\mathrm{def}}{=} & \{\,(W, H_1, H_2) \mid\ W \in \mathrm{World}_n\} \\
\mathrm{HeapRel}_n & \stackrel{\mathrm{def}}{=} & \{\,\varphi_H \subseteq \mathrm{HeapAtom}_n \mid\ \forall (W, H_1, H_2) \in \varphi_H.\ \forall W' \sqsupseteq W.\ (W', H_1, H_2) \in \varphi_H\,\} \\
\mathrm{TermAtom}_n[\tau_1, \tau_2] & \stackrel{\mathrm{def}}{=} & \{\,(W, e_1, e_2) \mid\ W \in \mathrm{World}_n\ \wedge\ W.\Psi_1; \cdot; \cdot \vdash e_1 : \tau_1\ \wedge\ W.\Psi_2; \cdot; \cdot \vdash e_2 : \tau_2\} \\
\mathrm{ValAtom}_n[\tau_1, \tau_2] & \stackrel{\mathrm{def}}{=} & \{\,(W, v_1, v_2) \in \mathrm{TermAtom}_n[\tau_1, \tau_2]\} \\
\mathrm{ContAtom}[\tau_1, \tau_2] \rightsquigarrow [\tau_1', \tau_2'] & \stackrel{\mathrm{def}}{=} & \{\,(W, E_1, E_2) \mid\ W \in \mathrm{World}\ \wedge\ \exists \Psi_1', \Psi_2'\ . \\
 & & \quad \vdash E_1 : (W.\Psi_1; \cdot; \cdot \vdash \tau_1) \rightsquigarrow (\Psi_1'; \cdot; \cdot \vdash \tau_1')\ \wedge \\
 & & \quad \vdash E_2 : (W.\Psi_2; \cdot; \cdot \vdash \tau_2) \rightsquigarrow (\Psi_2'; \cdot; \cdot \vdash \tau_2')\} \\
\mathrm{Island}_n & \stackrel{\mathrm{def}}{=} & \{\,\theta = (s, S, \delta, \pi, \mathrm{HR}, \mathrm{bij}) \mid\ s \in S\ \wedge\ S \in \mathrm{Set}\ \wedge\ \delta \subseteq S \times S\ \wedge\ \pi \subseteq \delta\ \wedge \\
 & & \qquad \delta, \pi\ \mathrm{reflexive}\ \wedge\ \delta, \pi\ \mathrm{transitive}\ \wedge\ \mathrm{HR} \in S \to \mathrm{HeapRel}_n\ \wedge\ \mathrm{bij} \in S \to \mathbb{P}(\mathrm{Val} \times \mathrm{Val})\,\} \\
\mathrm{World}_n & \stackrel{\mathrm{def}}{=} & \{\,W = (k, \Psi_1, \Psi_2, \Theta) \mid\ k < n\ \wedge\ \exists m.\ \Theta \in \mathrm{Island}_k^m\ \} \\
\varphi_H \otimes \varphi_H' & \stackrel{\mathrm{def}}{=} & \{\,(W, H_1 \uplus H_1', H_2 \uplus H_2') \mid (W, H_1, H_2) \in \varphi_H\ \wedge\ (W, H_1', H_2') \in \varphi_H'\}
\end{array}
$$

$$
\begin{array}{lcl}
\mathrm{ValRel}[\tau_1,\tau_2] & \stackrel{\mathrm{def}}{=} & \{\,\varphi \subseteq \mathrm{ValAtom}[\tau_1,\tau_2] \mid\ \forall (W,v_1,v_2)\in\varphi.\ (\forall W' \sqsupseteq W.\ (W',v_1,v_2)\in\varphi\,)\,\} \\
\mathcal{CM}_1(\tau_1,\varphi) & \stackrel{\mathrm{def}}{=} & \{\,(W,\mathbf{v_1},v_2) \mid\ (W,\mathsf{v}_1,v_2)\in\varphi \ \wedge\ \mathbf{CM}^{\tau_1}(\mathsf{v}_1)=\mathbf{v_1}\,\} \\
\mathcal{MC}_1(\tau_1,\varphi) & \stackrel{\mathrm{def}}{=} & \{\,(W,\mathsf{v}_1,v_2) \mid\ (W,\mathbf{v_1},v_2)\in\varphi \ \wedge\ {}^{\tau_1}\mathbf{MC}(\mathbf{v_1})=\mathsf{v}_1\,\} \\
\mathcal{CM}_2(\tau_2,\varphi) & \stackrel{\mathrm{def}}{=} & \{\,(W,v_1,\mathbf{v_2}) \mid\ (W,v_1,\mathsf{v}_2)\in\varphi \ \wedge\ \mathbf{CM}^{\tau_2}(\mathsf{v}_2)=\mathbf{v_2}\,\} \\
\mathcal{MC}_2(\tau_2,\varphi) & \stackrel{\mathrm{def}}{=} & \{\,(W,v_1,\mathsf{v}_2) \mid\ (W,v_1,\mathbf{v_2})\in\varphi \ \wedge\ {}^{\tau_2}\mathbf{MC}(\mathbf{v_2})=\mathsf{v}_2\,\} \\
\mathrm{MMValRel} & \stackrel{\mathrm{def}}{=} & \{\,\mathrm{VR}=(\tau_1,\tau_2,R) \mid\ R[M,M]\in\mathrm{ValRel}[\tau_1,\tau_2] \ \wedge\ R[C,M]\in\mathrm{ValRel}[\tau_1{}^{\langle\mathcal{C}\rangle},\tau_2] \ \wedge \\
 & & \quad R[M,C]\in\mathrm{ValRel}[\tau_1,\tau_2{}^{\langle\mathcal{C}\rangle}] \ \wedge\ R[C,C]\in\mathrm{ValRel}[\tau_1{}^{\langle\mathcal{C}\rangle},\tau_2{}^{\langle\mathcal{C}\rangle}] \ \wedge \\
 & & \quad \mathcal{CM}_1(\tau_1,R[M,M])\subseteq R[C,M] \ \wedge\ \mathcal{MC}_1(\tau_1,R[C,M])\subseteq R[M,M] \ \wedge \\
 & & \quad \mathcal{CM}_1(\tau_1,R[M,C])\subseteq R[C,C] \ \wedge\ \mathcal{MC}_1(\tau_1,R[C,C])\subseteq R[M,C] \ \wedge \\
 & & \quad \mathcal{CM}_2(\tau_2,R[M,M])\subseteq R[M,C] \ \wedge\ \mathcal{MC}_2(\tau_2,R[M,C])\subseteq R[M,M] \ \wedge \\
 & & \quad \mathcal{CM}_2(\tau_2,R[C,M])\subseteq R[C,C] \ \wedge\ \mathcal{MC}_2(\tau_2,R[C,C])\subseteq R[C,M]\,\} \\
\mathrm{CCValRel} & \stackrel{\mathrm{def}}{=} & \{\,\mathrm{VR}=(\boldsymbol{\tau_1},\boldsymbol{\tau_2},R) \mid\ R[C,C]\in\mathrm{ValRel}[\boldsymbol{\tau_1},\boldsymbol{\tau_2}]\,\}
\end{array}
$$

$$
\begin{array}{lcl}
\lfloor(\theta_1,\ldots,\theta_m)\rfloor_k & \stackrel{\mathrm{def}}{=} & (\lfloor\theta_1\rfloor_k,\ldots,\lfloor\theta_m\rfloor_k) \\
\lfloor(s,S,\delta,\pi,\mathrm{HR},\mathrm{bij})\rfloor_k & \stackrel{\mathrm{def}}{=} & (s,S,\delta,\pi,\lfloor\mathrm{HR}\rfloor_k,\mathrm{bij}) \\
\lfloor\mathrm{HR}\rfloor_k & \stackrel{\mathrm{def}}{=} & \lambda s.\ \lfloor\mathrm{HR}(s)\rfloor_k \\
\lfloor\varphi_H\rfloor_k & \stackrel{\mathrm{def}}{=} & \{\,(W,H_1,H_2)\in\varphi_H \mid\ W.k<k\,\}
\end{array}
$$

$$
\begin{array}{lcl}
\triangleright(k+1,\Psi_1,\Psi_2,\Theta) & \stackrel{\mathrm{def}}{=} & (k,\Psi_1,\Psi_2,\lfloor\Theta\rfloor_k) \\
\triangleright\varphi & \stackrel{\mathrm{def}}{=} & \{\,(W,e_1,e_2)\mid W.k>0 \implies (\triangleright W,e_1,e_2)\in\varphi\,\}
\end{array}
$$

$$
\begin{array}{lcl}
(k',\Psi_1',\Psi_2',\Theta')\sqsupseteq(k,\Psi_1,\Psi_2,\Theta) & \stackrel{\mathrm{def}}{=} & k'\le k \ \wedge\ \Psi_1'\supseteq\Psi_1 \ \wedge\ \Psi_2'\supseteq\Psi_2 \ \wedge\ \Theta'\sqsupseteq\lfloor\Theta\rfloor_{k'} \\
(\theta_1',\ldots,\theta_{m'}')\sqsupseteq(\theta_1,\ldots,\theta_m) & \stackrel{\mathrm{def}}{=} & m'\ge m \ \wedge\ \forall j\in\{1,\ldots,m\}.\ \theta_j'\sqsupseteq\theta_j \\
(s',S',\delta',\pi',\mathrm{HR}',\mathrm{bij}')\sqsupseteq(s,S,\delta,\pi,\mathrm{HR},\mathrm{bij}) & \stackrel{\mathrm{def}}{=} & (S',\delta',\pi',\mathrm{HR}',\mathrm{bij}')=(S,\delta,\pi,\mathrm{HR},\mathrm{bij}) \ \wedge\ (s,s')\in\delta
\end{array}
$$

$$
\begin{array}{lcl}
(k',\Psi_1',\Psi_2',\Theta')\sqsupseteq_{\mathrm{pub}}(k,\Psi_1,\Psi_2,\Theta) & \stackrel{\mathrm{def}}{=} & k'\le k \ \wedge\ \Psi_1'\supseteq\Psi_1 \ \wedge\ \Psi_2'\supseteq\Psi_2 \ \wedge\ \Theta'\sqsupseteq_{\mathrm{pub}}\lfloor\Theta\rfloor_{k'} \\
(\theta_1',\ldots,\theta_{m'}')\sqsupseteq_{\mathrm{pub}}(\theta_1,\ldots,\theta_m) & \stackrel{\mathrm{def}}{=} & m'\ge m \ \wedge\ \forall j\in\{1,\ldots,m\}.\ \theta_j'\sqsupseteq_{\mathrm{pub}}\theta_j \\
(s',S',\delta',\pi',\mathrm{HR}',\mathrm{bij}')\sqsupseteq_{\mathrm{pub}}(s,S,\delta,\pi,\mathrm{HR},\mathrm{bij}) & \stackrel{\mathrm{def}}{=} & (S',\delta',\pi',\mathrm{HR}',\mathrm{bij}')=(S,\delta,\pi,\mathrm{HR},\mathrm{bij}) \ \wedge\ (s,s')\in\pi
\end{array}
$$

$$
\begin{array}{lcl}
\mathrm{ValAtom}[\tau]\rho & \stackrel{\mathrm{def}}{=} & \mathrm{ValAtom}[\rho_1(\tau),\rho_2(\tau)] \\
\mathrm{ContAtom}[\tau_1,\tau_2]\rho \leadsto [\tau_1',\tau_2']\rho' & \stackrel{\mathrm{def}}{=} & \mathrm{ContAtom}[\rho_1(\tau_1),\rho_2(\tau_2)] \leadsto [\rho_1'(\tau_1'),\rho_2'(\tau_2')]
\end{array}
$$

$$
\begin{array}{lcl}
\mathcal{V}[\![\tau,\tau^{\langle\mathcal{C}\rangle}]\!]\rho & \stackrel{\mathrm{def}}{=} & \{(W,\mathsf{v}_1,\mathbf{v_2})\in\mathrm{ValAtom}[\tau,\tau^{\langle\mathcal{C}\rangle}]\rho \mid (W,\mathsf{v}_1,{}^{\rho_2(\tau)}\mathbf{MC}(\mathbf{v_2}))\in\mathcal{V}[\![\tau]\!]\rho \ \wedge \\
 & & \qquad (W,\mathbf{CM}^{\rho_1(\tau)}(\mathsf{v}_1),\mathbf{v_2})\in\mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}]\!]\rho\} \\
\mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle},\tau]\!]\rho & \stackrel{\mathrm{def}}{=} & \{(W,\mathbf{v_1},\mathsf{v}_2)\in\mathrm{ValAtom}[\tau^{\langle\mathcal{C}\rangle},\tau]\rho \mid (W,{}^{\rho_1(\tau)}\mathbf{MC}(\mathbf{v_1}),\mathsf{v}_2)\in\mathcal{V}[\![\tau]\!]\rho \ \wedge \\
 & & \qquad (W,\mathbf{v_1},\mathbf{CM}^{\rho_2(\tau)}(\mathsf{v}_2))\in\mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}]\!]\rho\} \\
\mathcal{V}[\![\tau,\tau]\!]\rho & \stackrel{\mathrm{def}}{=} & \mathcal{V}[\![\tau]\!]\rho
\end{array}
$$

$$
\begin{array}{lcl}
\mathcal{V}[\![\alpha]\!]\rho & \stackrel{\text{def}}{=} & \rho(\alpha).R[M,M] \\
\mathcal{V}[\![\mathsf{unit}]\!]\rho & \stackrel{\text{def}}{=} & \{\,(W,(),()) \in \mathrm{ValAtom}[\mathsf{unit}]\rho\,\} \\
\mathcal{V}[\![\mathsf{int}]\!]\rho & \stackrel{\text{def}}{=} & \{\,(W,\mathsf{n},\mathsf{n}) \in \mathrm{ValAtom}[\mathsf{int}]\rho\,\} \\
\mathcal{V}[\![\forall[\overline{\alpha}].(\overline{\tau}) \to \tau']\!]\rho & \stackrel{\text{def}}{=} & \{\,(W,\mathsf{v}_1,\mathsf{v}_2) \in \mathrm{ValAtom}[\forall[\overline{\alpha}].(\overline{\tau}) \to \tau']\rho \mid \\
& & \quad \forall W' \sqsupseteq W.\ \forall \overline{\mathrm{VR} \in \mathrm{MMValRel}}\ .\ \ \forall \overline{\mathsf{v}_1'}, \overline{\mathsf{v}_2'}\ .\ \overline{(W',\mathsf{v}_1',\mathsf{v}_2') \in \mathcal{V}[\![\tau]\!]\rho\overline{[\alpha \mapsto \mathrm{VR}]}} \\
& & \quad \implies (W',\mathsf{v}_1\,[\overline{\mathrm{VR}.\tau_1}]\,\overline{\mathsf{v}_1'},\mathsf{v}_2\,[\overline{\mathrm{VR}.\tau_2}]\,\overline{\mathsf{v}_2'}) \in \mathcal{E}[\![\tau',\tau']\!]\rho\overline{[\alpha \mapsto \mathrm{VR}]}\,\} \\
\mathcal{V}[\![\exists\alpha.\tau]\!]\rho & \stackrel{\text{def}}{=} & \{\,(W,\mathsf{pack}\,\langle\tau_1,\mathsf{v}_1\rangle\,\mathsf{as}\,\rho_1(\exists\alpha.\tau),\mathsf{pack}\,\langle\tau_2,\mathsf{v}_2\rangle\,\mathsf{as}\,\rho_2(\exists\alpha.\tau)) \in \mathrm{ValAtom}[\exists\alpha.\tau]\rho \mid \\
& & \quad \exists \mathrm{VR} \in \mathrm{MMValRel} \wedge \mathrm{VR}.\tau_1 = \tau_1\ \wedge\ \mathrm{VR}.\tau_2 = \tau_2\ \wedge\ (W,\mathsf{v}_1,\mathsf{v}_2) \in \mathcal{V}[\![\tau]\!]\rho[\alpha \mapsto \mathrm{VR}]\,\} \\
\mathcal{V}[\![\mu\alpha.\tau]\!]\rho & \stackrel{\text{def}}{=} & \{\,(W,\mathsf{fold}_{\rho_1(\mu\alpha.\tau)}\,\mathsf{v}_1,\mathsf{fold}_{\rho_2(\mu\alpha.\tau)}\,\mathsf{v}_2) \in \mathrm{ValAtom}[\mu\alpha.\tau]\rho \mid \\
& & \qquad (W,\mathsf{v}_1,\mathsf{v}_2) \in \triangleright\mathcal{V}[\![\tau[\mu\alpha.\tau/\alpha]]\!]\rho\,\} \\
\mathcal{V}[\![\mathsf{ref}\,\tau]\!]\rho & \stackrel{\text{def}}{=} & \{\,(W,\mathsf{v}_1,\mathsf{v}_2) \in \mathrm{ValAtom}[\mathsf{ref}\,\tau]\rho \mid\ \exists i.\ \forall W' \sqsupseteq W. \\
& & \qquad (\mathrm{loc}(\mathsf{v}_1),\mathrm{loc}(\mathsf{v}_2)) \in W'(i).\mathrm{bij}(W'(i).s) \wedge \exists\varphi_H.\ W'(i).\mathrm{HR}(W'(i).s) = \varphi_H \otimes \\
& & \qquad \{\,(\widetilde{W},H_1,H_2) \in \mathrm{HeapAtom} \mid \forall \tau_1,\tau_2,v_1',v_2'. \\
& & \qquad\quad \mathrm{lookup}^{\mathcal{T}}(\mathsf{v}_1,H_1) \hookrightarrow (v_1',\tau_1)\ \wedge \mathrm{lookup}^{\mathcal{T}}(\mathsf{v}_2,H_2) \hookrightarrow (v_2',\tau_2)\ \implies \\
& & \qquad\quad (\widetilde{W},v_1',v_2') \in \mathcal{V}[\![\tau_1,\tau_2]\!]\rho\,\}\,\} \\
\mathcal{V}[\![\langle\tau_1,\ldots,\tau_{\mathsf{n}}\rangle]\!]\rho & \stackrel{\text{def}}{=} & \{\,(W,\langle\mathsf{v}_{11},\ldots,\mathsf{v}_{1\mathsf{n}}\rangle,\langle\mathsf{v}_{21},\ldots,\mathsf{v}_{2\mathsf{n}}\rangle) \in \mathrm{ValAtom}[\langle\tau_1,\ldots,\tau_{\mathsf{n}}\rangle]\rho \mid \\
& & \qquad \forall \mathsf{j} \in \{1,\ldots,\mathsf{n}\}.\ (W,\mathsf{v}_{1\mathsf{j}},\mathsf{v}_{2\mathsf{j}}) \in \mathcal{V}[\![\tau_{\mathsf{j}}]\!]\rho\,\} \\
\mathcal{V}[\![\mathsf{L}\langle\boldsymbol{\tau}\rangle]\!]\rho & \stackrel{\text{def}}{=} & \{\,(W,{}^{\rho_1(\mathsf{L}\langle\boldsymbol{\tau}\rangle)}\mathcal{MC}\,\mathbf{v_1},{}^{\rho_2(\mathsf{L}\langle\boldsymbol{\tau}\rangle)}\mathcal{MC}\,\mathbf{v_2}) \in \mathrm{ValAtom}[\mathsf{L}\langle\boldsymbol{\tau}\rangle]\rho \mid (W,\mathbf{v_1},\mathbf{v_2}) \in \mathcal{V}[\![\boldsymbol{\tau}]\!]\rho\,\}
\end{array}
$$

$$
\begin{array}{lllcl}
\mathrm{lookup}^{\mathcal{T}} & (\ell, & \{\,\ell \mapsto \mathsf{v}\,\}) & \hookrightarrow & (\mathsf{v},\tau) \\
\mathrm{lookup}^{\mathcal{T}} & ({}^{\mathsf{ref}\,\tau'}\mathcal{MC}\,\boldsymbol{\ell}, & \{\,\boldsymbol{\ell} \mapsto \mathbf{v}\,\}) & \hookrightarrow & (\mathbf{v},\tau^{\langle\mathcal{C}\rangle})
\end{array}
$$

$$\mathcal{V}[\![\alpha]\!]\rho \stackrel{\text{def}}{=} \rho(\alpha).R[C,C]$$

$$\mathcal{V}[\![\mathbf{unit}]\!]\rho \stackrel{\text{def}}{=} \{(W,(),()) \in \text{ValAtom}[\mathbf{unit}]\rho\}$$

$$\mathcal{V}[\![\mathbf{int}]\!]\rho \stackrel{\text{def}}{=} \{(W,\mathbf{n},\mathbf{n}) \in \text{ValAtom}[\mathbf{int}]\rho\}$$

$$\begin{aligned}\mathcal{V}[\![\forall[\overline{\alpha}].(\overline{\tau}) \to \tau']\!]\rho \stackrel{\text{def}}{=} & \{(W,\mathbf{v_1},\mathbf{v_2}) \in \text{ValAtom}[\forall[\overline{\alpha}].(\overline{\tau}) \to \tau']\rho \mid \\ & \forall W' \sqsupseteq W.\ \forall\overline{\text{VR} \in \text{CCValRel}}\ .\ \forall\overline{\mathbf{v}_1'}, \overline{\mathbf{v}_2'}\ .\ \overline{(W',\mathbf{v}_1',\mathbf{v}_2') \in \mathcal{V}[\![\tau]\!]\rho\overline{[\alpha \mapsto \text{VR}]}} \\ & \implies (W',\mathbf{v_1}\ [\overline{\text{VR}.\tau_1}]\ \overline{\mathbf{v}_1'}, \mathbf{v_2}\ [\overline{\text{VR}.\tau_2}]\ \overline{\mathbf{v}_2'}) \in \mathcal{E}[\![\tau',\tau']\!]\rho\overline{[\alpha \mapsto \text{VR}]}\}\end{aligned}$$

$$\begin{aligned}\mathcal{V}[\![\exists\alpha.\tau]\!]\rho \stackrel{\text{def}}{=} & \{(W, \mathbf{pack}\ \langle\tau_1,\mathbf{v_1}\rangle\ \mathbf{as}\ \rho_1(\exists\alpha.\tau), \\ & \quad \mathbf{pack}\ \langle\tau_2,\mathbf{v_2}\rangle\ \mathbf{as}\ \rho_2(\exists\alpha.\tau)) \in \text{ValAtom}[\exists\alpha.\tau]\rho \mid \\ & \exists \text{VR} \in \text{CCValRel} \wedge \text{VR}.\tau_1 = \tau_1\ \wedge\ \text{VR}.\tau_2 = \tau_2\ \wedge\ (W,\mathbf{v_1},\mathbf{v_2}) \in \mathcal{V}[\![\tau]\!]\rho[\alpha \mapsto \text{VR}]\}\end{aligned}$$

$$\begin{aligned}\mathcal{V}[\![\mu\alpha.\tau]\!]\rho \stackrel{\text{def}}{=} & \{(W, \mathbf{fold}_{\rho_1(\mu\alpha.\tau)}\ \mathbf{v}_1, \mathbf{fold}_{\rho_2(\mu\alpha.\tau)}\ \mathbf{v_2}) \in \text{ValAtom}[\mu\alpha.\tau]\rho \mid \\ & (W,\mathbf{v_1},\mathbf{v_2}) \in \triangleright\mathcal{V}[\![\tau[\mu\alpha.\tau/\alpha]]\!]\rho\}\end{aligned}$$

$$\begin{aligned}\mathcal{V}[\![\mathbf{ref}\ \tau]\!]\rho \stackrel{\text{def}}{=} & \{(W,\mathbf{v_1},\mathbf{v_2}) \in \text{ValAtom}[\mathbf{ref}\ \tau]\rho \mid \exists i.\ \forall W' \sqsupseteq W. \\ & (\text{loc}(\mathbf{v_1}), \text{loc}(\mathbf{v_2})) \in W'(i).\text{bij}(W'(i).s) \wedge \exists\varphi_H.\ W'(i).\text{HR}(W'(i).s) = \varphi_H \otimes \\ & \{(\widetilde{W}, H_1, H_2) \in \text{HeapAtom} \mid \forall\tau_1,\tau_2,v_1',v_2'. \\ & \quad \text{lookup}^{\tau}(\mathbf{v_1}, H_1) \hookrightarrow (v_1',\tau_1)\ \wedge \text{lookup}^{\tau}(\mathbf{v_2}, H_2) \hookrightarrow (v_2',\tau_2) \implies \\ & \quad (\widetilde{W}, v_1', v_2') \in \mathcal{V}[\![\tau_1,\tau_2]\!]\rho\}\}\end{aligned}$$

$$\begin{aligned}\mathcal{V}[\![\langle\tau_1,\ldots,\tau_n\rangle]\!]\rho \stackrel{\text{def}}{=} & \{(W, \langle\mathbf{v_{11}},\ldots,\mathbf{v_{1n}}\rangle, \langle\mathbf{v_{21}},\ldots,\mathbf{v_{2n}}\rangle) \in \text{ValAtom}[\langle\tau_1,\ldots,\tau_n\rangle]\rho \mid \\ & \forall \mathbf{j} \in \{\mathbf{1},\ldots,\mathbf{n}\}.\ (W,\mathbf{v_{1j}},\mathbf{v_{2j}}) \in \mathcal{V}[\![\tau_j]\!]\rho\}\end{aligned}$$

$$\mathcal{V}[\![\lceil\alpha\rceil]\!]\rho \stackrel{\text{def}}{=} \rho(\alpha).R[C,C]$$

$$\begin{aligned}&\text{lookup}^{\tau}\ (\mathcal{CM}^{\mathsf{ref}\ \tau'}\ \ell,\ \{\ell \mapsto \mathsf{v}\}) \hookrightarrow (\mathsf{v},\hat{\tau})\ \text{ where }\ \exists\hat{\tau}\ .\ \hat{\tau}^{\langle\mathcal{C}\rangle} = \tau \\ &\text{lookup}^{\tau}\ (\ell,\ \{\ell \mapsto \mathbf{v}\}) \hookrightarrow (\mathbf{v},\tau)\end{aligned}$$

$$\begin{aligned}(H_1,H_2) : W \stackrel{\text{def}}{=} & \vdash H_1 : W.\Phi_1\ \wedge\ \vdash H_2 : W.\Phi_2\ \wedge \\ & (W.k > 0 \implies (\triangleright W, H_1, H_2) \in \bigotimes\{\theta.\text{HR}(\theta.s) \mid\ \theta \in W.\Theta\})\end{aligned}$$

$$\text{running}(k, \langle H \mid e\rangle) \stackrel{\text{def}}{=} \exists H', e'.\langle H \mid e\rangle \longmapsto^k \langle H' \mid e'\rangle$$

$$\begin{aligned}\mathcal{O} \stackrel{\text{def}}{=} & \{(W,e_1,e_2) \mid\ \forall(H_1,H_2) : W.\ ((\langle H_1 \mid e_1\rangle \Downarrow\ \wedge\ \langle H_2 \mid e_2\rangle \Downarrow) \vee \\ & \qquad (\text{running}(W.k, \langle H_1 \mid e_1\rangle)\ \wedge\ \text{running}(W.k, \langle H_2 \mid e_2\rangle))\}\end{aligned}$$

$$\begin{aligned}\mathcal{K}[\![\tau_1,\tau_2]\!]\rho \stackrel{\text{def}}{=} & \{(W,E_1,E_2) \in \text{ContAtom}[\tau_1,\tau_2]\rho \rightsquigarrow [\tau_1',\tau_2']\rho' \mid \\ & \forall W', v_1, v_2.\ W' \sqsupseteq_{\text{pub}} W\ \wedge \\ & \quad (W',v_1,v_2) \in \mathcal{V}[\![\tau_1,\tau_2]\!]\rho \implies (W', E_1[v_1], E_2[v_2]) \in \mathcal{O}\}\end{aligned}$$

$$\begin{aligned}\mathcal{E}[\![\tau_1,\tau_2]\!]\rho \stackrel{\text{def}}{=} & \{(W,e_1,e_2) \in \text{TermAtom}[\rho_1(\tau_1),\rho_2(\tau_2)] \mid \\ & \forall E_1,E_2.\ (W,E_1,E_2) \in \mathcal{K}[\![\tau_1,\tau_2]\!]\rho \implies (W, E_1[e_1], E_2[e_2]) \in \mathcal{O}\}\end{aligned}$$

$$
\begin{array}{lcl}
\mathcal{D}[\![\cdot]\!] & \stackrel{\text{def}}{=} & \{\,\emptyset\,\} \\
\mathcal{D}[\![\Delta, \alpha]\!] & \stackrel{\text{def}}{=} & \{\,\rho[\alpha \mapsto \text{VR}] \mid \ \rho \in \mathcal{D}[\![\Delta]\!] \ \wedge \ \text{VR} \in \text{MMValRel}\,\} \\
\mathcal{D}[\![\Delta, \boldsymbol{\alpha}]\!] & \stackrel{\text{def}}{=} & \{\,\rho[\boldsymbol{\alpha} \mapsto \text{VR}] \mid \ \rho \in \mathcal{D}[\![\Delta]\!] \ \wedge \ \text{VR} \in \text{CCValRel}\,\}
\end{array}
$$

$$
\begin{array}{lcl}
\mathcal{G}[\![\cdot]\!]\rho & \stackrel{\text{def}}{=} & \{\,(W, \emptyset) \mid \ W \in \text{World}\,\} \\
\mathcal{G}[\![\Gamma, x\!:\!\tau]\!]\rho & \stackrel{\text{def}}{=} & \{\,(W, \gamma[x \mapsto (v_1, v_2)] \mid \ (W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho \ \wedge \ (W, v_1, v_2) \in \mathcal{V}[\![\tau]\!]\rho\,\}
\end{array}
$$

$$
\begin{array}{lcl}
\mathcal{H}[\![\{\cdot\}]\!] & \stackrel{\text{def}}{=} & \text{World} \\
\mathcal{H}[\![\Psi, \ell\!:\!\tau]\!] & \stackrel{\text{def}}{=} & \mathcal{H}[\![\Psi]\!] \cap \{\,W \in \text{World} \mid \ (W, \ell, \ell) \in \mathcal{V}[\![\mathsf{ref}\ \tau]\!]\emptyset\,\} \\
\mathcal{H}[\![\boldsymbol{\{\cdot\}}]\!] & \stackrel{\text{def}}{=} & \text{World} \\
\mathcal{H}[\![\boldsymbol{\Psi}, \boldsymbol{\ell}\!:\!\boldsymbol{\tau}]\!] & \stackrel{\text{def}}{=} & \mathcal{H}[\![\boldsymbol{\Psi}]\!] \cap \{\,W \in \text{World} \mid \ (W, \boldsymbol{\ell}, \boldsymbol{\ell}) \in \mathcal{V}[\![\mathbf{ref}\ \boldsymbol{\tau}]\!]\emptyset\,\} \\
\mathcal{H}[\![\Psi; \boldsymbol{\Psi}]\!] & \stackrel{\text{def}}{=} & \mathcal{H}[\![\Psi]\!] \cap \mathcal{H}[\![\boldsymbol{\Psi}]\!]
\end{array}
$$

$$
\begin{array}{lcl}
\Psi; \Delta; \Gamma \vdash e_1 \approx^{log}_{M+C} e_2 : \tau & \stackrel{\text{def}}{=} & \Psi; \Delta; \Gamma \vdash e_1 : \tau \ \wedge \ \Psi; \Delta; \Gamma \vdash e_2 : \tau \ \wedge \\
& & \quad \forall W, \rho, \gamma.\ W \in \mathcal{H}[\![\Psi]\!] \ \wedge \ \rho \in \mathcal{D}[\![\Delta]\!] \ \wedge \ (W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho \implies \\
& & \quad (W, \rho_1(\gamma_1(e_1)), \rho_2(\gamma_2(e_2))) \in \mathcal{E}[\![\tau, \tau]\!]\rho
\end{array}
$$

In the proofs that follow, we will frequently use the following shorthand:

$$
\begin{array}{lcl}
\mathcal{E}[\![\tau]\!]\rho & \stackrel{\text{def}}{=} & \mathcal{E}[\![\tau, \tau]\!]\rho \\
\mathcal{K}[\![\tau]\!]\rho & \stackrel{\text{def}}{=} & \mathcal{K}[\![\tau, \tau]\!]\rho
\end{array}
$$

$$
\begin{array}{lcl}
\mathcal{MC}_{1,2}(\tau_1, \tau_2, \varphi) & \stackrel{\text{def}}{=} & \mathcal{MC}_1(\tau_1, \mathcal{MC}_2(\tau_2, \varphi)) \\
\mathcal{CM}_{1,2}(\tau_1, \tau_2, \varphi) & \stackrel{\text{def}}{=} & \mathcal{CM}_1(\tau_1, \mathcal{CM}_2(\tau_2, \varphi))
\end{array}
$$

# 7 Proofs: Basic Properties

## 7.1 Properties of the Value Translations

**Lemma 7.1 (Value Translation)**
For any $\mathsf{v}$, $\mathbf{v}$, $\Psi$ and

$$\Psi; \cdot; \cdot \vdash \mathsf{v} : \tau, \quad \Psi; \cdot; \cdot \vdash \mathbf{v} : \boldsymbol{\tau}$$

the following hold:

- $\exists! \mathbf{v}'.\ \mathbf{CM}^{\tau}(\mathsf{v}) = \mathbf{v}'$.
- $\exists! \mathsf{v}'.\ {}^{\tau}\mathbf{MC}(\mathbf{v}) = \mathsf{v}'$.

**Proof**

By inspection of the translations.

□

**Lemma 7.2 (Value Translation Preserves Types)**
-

- If $\Psi; \cdot; \cdot \vdash \mathsf{v} : \tau$ and $\mathbf{CM}^{\tau}(\mathsf{v}) = \mathbf{v}$ then $\Psi; \cdot; \cdot \vdash \mathbf{v} : \tau^{\langle \mathcal{C} \rangle}$
- If $\Psi; \cdot; \cdot \vdash \mathbf{v} : \tau^{\langle \mathcal{C} \rangle}$ and ${}^{\tau}\mathbf{MC}(\mathbf{v}) = \mathbf{v}$ then $\Psi; \cdot; \cdot \vdash \mathsf{v} : \tau$

**Proof**

By induction on the type derivations and inspection of the translations. □

## 7.2 Operations on Worlds

**Lemma 7.3 (World Extension is Reflexive and Transitive)**
For any $W, W', W'' \in$ World, we have

- $W \sqsupseteq W$
- $W \sqsupseteq_{\text{pub}} W$
- if $W'' \sqsupseteq W'$ and $W' \sqsupseteq W$, then $W'' \sqsupseteq W$
- if $W'' \sqsupseteq_{\text{pub}} W'$ and $W' \sqsupseteq_{\text{pub}} W$, then $W'' \sqsupseteq_{\text{pub}} W$.

**Proof**

By definition of $\sqsupseteq$ and $\sqsupseteq_{\text{pub}}$ for worlds and islands, and by the reflexivity and transitivity of the transition relations in the definition of well-formed islands. □

**Lemma 7.4 (Properties of $\triangleright$)**
For any $W, W', W'' \in$ World, we have

- $\triangleright W \sqsupseteq W$
- $\triangleright W \sqsupseteq_{\text{pub}} W$
- If $(H_1, H_2) : W$, then $(H_1, H_2) : \triangleright W$.

**Proof**

1. By definition of $\triangleright$ and $\sqsupseteq$, it suffices to show that $\lfloor \theta \rfloor_{W.k-1} \sqsupseteq \lfloor \theta \rfloor_{W.k-1}$ for each island $\theta \in W.\Theta$. But this relation is reflexive, so we are done.
2. Similar.
3. Note that if $W.k = 0$, there is nothing to show. Otherwise, the claim follows from the definitions of HeapRel and $\lfloor \varphi_M \rfloor_k$.

□

## 7.3 Basic Properties of Value and Expression Relations

**Lemma 7.5 (Related Values are Related Expressions)**
If $(W, v_1, v_2) \in \mathcal{V}[\![\tau]\!]\rho$ then $(W, v_1, v_2) \in \mathcal{E}[\![\tau]\!]\rho$.

**Proof**

Let $(W, E_1, E_2) \in \mathcal{K}[\![\tau]\!]\rho$. We need to show that $(W, E_1[v_1], E_2[v_2]) \in \mathcal{O}$. But since $\sqsupseteq_{\text{pub}}$ is reflexive, unfolding the definition of $\mathcal{K}[\![\tau]\!]\rho$ and applying our hypotheses gives the result immediately. □

**Lemma 7.6 (Monotonicity)**
If $\rho \in \mathcal{D}[\![\Delta]\!]$, $\Delta \vdash \tau$, $\Delta \vdash \boldsymbol{\tau}$ and $W' \sqsupseteq W$

1. $(W, \mathsf{v_1}, \mathsf{v_2}) \in \mathcal{V}[\![\boldsymbol{\tau}]\!]\rho \implies (W', \mathsf{v_1}, \mathsf{v_2}) \in \mathcal{V}[\![\boldsymbol{\tau}]\!]\rho$
2. $(W, \mathsf{v_1}, \mathsf{v_2}) \in \mathcal{V}[\![\tau]\!]\rho \implies (W', \mathsf{v_1}, \mathsf{v_2}) \in \mathcal{V}[\![\tau]\!]\rho$
3. $(W, \mathsf{v_1}, \mathsf{v_2}) \in \mathcal{V}[\![\tau, \tau^{\langle \mathcal{C} \rangle}]\!]\rho \implies (W', \mathsf{v_1}, \mathsf{v_2}) \in \mathcal{V}[\![\tau, \tau^{\langle \mathcal{C} \rangle}]\!]\rho$
4. $(W, \mathsf{v_1}, \mathsf{v_2}) \in \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}, \tau]\!]\rho \implies (W', \mathsf{v_1}, \mathsf{v_2}) \in \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}, \tau]\!]\rho$

**Proof**

The proofs, like the claims, are presented working up from the target language. This is because the case for lump types in the source language depends on the property holding in the target language.

1. By induction on $W'.k$ and on the structure of $\boldsymbol{\tau}$.
   In each case, we will need to show $(W', \mathsf{v_1}, \mathsf{v_2}) \in \text{ValAtom}[\boldsymbol{\tau}]\rho$. This amounts to showing that $W'.\Psi_i; \cdot \vdash \mathsf{v_i} : \boldsymbol{\tau}$ for $i \in \{1, 2\}$. We have by assumption that $W.\Psi_i; \cdot \vdash \mathsf{v_i} : \boldsymbol{\tau}$. By definition of world extension, $W'.\Psi_i \supseteq W.\Psi_i$, so this property holds.
   To complete the proof, we consider the possible cases of $\boldsymbol{\tau}$:
   **Case** $\boldsymbol{\alpha}$ Follows from $\rho(\boldsymbol{\alpha}).R[C, C] \in \text{ValRel}[\rho(\boldsymbol{\alpha}).\boldsymbol{\tau_1}, \rho(\boldsymbol{\alpha}).\boldsymbol{\tau_2}]$, which holds by $\rho(\boldsymbol{\alpha}) \in \text{CCValRel}$.
   **Case** **unit** Immediate.
   **Case** **int** Immediate.
   **Case** $\forall[\overline{\alpha}].(\overline{\tau}) \to \boldsymbol{\tau'}$ We need to show that $(W', \mathsf{v_1}, \mathsf{v_2}) \in \mathcal{V}[\![\forall[\overline{\alpha}].(\overline{\tau}) \to \boldsymbol{\tau'}]\!]\rho$.
   Let $W'' \sqsupseteq W'$, $\overline{\text{VR} \in \text{CCValRel}}$ such that $\rho' = \rho[\overline{\alpha \mapsto \text{VR}}]$, and $\overline{(W'', \mathsf{v'_1}, \mathsf{v'_2}) \in \mathcal{V}[\![\boldsymbol{\tau}]\!]\rho'}$. We need to show that
   $$(W'', \mathsf{v_1}\ [\overline{\text{VR}.\boldsymbol{\tau_1}}]\ \overline{\mathsf{v'_1}}, \mathsf{v_2}\ [\overline{\text{VR}.\boldsymbol{\tau_2}}]\ \overline{\mathsf{v'_2}}) \in \mathcal{E}[\![\boldsymbol{\tau'}]\!]\rho'.$$
   But by the transitivity of world extension, $W'' \sqsupseteq W$, so all the above assumptions instantiate our hypothesis that $(W, \mathsf{v_1}, \mathsf{v_2}) \in \mathcal{V}[\![\forall[\overline{\alpha}].(\overline{\tau}) \to \boldsymbol{\tau'}]\!]\rho$ to give exactly this result.
   **Case** **ref** $\boldsymbol{\tau'}$ By transitivity of world extension.
   **Case** $\exists\boldsymbol{\alpha}.\boldsymbol{\tau'}$ Follows from the induction hypothesis for the type.
   **Case** $\mu\boldsymbol{\alpha}.\boldsymbol{\tau'}$ Follows from the induction hypothesis for the step index.
   **Case** $\langle \boldsymbol{\tau_1}, \ldots, \boldsymbol{\tau_n} \rangle$ Follows from the induction hypotheses for the type.
   **Case** $\lceil \alpha \rceil$ Follows from $\rho(\alpha).R[C, C] \in \text{ValRel}[(\rho_1(\alpha).\tau_1)^{\langle \mathcal{C} \rangle}, (\rho_2(\alpha).\tau_2)^{\langle \mathcal{C} \rangle}]$, since $\rho(\alpha) \in \text{MMValRel}$.
2. By induction on $W'.k$ and on the structure of $\tau$. We will need to show $(W', \mathsf{v_1}, \mathsf{v_2}) \in \text{ValAtom}[\tau]\rho$. This holds by analogously to Claim 1. To complete the proof, we consider the possible cases of $\tau$:
   **Case** $\alpha$ Follows from $\rho(\alpha).R[M, M] \in \text{ValRel}[\rho(\alpha).\tau_1, \rho(\alpha).\tau_2]$, which holds by $\rho(\alpha) \in \text{MMValRel}$.
   **Case** unit Immediate.
   **Case** int Immediate.
   **Case** $\forall[\overline{\alpha}].(\overline{\tau}) \to \tau'$ We need to show that $(W', \mathsf{v_1}, \mathsf{v_2}) \in \mathcal{V}[\![\forall[\overline{\alpha}].(\overline{\tau}) \to \tau']\!]\rho$.
   Let $W'' \sqsupseteq W'$, $\overline{\text{VR} \in \text{MMValRel}}$ such that $\rho' = \rho[\overline{\alpha \mapsto \text{VR}}]$, and $\overline{(W'', \mathsf{v'_1}, \mathsf{v'_2}) \in \mathcal{V}[\![\tau]\!]\rho'}$. We need to show that
   $$(W'', \mathsf{v_1}\ [\overline{\text{VR}.\tau_1}]\ \overline{\mathsf{v'_1}}, \mathsf{v_2}\ [\overline{\text{VR}.\tau_2}]\ \overline{\mathsf{v'_2}}) \in \mathcal{E}[\![\tau']\!]\rho'.$$
   But by the transitivity of world extension, $W'' \sqsupseteq W$, so all the above assumptions instantiate our hypothesis that $(W, \mathsf{v_1}, \mathsf{v_2}) \in \mathcal{V}[\![\forall[\overline{\alpha}].(\overline{\tau}) \to \tau']\!]\rho$ to give exactly this result.

**Case** $\mathsf{ref}\,\tau'$ By transitivity of world extension.
**Case** $\exists\alpha.\tau'$ Follows from the induction hypothesis for the type.
**Case** $\mu\alpha.\tau'$ Follows from the induction hypothesis for the step index.
**Case** $\langle\tau_1,\ldots,\tau_n\rangle$ Follows from the induction hypotheses for the type.
**Case** $\mathsf{L}\langle\tau\rangle$ Follows from Claim 1.

3. By Claims 1 and 2.
4. By Claims 1 and 2.

□

## 7.4 Reduction Lemmas

**Lemma 7.7** **($\mathcal{O}$ Closed under Anti-Reduction)**
Given $W' \sqsupseteq W$, if $W.k \le W'.k + k_1$, $W.k \le W'.k + k_2$, and

$$\forall (H_1, H_2) : W.\ \exists (H_1', H_2') : W'.\ \langle H_1 \mid e_1\rangle \longmapsto^{k_1} \langle H_1' \mid e_1'\rangle \ \wedge\ \langle H_2 \mid e_2\rangle \longmapsto^{k_2} \langle H_2' \mid e_2'\rangle,$$

then

$$(W', e_1', e_2') \in \mathcal{O} \implies (W, e_1, e_2) \in \mathcal{O}.$$

**Proof**

Let $(H_1, H_2) : W$. Then, by our assumption, $\langle H_1 \mid e_1\rangle \longmapsto^{k_1} \langle H_1' \mid e_1'\rangle$ and $\langle H_2 \mid e_2\rangle \longmapsto^{k_2} \langle H_2' \mid e_2'\rangle$ for some $(H_1', H_2') : W'$. Since $(W', e_1', e_2') \in \mathcal{O}$, we have either that $\langle H_1' \mid e_1'\rangle \Downarrow$ and $\langle H_2' \mid e_2'\rangle \Downarrow$ or that $\mathrm{running}(W'.k, \langle H_1' \mid e_1'\rangle)$ and $\mathrm{running}(W'.k, \langle H_2' \mid e_2'\rangle)$.

In the former case, we have $\langle H_1 \mid e_1\rangle \Downarrow$ and $\langle H_2 \mid e_2\rangle \Downarrow$ by assumption. In the latter case, we have $\mathrm{running}(W'.k + k_1, \langle H_1 \mid e_1\rangle)$ and $\mathrm{running}(W'.k + k_2, \langle H_2 \mid e_2\rangle)$. Since we have as assumptions that both of these are more steps than needed, we have the result. □

**Lemma 7.8** **($\mathcal{O}$ Closed under Generalized Anti-Reduction)**
If $\forall H_1, H_2\ .\ (H_1, H_2) : W\ \wedge\ \langle H_2 \mid e\rangle \longmapsto \langle H_2 \mid e'\rangle$
Then

$$(W, e_1, e_2[e'/x]) \in \mathcal{O} \implies (W, e_1, e_2[e/x]) \in \mathcal{O}.$$

**Proof**

Let $(W, e_1, e_2[e'/x]) \in \mathcal{O}$ and let $(H_1', H_2') : W$. We have either that $\langle H_1' \mid e_1\rangle \Downarrow$ and $\langle H_2' \mid e_2[e'/x]\rangle \Downarrow$, or that $\mathrm{running}(W.k, \langle H_1' \mid e_1\rangle)$ and $\mathrm{running}(W.k, \langle H_1' \mid e_2[e'/x]\rangle)$. In the first case, it suffices to show that $\langle H_2' \mid e_2[e/x]\rangle \Downarrow$. In the second case, it suffices to show that $\mathrm{running}(W.k, \langle H_1' \mid e_2[e/x]\rangle)$. These can both be proven by induction on the structure of $e_2$. □

**Lemma 7.9** **($\mathcal{E}[\![\tau]\!]\rho$ Closed under Type-Preserving Anti-Reduction)**
Let $(W, e_1, e_2) \in \mathrm{TermAtom}[\tau]\rho$. Given $W' \sqsupseteq W$, if $W.k \le W'.k + k_1$, $W.k \le W'.k + k_2$, and

$$\forall (H_1, H_2) : W.\ \exists (H_1', H_2') : W'.\ \langle H_1 \mid e_1\rangle \longmapsto^{k_1} \langle H_1' \mid e_1'\rangle \ \wedge\ \langle H_2 \mid e_2\rangle \longmapsto^{k_2} \langle H_2' \mid e_2'\rangle,$$

then

$$(W', e_1', e_2') \in \mathcal{E}[\![\tau]\!]\rho \implies (W, e_1, e_2) \in \mathcal{E}[\![\tau]\!]\rho.$$

**Proof**

Let $(W, E_1, E_2) \in \mathcal{K}[\![\tau]\!]\rho$. We need to show that $(W, E_1[e_1], E_2[e_2]) \in \mathcal{O}$. By our assumption, $(W', E_1[e_1'], E_2[e_2']) \in \mathcal{O}$. By inspection of the operational semantics and by assumption, for any $(H_1, H_2) : W$, there is an $(H_1', H_2') : W'$ such that

$$\langle H_1 \mid E_1[e_1]\rangle \longmapsto^{k_1} \langle H_1' \mid E_1[e_1']\rangle \quad \text{and} \quad \langle H_2 \mid E_2[e_2]\rangle \longmapsto^{k_2} \langle H_2' \mid E_2[e_2']\rangle.$$

The result follows by Lemma 7.7. □

**Lemma 7.10** **($\mathcal{E}[\![\tau]\!]\rho$ Closed under Heap-Invariant Anti-Reduction)**
Let $(W, e_1, e_2) \in \text{TermAtom}[\tau]\rho$.
If

$$\forall (H_1, H_2) : W.\ \langle H_1 \mid e_1 \rangle \longmapsto^* \langle H_1 \mid e_1' \rangle \ \wedge\ \langle H_2 \mid e_2 \rangle \longmapsto^* \langle H_2 \mid e_2' \rangle,$$

then

$$(W, e_1', e_2') \in \mathcal{E}[\![\tau]\!]\rho \implies (W, e_1, e_2) \in \mathcal{E}[\![\tau]\!]\rho.$$

**Proof**

Follows from Lemma 7.9 using $W' = W$, $H_1' = H_1$, and $H_2' = H_2$. □

**Lemma 7.11** **($\mathcal{E}[\![\tau]\!]\rho$ Closed under Generalized Anti-Reduction)**
If $\forall H_1, H_2\ .\ (H_1, H_2) : W \ \wedge\ \langle H_2 \mid e \rangle \longmapsto \langle H_2 \mid e' \rangle\ \wedge (W, e_1, e_2[e/x]) \in \text{TermAtom}[\tau]\rho$
Then

$$(W, e_1, e_2[e'/x]) \in \mathcal{E}[\![\tau]\!]\rho \implies (W, e_1, e_2[e/x]) \in \mathcal{E}[\![\tau]\!]\rho.$$

**Proof**

By Lemma 7.8. □

**Lemma 7.12** **(Plugging Continuations Preserves Atoms)**
Let $(W, E_1, E_2) \in \text{ContAtom}[\tau]\rho \rightsquigarrow [\tau']\rho'$.

- If $(W, e_1, e_2) \in \text{TermAtom}[\tau]\rho$, then $(W, E_1[e_1], E_2[e_2]) \in \text{TermAtom}[\tau']\rho'$.
- If $(W, E_1', E_2') \in \text{ContAtom}[\tau']\rho' \rightsquigarrow [\tau'']\rho''$, then $(W, E_1'[E_1], E_2'[E_2]) \in \text{ContAtom}[\tau]\rho \rightsquigarrow [\tau'']\rho''$.

**Proof**

By induction on the type derivations. □

**Lemma 7.13** **(Monadic Bind)**
If $(W, e_1, e_2) \in \mathcal{E}[\![\tau]\!]\rho$, $(W, E_1, E_2) \in \text{ContAtom}[\tau]\rho \rightsquigarrow [\tau']\rho'$ and

$$\forall W' \sqsupseteq_{\text{pub}} W. (W', v_1, v_2) \in \mathcal{V}[\![\tau]\!]\rho \implies (W', E_1[v_1], E_2[v_2]) \in \mathcal{E}[\![\tau']\!]\rho,$$

then $(W, E_1[e_1], E_2[e_2]) \in \mathcal{E}[\![\tau']\!]\rho$.

**Proof**

We first need to show that $(W, E_1[e_1], E_2[e_2]) \in \text{TermAtom}[\tau']\rho'$. But this follows from Lemma 7.12, since $(W, e_1, e_2) \in \text{TermAtom}[\tau]\rho$.

Let $(W, E_1', E_2') \in \mathcal{K}[\![\tau']\!]\rho'$. We need to show that $(W, E_1'[E_1[e_1]], E_2'[E_2[e_1]]) \in \mathcal{O}$. It suffices to show that

$$(W, E_1'[E_1], E_2'[E_2]) \in \mathcal{K}[\![\tau]\!]\rho.$$

To get this, we first need $(W, E_1'[E_1], E_2'[E_2]) \in \text{ContAtom}[\tau]\rho \rightsquigarrow [\tau'']\rho''$ for some $\tau''$ and $\rho''$, but this follows immediately from our assumption and Lemma 7.12.

Next, let $W' \sqsupseteq_{\text{pub}} W$ such that $(W', v_1, v_2) \in \mathcal{V}[\![\tau]\!]\rho$. We must show that

$$(W', E_1'[E_1[v_1]], E_2'[E_2[v_2]]) \in \mathcal{O}.$$

Applying our premise, we find that $(W', E_1[v_1], E_2[v_2]) \in \mathcal{E}[\![\tau']\!]\rho'$. Instantiating this with the fact that $(W, E_1', E_2') \in \mathcal{K}[\![\tau']\!]\rho'$ gives the result. □

## 7.5 Identities on Abstract Type Interpretations

**Lemma 7.14**
If $\rho[\alpha \mapsto \mathrm{VR}] \in \mathcal{D}[\![\Delta, \alpha]\!]$ and $\alpha \notin \mathrm{ftv}\,\boldsymbol{\tau}$, then

1. $\mathcal{V}[\![\boldsymbol{\tau}]\!]\rho = \mathcal{V}[\![\boldsymbol{\tau}]\!]\rho[\alpha \mapsto \mathrm{VR}]$
2. $\mathcal{E}[\![\boldsymbol{\tau}]\!]\rho = \mathcal{E}[\![\boldsymbol{\tau}]\!]\rho[\alpha \mapsto \mathrm{VR}]$
3. $\mathcal{K}[\![\boldsymbol{\tau}]\!]\rho = \mathcal{K}[\![\boldsymbol{\tau}]\!]\rho[\alpha \mapsto \mathrm{VR}]$

**Proof**

We prove all claims simultaneously, by induction on the step index and the well formedness judgement of $\boldsymbol{\tau}$.

1. Consider the possible cases of $\boldsymbol{\tau}$:

   **Case** $\boldsymbol{\alpha}$ Immediate, since $\alpha \neq \boldsymbol{\alpha}$.
   **Case** **unit** Immediate.
   **Case** **int** Immediate.
   **Case** $\forall[\overline{\boldsymbol{\alpha}}].(\overline{\boldsymbol{\tau}}) \rightarrow \boldsymbol{\tau}'$ Follows from the induction hypothesis for $\boldsymbol{\tau}$ and from claim 2 (also using the induction hypothesis for $\boldsymbol{\tau}$).
   **Case** **ref** $\boldsymbol{\tau}$ Follows from the induction hypothesis for $\boldsymbol{\tau}$.
   **Case** $\exists\boldsymbol{\alpha} :: \mathbf{s}\,.\,\boldsymbol{\tau}$ Follows from the induction hypothesis for $\boldsymbol{\tau}$.
   **Case** $\boldsymbol{\mu\alpha.\tau}$ Follows from the induction hypothesis for the step index.
   **Case** $\langle \boldsymbol{\tau_1}, \ldots, \boldsymbol{\tau_n} \rangle$ Follows from the induction hypothesis for $\boldsymbol{\tau}$.
   **Case** $\lceil \alpha \rceil$ Immediate, since $\alpha \neq \alpha$.
2. Follows from claim 3.
3. Follows from claim 1.

□

**Lemma 7.15**
If $\rho[\alpha \mapsto \mathrm{VR}] \in \mathcal{D}[\![\Delta, \alpha]\!]$ and $\alpha \notin \mathrm{ftv}\,\tau$, then

1. $\mathcal{V}[\![\tau]\!]\rho = \mathcal{V}[\![\tau]\!]\rho[\alpha \mapsto \mathrm{VR}]$
2. $\mathcal{E}[\![\tau]\!]\rho = \mathcal{E}[\![\tau]\!]\rho[\alpha \mapsto \mathrm{VR}]$
3. $\mathcal{K}[\![\tau]\!]\rho = \mathcal{K}[\![\tau]\!]\rho[\alpha \mapsto \mathrm{VR}]$

**Proof**

We prove all claims simultaneously, by induction on the step index and $\tau$.

1. Consider the possible cases of $\tau$:

   **Case** $\alpha$ Immediate, since $\alpha \neq \alpha$.
   **Case** unit Immediate.
   **Case** int Immediate.
   **Case** $\forall[\overline{\alpha}].(\overline{\tau}) \rightarrow \tau'$ Follows from the induction hypothesis for $\tau$ and from claim 2 (also using the induction hypothesis for $\tau$).
   **Case** ref $\tau$ Follows from the induction hypothesis for $\tau$.
   **Case** $\exists\alpha.\tau$ Follows from the induction hypothesis for $\tau$.
   **Case** $\mu\alpha.\tau$ Follows from the induction hypothesis for the step index.
   **Case** $\langle \tau_1, \ldots, \tau_n \rangle$ Follows from the induction hypothesis for $\tau$.

**Case** $\mathsf{L}\langle\tau\rangle$ Follows from Lemma 7.14.

2. Follows from claim 3.
3. Follows from claim 1.

□

**Definition 7.16**
$\lfloor(\tau_1, \tau_2, R)\rfloor_{C,C} \stackrel{\text{def}}{=} (\tau_1{}^{\langle\mathcal{C}\rangle}, \tau_2{}^{\langle\mathcal{C}\rangle}, [R[C, C]])$

**Lemma 7.17**
If VR $\in$ MMValRel then $\lfloor \text{VR} \rfloor_{C,C} \in$ CCValRel.

**Proof**

Immediate, by the definitions of MMValRel, CCValRel □

**Lemma 7.18**
If $\rho \in \mathcal{D}[\![\Delta]\!]$ and $\alpha \notin \text{ftv}\, \tau$, then

1. $\mathcal{V}[\![\tau]\!]\rho[\alpha \mapsto \text{VR}] = \mathcal{V}[\![\tau[\alpha/\lceil\alpha\rceil]]\!]\rho[\alpha \mapsto \lfloor\text{VR}\rfloor_{C,C}]$
2. $\mathcal{E}[\![\tau]\!]\rho[\alpha \mapsto \text{VR}] = \mathcal{E}[\![\tau[\alpha/\lceil\alpha\rceil]]\!]\rho[\alpha \mapsto \lfloor\text{VR}\rfloor_{C,C}]$
3. $\mathcal{K}[\![\tau]\!]\rho[\alpha \mapsto \text{VR}] = \mathcal{K}[\![\tau[\alpha/\lceil\alpha\rceil]]\!]\rho[\alpha \mapsto \lfloor\text{VR}\rfloor_{C,C}]$

**Proof**

The proof follows the same structure as Lemma 7.14. The only interesting case is in claim 1, when $\tau = \lceil\alpha\rceil$. In this case we have

$\mathcal{V}[\![\lceil\alpha\rceil]\!]\rho[\alpha \mapsto \text{VR}] = \text{VR}.R[C, C] = \lfloor\text{VR}\rfloor_{C,C}.R[C, C] = \mathcal{V}[\![\alpha]\!]\rho[\alpha \mapsto \lfloor\text{VR}\rfloor_{C,C}] = \mathcal{V}[\![\lceil\alpha\rceil[\alpha/\lceil\alpha\rceil]]\!]\rho[\alpha \mapsto \lfloor\text{VR}\rfloor_{C,C}]$.

□

**Definition 7.19**
$\mathsf{L}\langle(\tau_1, \tau_2, R\rangle \stackrel{\text{def}}{=} (\mathsf{L}\langle\tau_1\rangle, \mathsf{L}\langle\tau_2\rangle, R')$

where $R' = \begin{bmatrix} \{(W, {}^{\mathsf{L}\langle\tau_1\rangle}\mathcal{MC}\,\mathsf{v_1}, {}^{\mathsf{L}\langle\tau_2\rangle}\mathcal{MC}\,\mathsf{v_2}) \mid (W, \mathsf{v_1}, \mathsf{v_2}) \in R[C, C]\} & \{(W, {}^{\mathsf{L}\langle\tau_1\rangle}\mathcal{MC}\,\mathsf{v_1}, \mathsf{v_2}) \mid (W, \mathsf{v_1}, \mathsf{v_2}) \in R[C, C]\} \\ \{(W, \mathsf{v_1}, {}^{\mathsf{L}\langle\tau_2\rangle}\mathcal{MC}\,\mathsf{v_2}) \mid (W, \mathsf{v_1}, \mathsf{v_2}) \in R[C, C]\} & R[C, C] \end{bmatrix}$

**Lemma 7.20**
If VR $\in$ CCValRel then $\mathsf{L}\langle\text{VR}\rangle \in$ MMValRel.

**Proof**

Let VR $= (\tau_1, \tau_2, R)$, where the only entry in $R$ is $R[C, C]$.
After applying Definition 7.19, we are given an $R'$ built from $R$ and need to show:

- $R'[M, M] = \{(W, {}^{\mathsf{L}\langle\tau_1\rangle}\mathcal{MC}\,\mathsf{v_1}, {}^{\mathsf{L}\langle\tau_2\rangle}\mathcal{MC}\,\mathsf{v_2}) \mid (W, \mathsf{v_1}, \mathsf{v_2}) \in R[C, C]\} \in \text{ValRel}[\mathsf{L}\langle\tau_1\rangle, \mathsf{L}\langle\tau_2\rangle]$,
- $R'[C, M] = \{(W, \mathsf{v_1}, {}^{\mathsf{L}\langle\tau_2\rangle}\mathcal{MC}\,\mathsf{v_2}) \mid (W, \mathsf{v_1}, \mathsf{v_2}) \in R[C, C]\} \in \text{ValRel}[\tau_1, \mathsf{L}\langle\tau_2\rangle]$,
- $R'[M, C] = \{(W, {}^{\mathsf{L}\langle\tau_1\rangle}\mathcal{MC}\,\mathsf{v_1}, \mathsf{v_2} \mid (W, \mathsf{v_1}, \mathsf{v_2}) \in R[C, C]\} \in \text{ValRel}[\mathsf{L}\langle\tau_1\rangle, \tau_2]$,
- $\mathcal{CM}_1(\mathsf{L}\langle\tau_1\rangle, R'[M, M]) \subseteq R'[C, M]$
- $\mathcal{MC}_1(\mathsf{L}\langle\tau_1\rangle, R'[C, M]) \subseteq R'[M, M]$
- $\mathcal{CM}_1(\mathsf{L}\langle\tau_1\rangle, R'[M, C]) \subseteq R'[C, C]$
- $\mathcal{MC}_1(\mathsf{L}\langle\tau_1\rangle, R'[C, C]) \subseteq R'[M, C]$
- $\mathcal{CM}_2(\mathsf{L}\langle\tau_2\rangle, R'[M, M]) \subseteq R'[M, C]$
- $\mathcal{MC}_2(\mathsf{L}\langle\tau_2\rangle, R'[M, C]) \subseteq R'[M, M]$

- $\mathcal{CM}_2(\mathsf{L}\langle\boldsymbol{\tau_2}\rangle, R'[C,M]) \subseteq R'[C,C]$
- $\mathcal{MC}_2(\mathsf{L}\langle\boldsymbol{\tau_2}\rangle, R'[C,C]) \subseteq R'[C,M]$

All of these follow from the definition of $\mathsf{L}\langle\mathrm{VR}\rangle$ and lump value translations. □

**Lemma 7.21**
If $\rho \in \mathcal{D}[\![\Delta]\!]$ and $\boldsymbol{\alpha} \notin \mathrm{ftv}\,\boldsymbol{\tau}$, then

1. $\mathcal{V}[\![\boldsymbol{\tau}]\!]\rho[\alpha \mapsto \mathsf{L}\langle\mathrm{VR}\rangle] = \mathcal{V}[\![\boldsymbol{\tau}[\boldsymbol{\alpha}/\lceil\alpha\rceil]]\!]\rho[\boldsymbol{\alpha} \mapsto \mathrm{VR}]$
2. $\mathcal{E}[\![\boldsymbol{\tau}]\!]\rho[\alpha \mapsto \mathsf{L}\langle\mathrm{VR}\rangle] = \mathcal{E}[\![\boldsymbol{\tau}[\boldsymbol{\alpha}/\lceil\alpha\rceil]]\!]\rho[\boldsymbol{\alpha} \mapsto \mathrm{VR}]$
3. $\mathcal{K}[\![\boldsymbol{\tau}]\!]\rho[\alpha \mapsto \mathsf{L}\langle\mathrm{VR}\rangle] = \mathcal{K}[\![\boldsymbol{\tau}[\boldsymbol{\alpha}/\lceil\alpha\rceil]]\!]\rho[\boldsymbol{\alpha} \mapsto \mathrm{VR}]$

**Proof**

The proof follows the same structure as the proof of Lemma 7.14. The only interesting case is in claim 1, when $\boldsymbol{\tau} = \lceil\alpha\rceil$. In this case we have

$$\mathcal{V}[\![\lceil\alpha\rceil]\!]\rho[\alpha \mapsto \mathsf{L}\langle\mathrm{VR}\rangle] = \mathsf{L}\langle\mathrm{VR}\rangle.R[C,C] = \mathrm{VR}.R[C,C] = \mathcal{V}[\![\boldsymbol{\alpha}]\!]\rho[\boldsymbol{\alpha} \mapsto \mathrm{VR}] = \mathcal{V}[\![\lceil\alpha\rceil[\boldsymbol{\alpha}/\lceil\alpha\rceil]]\!]\rho[\boldsymbol{\alpha} \mapsto \mathrm{VR}].$$

□

Finally, this set of identities deals with a property of the lump type exploited by the value translations in our multi-language framework, which we need to reason about in order to prove boundary cancellation.

Since the translation of e.g. $\mathsf{L}\langle\tau^{\langle\mathcal{C}\rangle}\rangle$ is the same as the translation of $\tau$, we can transform the interpretation of an abstract type between these two instantiations. Intuitively, thanks to parametricity, nothing can be done with an abstract value except to return it from the term that was required to keep it abstract, and similarly, there are no operations on a lump except to send it over the boundary. If the return from an abstract view requires passing through a boundary, a lump of the translation of a value is indistinguishable from the original value. Thus, an interpretation VR of a type variable is equivalent to this transformation of it (opaqueR(VR)), as long as we are viewing it from a lower-level language, where we have to perform a translation to get to the underlying values.

**Definition 7.22**
Given $(\tau_1, \tau_2, R) \in \mathrm{MMValRel}$

$$\mathrm{opaqueR}(\tau_1, \tau_2, R) \stackrel{\mathrm{def}}{=} (\tau_1, \mathsf{L}\langle\tau_2{}^{\langle\mathcal{C}\rangle}\rangle, R')$$

where

$$\begin{aligned}
R'[M,M] &= \{(W, \mathsf{v}_1, {}^{\mathsf{L}\langle\tau_2{}^{\langle\mathcal{C}\rangle}\rangle}\mathcal{MC}\,\mathbf{v}_2') \mid (W, \mathsf{v}_1, \mathsf{v}_2) \in R[M,M] \;\wedge\; \mathbf{CM}^{\tau_2}(\mathsf{v}_2) = \mathbf{v}_2'\} \\
&\quad \cup \{(W, \mathsf{v}_1, {}^{\mathsf{L}\langle\tau_2{}^{\langle\mathcal{C}\rangle}\rangle}\mathcal{MC}\,\mathbf{v}_2) \mid (W, \mathsf{v}_1, \mathbf{v}_2) \in R[M,C]\} \\
R'[C,M] &= \{(W, \mathbf{v}_1, {}^{\mathsf{L}\langle\tau_2{}^{\langle\mathcal{C}\rangle}\rangle}\mathcal{MC}\,\mathbf{v}_2') \mid (W, \mathbf{v}_1, \mathsf{v}_2) \in R[C,M] \;\wedge\; \mathbf{CM}^{\tau_2}(\mathsf{v}_2) = \mathbf{v}_2'\} \\
&\quad \cup \{(W, \mathbf{v}_1, {}^{\mathsf{L}\langle\tau_2{}^{\langle\mathcal{C}\rangle}\rangle}\mathcal{MC}\,\mathbf{v}_2) \mid (W, \mathbf{v}_1, \mathbf{v}_2) \in R[C,C]\} \\
R'[M,C] &= R[M,C] \\
&\quad \cup \{(W, \mathsf{v}_1, \mathbf{v}_2') \mid (W, \mathsf{v}_1, \mathsf{v}_2) \in R[M,M] \;\wedge\; \mathbf{CM}^{\tau_2}(\mathsf{v}_2) = \mathbf{v}_2'\} \\
R'[C,C] &= R[C,C]
\end{aligned}$$

The curious reader might wonder why the definition of $R'[M,M]$ doesn't include references to $R[C,M]$ or $R[C,C]$. In the case of $R[C,M]$, the answer is that MMValRel gives us $\mathcal{MC}_1(\tau_1, R[C,M]) \subseteq R[M,M]$, which when combined with the first set in $R'[M,M]$ gives us $\{(W, \mathsf{v}_1', {}^{\mathsf{L}\langle\tau_2{}^{\langle\mathcal{C}\rangle}\rangle}\mathcal{MC}\,\mathbf{v}_2') \mid (W, \mathbf{v}_1, \mathsf{v}_2) \in R[C,M] \wedge {}^{\tau_1}\mathbf{MC}(\mathbf{v}_1) = \mathsf{v}_1' \;\wedge\; \mathbf{CM}^{\tau_2}(\mathsf{v}_2) = \mathbf{v}_2'\}$. Hence we don't need to manually encode it into the definition of opaqueR. A similar argument can be made for a $R[C,C]$ entry.

**Lemma 7.23**
If $\mathrm{VR} \in \mathrm{MMValRel}$ then $\mathrm{opaqueR}(\mathrm{VR}) \in \mathrm{MMValRel}$.

**Proof**

Let $\mathrm{VR} = (\tau_1, \tau_2, R)$. By applying opaqueR, a $R'$ is constructed, where we label each union with a subscript, for example:

$$R'[M,M]_1 = \{(W, \mathsf{v}_1, {}^{\mathsf{L}\langle(\tau_2{}^{\langle \mathcal{C} \rangle})\rangle}\mathcal{MC}\,\mathbf{v_2}) \mid (W, \mathsf{v}_1, \mathsf{v}_2) \in R[M,M] \ \wedge\ \mathbf{CM}^{\tau_2}(\mathsf{v}_2) = \mathbf{v_2}\},$$

We need to show:

- $R'[M,M] \in \mathrm{ValRel}[\tau_1, \mathsf{L}\langle\tau_2{}^{\langle \mathcal{C}\rangle}\rangle]$
- $R'[C,M] \in \mathrm{ValRel}[\tau_1{}^{\langle \mathcal{C}\rangle}, \mathsf{L}\langle\tau_2{}^{\langle \mathcal{C}\rangle}\rangle]$
- $R'[M,C] \in \mathrm{ValRel}[\tau_1, \tau_2{}^{\langle \mathcal{C}\rangle}]$

which are monotonicity properties that follow from $R \in \mathrm{MMValRel}$, and

- $\mathcal{CM}_1(\tau_1, R'[M,M]_1) \subseteq R'[C,M]_1$
- $\mathcal{CM}_1(\tau_1, R'[M,M]_2) \subseteq R'[C,M]_2$
- $\mathcal{MC}_1(\tau_1, R'[C,M]_1) \subseteq R'[M,M]_1$
- $\mathcal{MC}_1(\tau_1, R'[C,M]_2) \subseteq R'[M,M]_2$
- $\mathcal{CM}_1(\tau_1, R'[M,C]_1) \subseteq R'[C,C]$
- $\mathcal{CM}_1(\tau_1, R'[M,C]_2) \subseteq R'[C,C]$
- $\mathcal{MC}_1(\tau_1, R'[C,C]) \subseteq R'[M,C]_1$
- $\mathcal{CM}_2(\mathsf{L}\langle\tau_2{}^{\langle \mathcal{C}\rangle}\rangle, R'[M,M]_1) \subseteq R'[M,C]_1$
- $\mathcal{CM}_2(\mathsf{L}\langle\tau_2{}^{\langle \mathcal{C}\rangle}\rangle, R'[M,M]_2) \subseteq R'[M,C]_1$
- $\mathcal{MC}_2(\mathsf{L}\langle\tau_2{}^{\langle \mathcal{C}\rangle}\rangle, R'[M,C]_1) \subseteq R'[M,M]_2$
- $\mathcal{MC}_2(\mathsf{L}\langle\tau_2{}^{\langle \mathcal{C}\rangle}\rangle, R'[M,C]_2) \subseteq R'[M,M]_1$
- $\mathcal{CM}_2(\mathsf{L}\langle\tau_2{}^{\langle \mathcal{C}\rangle}\rangle, R'[C,M]_1) \subseteq R'[C,C]$
- $\mathcal{CM}_2(\mathsf{L}\langle\tau_2{}^{\langle \mathcal{C}\rangle}\rangle, R'[C,M]_2) \subseteq R'[C,C]$
- $\mathcal{MC}_2(\mathsf{L}\langle\tau_2{}^{\langle \mathcal{C}\rangle}\rangle, R'[C,C]) \subseteq R'[C,M]_2$

These follow from relation subset properties given by $R \in \mathrm{MMValRel}$ and the definition of relation and value boundary translations.
The proof of these cases are very similar so we'll just show $\mathcal{CM}_1(\tau_1, R'[M,M]_1) \subseteq R'[C,M]$, that is, $\forall i \in \{1,2\}\ .\ \mathcal{CM}_1(\tau_1, R'[M,M]_i) \subseteq R'[C,M]$.
Focusing on the $i = 1$ case, it suffices to show that $\mathcal{CM}_1(\tau_1, R'[M,M]_1) \subseteq R'[C,M]_1$:

To begin, assume $(W, \mathbf{v_1}, \mathsf{v}_2) \in \mathcal{CM}_1(\tau_1, R'[M,M]_1)$ and show $(W, \mathbf{v_1}, \mathsf{v}_2) \in R'[C,M]_1$

Expanding the premise gives us $\mathbf{CM}^{\tau_1}(\mathsf{v}_1) = \mathbf{v_1}$ and $(W, \mathsf{v}_1, \mathsf{v}_2) \in R'[M,M]_1$

Unfolding $R'[M,M]_1$ we get that $\mathsf{v}_2 = {}^{\mathsf{L}\langle\tau_2{}^{\langle \mathcal{C}\rangle}\rangle}\mathcal{MC}\,\mathbf{v_2'}$ and $\mathbf{CM}^{\tau_2}(\hat{\mathsf{v}}_2) = \mathbf{v_2'}$ and $(W, \mathsf{v}_1, \hat{\mathsf{v}}_2) \in R[M,M]$

We have from $R \in \mathrm{MMValRel}$ that $\mathcal{CM}_1(\tau_1, R[M,M]) \subseteq R[C,M]$
Hence, from $\mathbf{CM}^{\tau_2}(\hat{\mathsf{v}}_2) = \mathbf{v_2'}$ and $(W, \mathsf{v}_1, \hat{\mathsf{v}}_2) \in R[M,M]$ we can conclude that $(W, \mathsf{v}_1, \mathbf{v_2'}) \in R[C,M]$
Giving us that $(W, \mathsf{v}_1, {}^{\mathsf{L}\langle\tau_2{}^{\langle \mathcal{C}\rangle}\rangle}\mathcal{MC}\,\mathbf{v_2'}) \in R'[C,M]_1$ □

**Lemma 7.24**
Let $\mathrm{VR} \in \mathrm{ValRel}[\tau_1, \tau_2]$ and $\mathrm{VR}' = \mathrm{opaqueR}(\mathrm{VR})$. Let $R[M,M] = \mathrm{VR}.R[M,M]$ and $R'[M,M] = \mathrm{VR}'.R[M,M]$.

1. If $(W, \mathsf{v}_1, \mathsf{v}_2) \in R[M,M]$ and $\mathbf{CM}^{\tau_2}(\mathsf{v}_2) = \mathbf{v_2}$, then $(W, \mathsf{v}_1, {}^{\mathsf{L}\langle\tau_2{}^{\langle \mathcal{C}\rangle}\rangle}\mathcal{MC}\,\mathbf{v_2}) \in R'[M,M]$

2. If $(W, \mathsf{v}_1, {}^{\mathsf{L}\langle\tau_2{}^{\langle \mathcal{C}\rangle}\rangle}\mathcal{MC}\,\mathbf{v_2}) \in R'[M,M]$ and ${}^{\tau_2}\mathbf{MC}(\mathbf{v_2}) = \mathsf{v}_2$, then $(W, \mathsf{v}_1, \mathsf{v}_2) \in R[M,M]$

**Proof**

1. Immediate from the first part of the definition of $R'[M,M]$.
2. By the boundary cancellation properties encoded in $R'$.

   - Case $(W, \mathsf{v}_1, {}^{\mathsf{L}\langle\tau_2{}^{\langle\mathcal{C}\rangle}\rangle}\mathcal{MC}\,\mathbf{v_2}) \in R'[M,M]_1$:
     We have ${}^{\tau_2}\mathbf{MC}(\mathbf{v_2}) = \mathsf{v}_2$ and $R'[M,M]_1$ gives us $\exists \mathsf{v}_2'\ .\ \mathbf{CM}^{\tau_2}(\mathsf{v}_2') = \mathbf{v_2}$ and $(W, \mathsf{v}_1, \mathsf{v}_2') \in R[M,M]$
     By $\mathcal{CM}_2(\tau_2, R[M,M]) \subseteq R[M,C]$ we know that $(W, \mathsf{v}_1, \mathbf{v_2}) \in R[M,C]$
     Using to apply $\mathcal{MC}_2(\tau_2, R[M,C]) \subseteq R[M,M]$ gives us the desired conclusion.
   - Case $(W, \mathsf{v}_1, {}^{\mathsf{L}\langle\tau_2{}^{\langle\mathcal{C}\rangle}\rangle}\mathcal{MC}\,\mathbf{v_2}) \in R'[M,M]_2$:
     $R'[M,M]_2$ gives us $(W, \mathsf{v}_1, \mathbf{v_2}) \in R[M,C]$
     By ${}^{\tau_2}\mathbf{MC}(\mathbf{v_2}) = \mathsf{v}_2$ and $\mathcal{MC}_2(\tau_2, R[M,C]) \subseteq R[M,M]$ we get the desired conclusion.

□

**Lemma 7.25**
-

1. $\mathcal{V}[\![\boldsymbol{\tau}]\!]\rho[\alpha \mapsto \mathrm{VR}] = \mathcal{V}[\![\boldsymbol{\tau}]\!]\rho[\alpha \mapsto \mathrm{opaqueR}(\mathrm{VR})]$
2. $\mathcal{E}[\![\boldsymbol{\tau}]\!]\rho[\alpha \mapsto \mathrm{VR}] = \mathcal{E}[\![\boldsymbol{\tau}]\!]\rho[\alpha \mapsto \mathrm{opaqueR}(\mathrm{VR})]$
3. $\mathcal{K}[\![\boldsymbol{\tau}]\!]\rho[\alpha \mapsto \mathrm{VR}] = \mathcal{K}[\![\boldsymbol{\tau}]\!]\rho[\alpha \mapsto \mathrm{opaqueR}(\mathrm{VR})]$

**Proof**

Follows the structure of Lemma 7.14. The only interesting case is in claim (1) with $\boldsymbol{\tau} = \lceil\alpha\rceil$, where

$$\mathcal{V}[\![\lceil\alpha\rceil]\!]\rho[\alpha \mapsto \mathrm{VR}] = \mathrm{VR}.R[C,C] = \mathrm{opaqueR}(\mathrm{VR}).R[C,C] = \mathcal{V}[\![\lceil\alpha\rceil]\!]\rho[\alpha \mapsto \mathrm{opaqueR}(\mathrm{VR})]$$

by definition. □

# 8 Proofs: Boundary Cancellation

**Lemma 8.1** (ValAtom **closed under MC/CM Boundaries Translations)**
Given $W$, $\tau$, and $\Delta$, let $\rho \in \mathcal{D}[\![\Delta, \overline{\beta}]\!]$ such that $\widetilde{\rho} \in \mathcal{D}[\![\Delta]\!]$, $\rho = \widetilde{\rho}\overline{[\beta \mapsto \mathrm{VR}]}$, and $\rho' = \widetilde{\rho}\overline{[\beta \mapsto \mathrm{VR}']}$, where for each $\mathrm{VR}_i$, $\mathrm{VR}'_i$, either $\mathrm{VR}_i = \mathrm{opaqueR}(\mathrm{VR}'_i)$ or $\mathrm{opaqueR}(\mathrm{VR}_i) = \mathrm{VR}'_i$. Also let $(W, \mathsf{v}_1, \mathsf{v}_2) \in \mathrm{ValAtom}[\tau]\rho$ and ${}^{\rho'_2(\tau)}\mathbf{MC}(\mathbf{CM}^{\rho_2(\tau)}(\mathsf{v}_2)) = \mathsf{v}'_2$. Then $(W, \mathsf{v}_1, \mathsf{v}'_2) \in \mathrm{ValAtom}[\tau]\rho'$.

**Proof**

We need to show that $(W, \mathsf{v}_1, \mathsf{v}'_2) \in \mathrm{TermAtom}[\tau]\rho'$. From $(W, \mathsf{v}_1, \mathsf{v}_2) \in \mathrm{ValAtom}[\tau]\rho$, we know that $W \in \mathrm{World}$, $W.\Psi_1; \cdot; \cdot \vdash \mathsf{v}_1 : \rho_1(\tau)$, and $W.\Psi_2; \cdot; \cdot \vdash \mathsf{v}_2 : \rho_2(\tau)$. By definition of opaqueR, $\rho'_1 = \rho_1$, so it suffices to show that $W.\Psi_2; \cdot; \cdot \vdash \mathsf{v}'_2 : \rho'_2(\tau)$. But now we need only use our hypothesis that ${}^{\rho'_2(\tau)}\mathbf{MC}(\mathbf{CM}^{\rho_2(\tau)}(\mathsf{v}_2)) = \mathsf{v}'_2$ to apply Lemma 7.2 twice. □

**Lemma 8.2 (MC/CM Right Boundary Cancellation)**
Given $W$, $\tau$, and $\Delta$, let $\rho \in \mathcal{D}[\![\Delta, \overline{\alpha}]\!]$ such that $\widetilde{\rho} \in \mathcal{D}[\![\Delta]\!]$, $\rho = \widetilde{\rho}\overline{[\alpha \mapsto \mathrm{VR}]}$, and $\rho' = \widetilde{\rho}\overline{[\alpha \mapsto \mathrm{VR}']}$, where for each $\mathrm{VR}_i$, $\mathrm{VR}'_i$, either $\mathrm{VR}_i = \mathrm{opaqueR}(\mathrm{VR}'_i)$ or $\mathrm{opaqueR}(\mathrm{VR}_i) = \mathrm{VR}'_i$. Then

1. If $(W, \mathsf{e}_1, \mathsf{e}_2) \in \mathcal{E}[\![\tau]\!]\rho$, then $(W, \mathsf{e}_1, {}^{\rho'_2(\tau)}\mathcal{MC}\,\mathcal{CM}^{\rho_2(\tau)}\ \mathsf{e}_2) \in \mathcal{E}[\![\tau]\!]\rho'$.
2. If $(W, \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\tau]\!]\rho$, and ${}^{\rho'_2(\tau)}\mathbf{MC}(\mathbf{CM}^{\rho_2(\tau)}(\mathsf{v}_2)) = \mathsf{v}'_2$, then $(W, \mathsf{v}_1, \mathsf{v}'_2) \in \mathcal{V}[\![\tau]\!]\rho'$.

**Proof**

We prove the two claims simultaneously by induction on $W.k$ and then on the structure of $\tau$.

For claim (1), we apply lemma 7.13 with $(W', [\cdot], {}^{\tau}\mathcal{MC}\,\mathcal{CM}^{\tau}\ [\cdot]) \in \mathrm{ContAtom}[\tau]\rho \rightsquigarrow [\tau]\rho'$ and the premise. Assuming $W' \sqsupseteq_{\mathrm{pub}} W$ and $(W', \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\tau]\!]\rho$, we now must show

$$(W', \mathsf{v}_1, {}^{\tau}\mathcal{MC}\,\mathcal{CM}^{\tau}\ \mathsf{v}_2) \in \mathcal{E}[\![\tau]\!]\rho'.$$

We can obtain $(H_1, H_2) : W'$ by unfolding the line above down to the $\mathcal{O}$ relation. By the semantics of the value translation there is a $\mathsf{v}'_2$ such that

$$\langle H_2 \mid {}^{\tau}\mathcal{MC}\,\mathcal{CM}^{\tau}\ \mathsf{v}_2\rangle \longmapsto^2 \langle H_2 \mid \mathsf{v}'_2\rangle.$$

Re-folding up to the E relation level, it suffices to show, by Lemma 7.10, $(W', \mathsf{v}_1, \mathsf{v}'_2) \in \mathcal{E}[\![\tau]\!]\rho'$, and finally, by Lemma 7.5, we need only show $(W', \mathsf{v}_1, \mathsf{v}'_2) \in \mathcal{V}[\![\tau]\!]\rho'$ which we have by claim (2).

We prove case (2) by considering the possible cases of $\tau$.

**Case** $\alpha$

Since $\rho \in \mathcal{D}[\![\Delta, \overline{\alpha}]\!]$, we know that $\rho(\alpha) \in \mathrm{MMValRel}$. By Lemma 7.23, $\rho'(\alpha) \in \mathrm{MMValRel}$ as well. Consider the three possible cases of $\rho(\alpha)$ and $\rho'(\alpha)$:

- If $\alpha \in \Delta$ then $\rho(\alpha) = \rho'(\alpha) = (\tau_1, \tau_2, R)$ and the result is immediate since $\rho(\alpha) \in \mathrm{ValRel}[\tau_1, \tau_2]$.
- If $\rho(\alpha) = (\tau_1, \tau_2, R)$ and $\rho'(\alpha) = \mathrm{opaqueR}(\rho(\alpha)) = (\tau_1, \mathsf{L}\langle {\tau_2}^{\langle \mathcal{C}\rangle}\rangle, R')$, where $R'$ is built from $R$ using definition of opaqueR (Definitions 7.22), then by Lemma 7.2, $\mathsf{v}'_2 = {}^{\mathsf{L}\langle \tau_2{}^{\langle \mathcal{C}\rangle}\rangle}\mathcal{MC}\ \mathbf{v_2}$ for some $\mathbf{v_2}$ such that $\mathbf{CM}^{\tau_2}(\mathsf{v}_2) = \mathbf{v_2}$. The result follows from Lemma 7.24.
- Finally, if $\rho'(\alpha) = (\tau_1, \tau_2, R)$ and

$$\rho(\alpha) = \mathrm{opaqueR}(\rho'(\alpha)) = (\tau_1, \mathsf{L}\langle {\tau_2}^{\langle \mathcal{C}\rangle}\rangle, R'),$$

then by the premise of the statement there exists some $\mathbf{v_2}$ such that $\mathsf{v}_2 = {}^{\mathsf{L}\langle \tau_2{}^{\langle \mathcal{C}\rangle}\rangle}\mathcal{MC}\ \mathbf{v_2}$, and ${}^{\tau_2}\mathbf{MC}(\mathbf{v_2}) = \mathsf{v}'_2$. Here again $R'$ is built from $R$ using definition of opaqueR (Definitions 7.22). The result follows from Lemma 7.24.

**Case** $\mathsf{unit}$ By inspection of the boundary value translation $\mathsf{v}_2 = \mathsf{v}'_2 = ()$

**Case** $\mathsf{int}$ By inspection of the boundary value translation $\mathsf{v}_2 = \mathsf{v}'_2 = \mathsf{n}$

**Case** $\forall[\overline{\alpha}].(\overline{\tau}) \to \tau'$ By Lemma 8.1, we know that $(W, \mathsf{v}_1, \mathsf{v}_2') \in \mathrm{ValAtom}[\forall[\overline{\alpha}].(\overline{\tau}) \to \tau']\rho'$.

Let $W' \sqsupseteq W$, $\overline{\mathrm{VR} \in \mathrm{MMValRel}}$ and $\overline{(W', \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2) \in \mathcal{V}[\![\tau]\!]\rho'\overline{[\alpha \mapsto \mathrm{VR}]}}$. We need to show that

$$(W', \mathsf{v}_1\,[\overline{\mathrm{VR}.\tau_1}]\,\overline{\hat{\mathsf{v}}_1}, \mathsf{v}_2'\,[\overline{\mathrm{VR}.\tau_2}]\,\overline{\hat{\mathsf{v}}_2}) \in \mathcal{E}[\![\tau']\!]\rho'\overline{[\alpha \mapsto \mathrm{VR}]}.$$

For convenience, let $\overline{\tau_1 = \mathrm{VR}.\tau_1}$, $\overline{\tau_2 = \mathrm{VR}.\tau_2}$, $\hat{\rho} = \rho\overline{[\alpha \mapsto \mathrm{opaqueR}(\mathrm{VR})]}$, and $\hat{\rho}' = \rho'\overline{[\alpha \mapsto \mathrm{VR}]}$. Thus we can restate our assumptions as $\overline{(W', \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2) \in \mathcal{V}[\![\tau]\!]\hat{\rho}'}$, and we can restate our proof obligation as $(W', \mathsf{v}_1\,[\overline{\tau_1}]\,\overline{\hat{\mathsf{v}}_1}, \mathsf{v}_2'\,[\overline{\tau_2}]\,\overline{\hat{\mathsf{v}}_2}) \in \mathcal{E}[\![\tau']\!]\hat{\rho}'$.

By the value translation there are some $\hat{\mathbf{v}}$ and $\overline{\hat{\mathsf{v}}_2'}$ such that

$$\overline{\mathbf{CM}^{\hat{\rho}_2'(\tau)}(\hat{\mathsf{v}}_2) = \hat{\mathbf{v}}} \qquad \text{and} \qquad \overline{{}^{\hat{\rho}_2(\tau)}\mathbf{MC}(\hat{\mathbf{v}}) = \hat{\mathsf{v}}_2'}.$$

By the induction hypothesis,

$$\overline{(W', \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2') \in \mathcal{V}[\![\tau]\!]\hat{\rho}}.$$

Hence, by our assumption that $(W, \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\forall[\overline{\alpha}].(\overline{\tau}) \to \tau']\!]\rho$, we have

$$(W', \mathsf{v}_1\,[\overline{\tau_1}]\,\overline{\hat{\mathsf{v}}_1}, \mathsf{v}_2\,[\overline{\mathsf{L}\langle\tau_2{}^{\langle\mathcal{C}\rangle}\rangle}]\,\overline{\hat{\mathsf{v}}_2'}) \in \mathcal{E}[\![\tau']\!]\hat{\rho}.$$

By the induction hypothesis and by claim (1),

$$(W', \mathsf{v}_1\,[\overline{\tau_1}]\,\overline{\hat{\mathsf{v}}_1}, {}^{\hat{\rho}_2'(\tau')}\mathcal{MC}\,\mathcal{CM}^{\hat{\rho}_2(\tau')}\,\mathsf{v}_2\,[\overline{\mathsf{L}\langle\tau_2{}^{\langle\mathcal{C}\rangle}\rangle}]\,\overline{\hat{\mathsf{v}}_2'}) \in \mathcal{E}[\![\tau']\!]\hat{\rho}'.$$

Let $(H_1, H_2) : W'$. By Lemma 7.10, it suffices to show that

$$\langle H_2 \mid \mathsf{v}_2'\,[\overline{\tau_2}]\,\overline{\hat{\mathsf{v}}_2}\rangle \longmapsto^* \langle H_2 \mid {}^{\hat{\rho}_2'(\tau')}\mathcal{MC}\,\mathcal{CM}^{\hat{\rho}_2(\tau')}\,\mathsf{v}_2\,[\overline{\mathsf{L}\langle\tau_2{}^{\langle\mathcal{C}\rangle}\rangle}]\,\overline{\hat{\mathsf{v}}_2'}\rangle.$$

To show this, we derive the shape of $\mathsf{v}_2'$. By definition,

$$\mathbf{CM}^{\rho_2(\forall[\overline{\alpha}].(\overline{\tau}) \to \tau')}(\mathsf{v}_2) = \mathbf{pack}\,\langle\mathbf{unit}, \langle\mathbf{v}, ()\rangle\rangle\ \mathbf{as}\ (\rho_2(\forall[\overline{\alpha}].(\overline{\tau}) \to \tau'))^{\langle\mathcal{C}\rangle},$$

where

$$\mathbf{v} = \boldsymbol{\lambda}[\overline{\alpha}](\mathbf{z} : \mathbf{unit}, \overline{\mathbf{x} : \rho_2(\tau)^{\langle\mathcal{C}\rangle}\,\overline{[\alpha/\lceil\alpha\rceil]}}).\mathcal{CM}^{\rho_2(\tau')\overline{[\mathsf{L}\langle\alpha\rangle/\alpha]}}\,\mathsf{v}_2\,[\overline{\mathsf{L}\langle\alpha\rangle}]\,\overline{{}^{\rho_2(\tau)\overline{[\mathsf{L}\langle\alpha\rangle/\alpha]}}\mathcal{MC}\,\mathbf{x}}.$$

Also by definition,

$${}^{\rho_2'(\forall[\overline{\alpha}].(\overline{\tau}) \to \tau')}\mathbf{MC}(\mathbf{pack}\,\langle\mathbf{unit}, \langle\mathbf{v}, ()\rangle\rangle) = \lambda[\overline{\alpha}](\overline{\mathsf{x} : \tau}).{}^{\rho_2'(\tau')}\mathcal{MC}\,\mathbf{e},$$

where

$$\mathbf{e} = \mathbf{unpack}\,\langle\beta, \mathbf{y}\rangle = \mathbf{pack}\,\langle\mathbf{unit}, \langle\mathbf{v}, ()\rangle\rangle\ \mathbf{in}\ (\pi_1(\mathbf{y}))\,[\overline{\lceil\alpha\rceil}]\,\pi_2(\mathbf{y}), \overline{\mathcal{CM}^{\rho_2'(\tau)}\,\mathsf{x}}.$$

Thus $\mathsf{v}_2' = \lambda[\overline{\alpha}](\overline{\mathsf{x} : \tau}).{}^{\rho_2'(\tau')}\mathcal{MC}\,\mathbf{e}$.

By the operational semantics, we have

$$\begin{array}{lll}
\langle H_2 \mid \mathsf{v}_2'\,[\overline{\tau_2}]\,\overline{\hat{\mathsf{v}}_2}\rangle & \longmapsto & \langle H_2 \mid ({}^{\rho_2'(\tau')}\mathcal{MC}\,\mathbf{e})\overline{[\tau_2/\alpha]}\overline{[\hat{\mathsf{v}}_2/\mathsf{x}]}\rangle \\
& \longmapsto^* & \langle H_2 \mid {}^{\hat{\rho}_2'(\tau')}\mathcal{MC}\,(\mathbf{v}\,[\overline{\tau_2{}^{\langle\mathcal{C}\rangle}}]\,(), \overline{\mathcal{CM}^{\hat{\rho}_2'(\tau)}\,\hat{\mathsf{v}}_2})\rangle \\
& \longmapsto^* & \langle H_2 \mid {}^{\hat{\rho}_2'(\tau')}\mathcal{MC}\,(\mathbf{v}\,[\overline{\tau_2{}^{\langle\mathcal{C}\rangle}}]\,(), \overline{\hat{\mathbf{v}}})\rangle \\
& \longmapsto & \langle H_2 \mid {}^{\hat{\rho}_2'(\tau')}\mathcal{MC}\,\mathcal{CM}^{\hat{\rho}_2(\tau')}\,\mathsf{v}_2\,[\overline{\mathsf{L}\langle\tau_2{}^{\langle\mathcal{C}\rangle}\rangle}]\,\overline{{}^{\hat{\rho}_2(\tau)}\mathcal{MC}\,\hat{\mathbf{v}}}\rangle \\
& \longmapsto^* & \langle H_2 \mid {}^{\hat{\rho}_2'(\tau')}\mathcal{MC}\,\mathcal{CM}^{\hat{\rho}_2(\tau')}\,\mathsf{v}_2\,[\overline{\mathsf{L}\langle\tau_2{}^{\langle\mathcal{C}\rangle}\rangle}]\,\overline{\hat{\mathsf{v}}_2'}\rangle
\end{array}$$

as desired.

**Case** $\mathsf{ref}\,\tau$

$\mathsf{v}_2$ can be two possible values

- $\mathsf{v}_2 = \ell$:
  ${}^{\mathsf{ref}\,\tau}\mathbf{MC}(\mathbf{CM}^{\mathsf{ref}\,\tau}(\ell)) \longmapsto {}^{\mathsf{ref}\,\tau}\mathbf{MC}(\mathcal{CM}^{\mathsf{ref}\,\tau}\,\ell) \longmapsto \ell$.
  Hence, $\mathsf{v}_2 = \mathsf{v}_2' = \ell$
- $\mathsf{v}_2 = {}^{\mathsf{ref}\,\tau}\mathcal{MC}\,\boldsymbol{\ell}$:
  ${}^{\mathsf{ref}\,\tau}\mathbf{MC}(\mathbf{CM}^{\mathsf{ref}\,\tau}({}^{\mathsf{ref}\,\tau}\mathcal{MC}\,\boldsymbol{\ell})) \longmapsto {}^{\mathsf{ref}\,\tau}\mathbf{MC}(\boldsymbol{\ell}) \longmapsto {}^{\mathsf{ref}\,\tau}\mathcal{MC}\,\boldsymbol{\ell}$.
  Hence, $\mathsf{v}_2 = \mathsf{v}_2' = {}^{\mathsf{ref}\,\tau}\mathcal{MC}\,\boldsymbol{\ell}$

**Case** $\exists\alpha.\tau$

By Lemma 8.1, we know that $(W, \mathsf{v}_1, \mathsf{v}_2') \in \mathrm{ValAtom}[\exists\alpha.\tau]\rho'$.
Our hypothesis, $(W, \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\exists\alpha.\tau]\!]\rho$, gives us

$$\mathsf{v}_1 = \mathsf{pack}\ \langle\hat{\tau}_1, \hat{\mathsf{v}}_1\rangle\ \mathsf{as}\ \rho_1(\exists\alpha.\tau) \quad \text{and} \quad \mathsf{v}_2 = \mathsf{pack}\ \langle\hat{\tau}_2, \hat{\mathsf{v}}_2\rangle\ \mathsf{as}\ \rho_2(\exists\alpha.\tau)$$

and some VR such that

$$\mathrm{VR} \in \mathrm{MMValRel} \quad \mathrm{VR}.\tau_1 = \tau_1 \quad \mathrm{VR}.\tau_2 = \tau_2 \quad (W, \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2) \in \mathcal{V}[\![\tau]\!]\rho[\alpha \mapsto \mathrm{VR}]$$

We need to show $(W, \mathsf{v}_1, \mathsf{v}_2') \in \mathcal{V}[\![\exists\alpha.\tau]\!]\rho'$ where

$$\mathbf{v_2} = \mathbf{pack}\ \langle\hat{\tau}_2^{\langle\mathcal{C}\rangle}, \hat{\mathbf{v}}_2\rangle\ \mathbf{as}\ \rho_2(\exists\alpha.\tau)^{\langle\mathcal{C}\rangle} = \mathbf{CM}^{\rho_2(\exists\alpha.\tau)}(\mathsf{pack}\ \langle\hat{\tau}_2, \hat{\mathsf{v}}_2\rangle\ \mathsf{as}\ \rho_2(\exists\alpha.\tau))$$

$$\mathsf{v}_2' = \mathsf{pack}\ \langle\mathsf{L}\langle\hat{\tau}_2^{\langle\mathcal{C}\rangle}\rangle, \hat{\mathsf{v}}_2'\rangle\ \mathsf{as}\ \rho_2'(\exists\alpha.\tau) = {}^{\rho_2'(\exists\alpha.\tau)}\mathbf{MC}(\mathbf{v_2})$$

$$\text{given} \quad \hat{\mathbf{v}}_2 = \mathbf{CM}^{\rho_2(\tau[\hat{\tau}_2/\alpha])}(\hat{\mathsf{v}}_2) \quad \text{and} \quad \hat{\mathsf{v}}_2' = {}^{\rho_2'(\tau[\mathsf{L}\langle\hat{\tau}_2^{\langle\mathcal{C}\rangle}\rangle/\alpha])}\mathbf{MC}(\hat{\mathbf{v}}_2)$$

This comes down to showing that there exists some VR′ for which the following hold:

$$\mathrm{VR}' \in \mathrm{MMValRel} \quad \mathrm{VR}'.\tau_1 = \tau_1 \quad \mathrm{VR}'.\tau_2 = \mathsf{L}\langle\boldsymbol{\tau_2}\rangle \quad (W, \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2') \in \mathcal{V}[\![\tau]\!]\rho'[\alpha \mapsto \mathrm{VR}']$$

To satisfy these obligations we let $\mathrm{VR}' = \mathrm{opaqueR}(\mathrm{VR})$ and instantiate the inductive hypothesis with

$$(W, \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2) \in \mathcal{V}[\![\tau]\!]\rho[\alpha \mapsto \mathrm{VR}] \quad \text{and} \quad \hat{\mathsf{v}}_2' = {}^{\rho_2'(\tau[\mathsf{L}\langle\hat{\tau}_2^{\langle\mathcal{C}\rangle}\rangle/\alpha])}\mathbf{MC}(\mathbf{CM}^{\rho_2(\tau[\hat{\tau}_2/\alpha])}(\hat{\mathsf{v}}_2))$$

**Case** $\mu\alpha.\tau$

By Lemma 8.1, we know that $(W, \mathsf{v}_1, \mathsf{v}_2') \in \mathrm{ValAtom}[\mu\alpha.\tau]\rho'$.
By the relation part of our hypothesis, $(W, \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\mu\alpha.\tau]\!]\rho$, we know that

$$\mathsf{v}_1 = \mathsf{fold}_{\rho_1(\mu\alpha.\tau)}\ \hat{\mathsf{v}}_1, \qquad \mathsf{v}_2 = \mathsf{fold}_{\rho_2(\mu\alpha.\tau)}\ \hat{\mathsf{v}}_2,$$

and that $(W, \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2) \in \triangleright\mathcal{V}[\![\tau[\mu\alpha.\tau/\alpha]]\!]\rho$.
Inspecting the $\mathsf{v}_2$ value translation part of our hypothesis gives us $\mathsf{v}_2' = \mathsf{fold}_{\rho_2'(\mu\alpha.\tau)}\ \hat{\mathsf{v}}_2'$ where

$$\hat{\mathsf{v}}_2' = {}^{\rho_2'(\tau[\mu\alpha.\tau/\alpha])}\mathbf{MC}(\hat{\mathbf{v}}_2) \qquad \hat{\mathbf{v}}_2 = \mathbf{CM}^{\rho_2(\tau[\mu\alpha.\tau/\alpha])}(\hat{\mathsf{v}}_2)$$

We are left to show $(W, \mathsf{v}_1, \mathsf{v}_2') \in \mathcal{V}[\![\mu\alpha.\tau]\!]\rho'$. That is, $(W, \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2') \in \triangleright\mathcal{V}[\![\tau[\mu\alpha.\tau/\alpha]]\!]\rho'$
Providing the inductive hypothesis with

$$(W, \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2) \in \triangleright\mathcal{V}[\![\tau[\mu\alpha.\tau/\alpha]]\!]\rho' \qquad {}^{\rho_2'(\tau[\mu\alpha.\tau/\alpha])}\mathbf{MC}(\mathbf{CM}^{\rho_2(\tau[\mu\alpha.\tau/\alpha])}(\hat{\mathsf{v}}_2)) = \hat{\mathsf{v}}_2'$$

gives us exactly this.

**Case** $\langle\overline{\tau}\rangle$

By the definition of the value translations and the induction hypothesis.

**Case** $\mathsf{L}\langle\tau\rangle$

By Lemma 8.1, we know that $(W, \mathsf{v}_1, \mathsf{v}_2') \in \text{ValAtom}[\mathsf{L}\langle\tau\rangle]\rho'$.

By assumption, we know that

$$\mathsf{v}_1 = {}^{\rho_1(\mathsf{L}\langle\tau\rangle)}\mathcal{MC}\,\hat{\mathbf{v}}_1, \qquad \mathsf{v}_2 = {}^{\rho_2(\mathsf{L}\langle\tau\rangle)}\mathcal{MC}\,\hat{\mathbf{v}}_2,$$

and $(W, \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2) \in \mathcal{V}[\![\boldsymbol{\tau}]\!]\rho$. By inspection of the value translations, we know that $\mathsf{v}_2' = {}^{\rho_2'(\mathsf{L}\langle\tau\rangle)}\mathcal{MC}\,\hat{\mathbf{v}}_2$, so we need to show only that $(W, \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2) \in \mathcal{V}[\![\boldsymbol{\tau}]\!]\rho'$. But this follows by Lemma 7.25.

□

**Lemma 8.3 (MC/CM Left Boundary Cancellation)**
Given $W$, $\tau$, and $\Delta$, let $\rho \in \mathcal{D}[\![\Delta, \overline{\alpha}]\!]$ such that $\widetilde{\rho} \in \mathcal{D}[\![\Delta]\!]$, $\rho = \widetilde{\rho}\overline{[\alpha \mapsto \text{VR}]}$, and $\rho' = \widetilde{\rho}\overline{[\alpha \mapsto \text{VR}']}$, where for each $\text{VR}_i$, $\text{VR}_i'$, either $\text{VR}_i = \text{opaqueL}(\text{VR}_i')$ or $\text{opaqueL}(\text{VR}_i) = \text{VR}_i'$. Then

1. If $(W, \mathsf{e}_1, \mathsf{e}_2) \in \mathcal{E}[\![\tau]\!]\rho$, then $(W, {}^{\rho_2'(\tau)}\mathcal{MC}\,\mathcal{CM}^{\rho_1(\tau)}\,\mathsf{e}_1, \mathsf{e}_1) \in \mathcal{E}[\![\tau]\!]\rho'$.
2. If $(W, \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\tau]\!]\rho$, and ${}^{\rho_1'(\tau)}\mathbf{MC}(\mathbf{CM}^{\rho_1(\tau)}(\mathsf{v}_1)) = \mathsf{v}_1'$, then $(W, \mathsf{v}_1', \mathsf{v}_2) \in \mathcal{V}[\![\tau]\!]\rho'$.

**Proof**

The same as the proof of Lemma 8.2, using symmetry of the logical relation to flip the terms. Assumes a left side version of Lemma 8.1 and a definition of opaqueL that manipulates left side terms similar to how opaqueR manipulates terms on the right of a binary relation. □

**Lemma 8.4** (ValAtom **closed under CM/MC Right Boundaries Translations)**
Given $W$, $\tau$, and $\Delta$, let $\rho \in \mathcal{D}[\![\Delta, \overline{\beta}]\!]$ such that $\widetilde{\rho} \in \mathcal{D}[\![\Delta]\!]$, $\rho = \widetilde{\rho}\overline{[\beta \mapsto \text{VR}]}$, and $\rho' = \widetilde{\rho}\overline{[\beta \mapsto \text{opaqueR}(\text{VR})]}$. Also let $(W, \mathbf{v}_1, \mathbf{v}_2) \in \text{ValAtom}[\tau^{\langle\mathcal{C}\rangle}]\rho$ and $\mathbf{CM}^{\rho_2'(\tau)}({}^{\rho_2'(\tau)}\mathbf{MC}(\mathbf{v}_2)) = \mathbf{v}_2'$. Then

$$(W, \mathbf{v}_1, \mathbf{v}_2') \in \text{ValAtom}[\tau^{\langle\mathcal{C}\rangle}]\rho.$$

**Proof**

We need to show that $(W, \mathbf{v}_1, \mathbf{v}_2') \in \text{TermAtom}[\rho_1(\tau^{\langle\mathcal{C}\rangle}), \rho_2(\tau^{\langle\mathcal{C}\rangle})]$. By $(W, \mathbf{v}_1, \mathbf{v}_2) \in \text{ValAtom}[\tau^{\langle\mathcal{C}\rangle}]\rho$, we know that $W \in \text{World}$, $W.\Psi; \cdot; \cdot \vdash \mathbf{v}_1 : \rho_1(\tau^{\langle\mathcal{C}\rangle})$, and $W.\Psi; \cdot; \cdot \vdash \mathbf{v}_2 : \rho_2(\tau^{\langle\mathcal{C}\rangle})$. It suffices to show that $W.\Psi; \cdot; \cdot \vdash \mathbf{v}_2' : \rho_2(\tau^{\langle\mathcal{C}\rangle})$. But we can simply use $\mathbf{CM}^{\rho_2'(\tau)}({}^{\rho_2'(\tau)}\mathbf{MC}(\mathbf{v}_2)) = \mathbf{v}_2'$ to apply Lemma 7.2 twice. □

**Lemma 8.5 (CM/MC Right Boundary Cancellation)**
Given $W$, $\tau$, and $\Delta$, let $\rho \in \mathcal{D}[\![\Delta, \overline{\beta}]\!]$ such that $\widetilde{\rho} \in \mathcal{D}[\![\Delta]\!]$, $\rho = \widetilde{\rho}\overline{[\beta \mapsto \text{VR}]}$, and $\rho' = \widetilde{\rho}\overline{[\beta \mapsto \text{opaqueR}(\text{VR})]}$. Then

1. If $(W, \mathbf{e}_1, \mathbf{e}_2) \in \mathcal{E}[\![\tau^{\langle\mathcal{C}\rangle}]\!]\rho$, then $(W, \mathbf{e}_1, \mathcal{CM}^{\rho_2'(\tau)}\,{}^{\rho_2'(\tau)}\mathcal{MC}\,\mathbf{e}_2) \in \mathcal{E}[\![\tau^{\langle\mathcal{C}\rangle}]\!]\rho$.
2. If $(W, \mathbf{v}_1, \mathbf{v}_2) \in \mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}]\!]\rho$ and $\mathbf{CM}^{\rho_2'(\tau)}({}^{\rho_2'(\tau)}\mathbf{MC}(\mathbf{v}_2)) = \mathbf{v}_2'$, then

$$(W, \mathbf{v}_1, \mathbf{v}_2') \in \mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}]\!]\rho.$$

**Proof**

We prove both claims simultaneously by induction on $W.k$ and then on the structure of $\tau$.

For claim (1), let $W' \sqsupseteq_{\text{pub}} W$ and $(W', \mathbf{v}_1, \mathbf{v}_2) \in \mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}]\!]\rho$. Note that $(W, [\cdot], \mathcal{CM}^{\rho_2'(\tau)}\,{}^{\rho_2'(\tau)}\mathcal{MC}\,[\cdot]) \in \text{ContAtom}[\tau^{\langle\mathcal{C}\rangle}]\rho \rightsquigarrow [\tau^{\langle\mathcal{C}\rangle}]\rho$. By Lemma 7.13, it suffices to show

$$(W', \mathbf{v}_1, \mathcal{CM}^{\rho_2'(\tau)}\,{}^{\rho_2'(\tau)}\mathcal{MC}\,\mathbf{v}_2) \in \mathcal{E}[\![\tau^{\langle\mathcal{C}\rangle}]\!]\rho.$$

By value boundary translation, for any $H$, there is a $\mathbf{v}_2'$ such that $\langle H \mid \mathcal{CM}^{\rho_2'(\tau)}\,{}^{\rho_2'(\tau)}\mathcal{MC}\,\mathbf{v}_2\rangle \longmapsto^* \langle H \mid \mathbf{v}_2'\rangle$. Thus, by Lemma 7.10, it suffices to show $(W', \mathbf{v}_1, \mathbf{v}_2') \in \mathcal{E}[\![\tau^{\langle\mathcal{C}\rangle}]\!]\rho$, and finally, by Lemma 7.5, we need only show $(W', \mathbf{v}_1, \mathbf{v}_2') \in \mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}]\!]\rho$. We have this by claim (2).

We prove claim (2) by cases of $\tau$:

**Case** $\alpha$

By the definition of opaqueR we have $\rho(\alpha).R[C,C] = \rho'(\alpha).R[C,C]$.
$\rho'(\alpha) \in$ MMValRel gives us $\mathcal{MC}_2(\tau_2, R[C,C]) \subseteq R[C,M]$ and $\mathcal{CM}_2(\tau_2, R[C,M]) \subseteq R[C,C]$ by which the result immediately follows.

**Case** unit

By inspection of the translation, $\mathbf{v}_2' = \mathbf{v}_2 = ()$, so we are done.

**Case** int

By inspection of the translation, $\mathbf{v}_2' = \mathbf{v}_2 = \mathbf{n}$, so we are done.

**Case** $\forall[\overline{\alpha}].(\overline{\tau}) \to \tau'$

By Lemma 8.4, we know that $(W, \mathbf{v}_1, \mathbf{v}_2') \in \text{ValAtom}[\forall[\overline{\alpha}].(\overline{\tau}) \to \tau'^{\langle C \rangle}]\rho$.
Recall that

$$\forall[\overline{\alpha}].(\overline{\tau}) \to \tau'^{\langle C \rangle} = \exists \beta. \langle \left( \forall[\overline{\alpha}].(\beta, \overline{\tau^{\langle C \rangle}[\alpha/\lceil \alpha \rceil]}) \to \tau'^{\langle C \rangle} \overline{[\alpha/\lceil \alpha \rceil]} \right), \beta \rangle$$

Let $\tau_f = \forall[\overline{\alpha}].(\beta, \overline{\tau^{\langle C \rangle}[\alpha/\lceil \alpha \rceil]}) \to \tau'^{\langle C \rangle} \overline{[\alpha/\lceil \alpha \rceil]}$. Since $(W, \mathbf{v}_1, \mathbf{v}_2) \in \mathcal{V}[\![\exists \beta. \langle \tau_f, \beta \rangle]\!]\rho$, we know that

$$\mathbf{v}_1 = \mathbf{pack}\ \langle \tau_1, \langle \mathbf{v}_{f1}, \mathbf{v}_{env1} \rangle \rangle\ \mathbf{as}\ \exists \beta. \langle \rho_1(\tau_f), \beta \rangle$$

$$\mathbf{v}_2 = \mathbf{pack}\ \langle \tau_2, \langle \mathbf{v}_{f2}, \mathbf{v}_{env2} \rangle \rangle\ \mathbf{as}\ \exists \beta. \langle \rho_2(\tau_f), \beta \rangle,$$

and there is some VR $\in$ CCValRel such that VR.$\tau_1 = \tau_1$, VR.$\tau_2 = \tau_2$,

$$(W, \mathbf{v}_{f1}, \mathbf{v}_{f2}) \in \mathcal{V}[\![\tau_f]\!]\rho[\beta \mapsto \text{VR}], \quad \text{and} \quad (W, \mathbf{v}_{env1}, \mathbf{v}_{env2}) \in \text{VR}.R[C,C].$$

By inpection of the translations, $\mathbf{v}_2' = \mathbf{pack}\ \langle \mathbf{unit}, \langle \mathbf{v}_{f2}', () \rangle \rangle\ \mathbf{as}\ \exists \beta. \langle \rho_2(\tau_f), \beta \rangle$. We need to find some VR$' \in$ CCValRel such that VR$'.\tau_1 = \tau_1$, VR$'.\tau_2 = \mathbf{unit}$, $(W, \mathbf{v}_{env1}, ()) \in \text{VR}'.R[C,C]$ and for any $\mathbf{v}_1$ the following holds

$$(W', \mathbf{v}_1, ()) \in \text{VR}'.R[C,C] \implies (W', \mathbf{v}_1, \mathbf{v}_{env2}) \in \text{VR}.R[C,C]$$

Let VR$' = (\tau_1, \mathbf{unit}, [\{(W', \mathbf{v}_{env1}, ()) \mid \forall W' \,.\, W' \sqsupseteq W\}])$ , where the $[\ldots]$ signifies a 1-dimensional relation matrix, since we are working in C, the lowest language.
It is straight forward to see that VR$'$ satifies the aformentioned properties.
From these properties we now show that $(W, \mathbf{v}_{f1}, \mathbf{v}_{f2}') \in \mathcal{V}[\![\tau_f]\!]\rho[\beta \mapsto \text{VR}']$
Let $W' \sqsupseteq W$, $\overline{\text{VR}^* \in \text{CCValRel}}$, $\rho^* = \rho[\beta \mapsto \text{VR}']\overline{[\alpha \mapsto \text{VR}^*]}$, $(W', \mathbf{v}_{env1}^*, \mathbf{v}_{env2}^*) \in \text{VR}'.R[C,C]$, and $\overline{(W', \mathbf{v}_1^*, \mathbf{v}_2^*) \in \mathcal{V}[\![\tau^{\langle C \rangle}[\alpha/\lceil \alpha \rceil]]\!]\rho^*}$.
For convenience, also let $\overline{\tau_1^* = \text{VR}^*.\tau_1}$ and $\overline{\tau_2^* = \text{VR}^*.\tau_2}$. We need to show that

$$(W', \mathbf{v}_{f1}\ [\overline{\tau_1^*}]\ \mathbf{v}_{env1}^*, \overline{\mathbf{v}_1^*}, \mathbf{v}_{f2}'\ [\overline{\tau_2^*}]\ \mathbf{v}_{env2}^*, \overline{\mathbf{v}_2^*}) \in \mathcal{E}[\![\tau'^{\langle C \rangle} \overline{[\alpha/\lceil \alpha \rceil]}]\!]\rho^*.$$

Let $\hat{\rho} = \rho[\beta \mapsto \text{VR}]\overline{[\alpha \mapsto \text{VR}^*]}$. By value boundary translation there exist $\overline{\mathsf{v}_2^*}$ and $\overline{\mathbf{v}_2^{*\prime}}$ such that

$$\overline{{}^{\rho_2'(\tau)\overline{[\mathsf{L}\langle \tau_2^* \rangle/\alpha]}}\mathbf{MC}(\mathbf{v}_2^*) = \mathsf{v}_2^*} \quad \text{and} \quad \overline{\mathbf{CM}^{\rho_2'(\tau)\overline{[\mathsf{L}\langle \tau_2^* \rangle/\alpha]}}(\mathsf{v}_2^*) = \mathbf{v}_2^{*\prime}}.$$

By Lemma 7.21, $\overline{\mathcal{V}[\![\tau^{\langle C \rangle}\overline{[\alpha/\lceil \alpha \rceil]}]\!]\rho^* = \mathcal{V}[\![\tau^{\langle C \rangle}]\!]\rho[\beta \mapsto \text{VR}']\overline{[\alpha \mapsto \mathsf{L}\langle \text{VR}^* \rangle]}}$, so we can apply the induction hypothesis and get $\overline{(W', \mathbf{v}_1^*, \mathbf{v}_2^{*\prime}) \in \mathcal{V}[\![\tau^{\langle C \rangle}\overline{[\alpha/\lceil \alpha \rceil]}]\!]\rho^*}$. Note that $\rho^*$ and $\hat{\rho}$ only differ at $\beta$, and that $\overline{\beta \notin \text{ftv}(\tau^{\langle C \rangle}\overline{[\alpha/\lceil \alpha \rceil]})}$. Therefore, by Lemma 7.14,

$$\overline{(W', \mathbf{v}_1^*, \mathbf{v}_2^{*\prime}) \in \mathcal{V}[\![\tau^{\langle C \rangle}\overline{[\alpha/\lceil \alpha \rceil]}]\!]\hat{\rho}}.$$

Since VR$'.\tau_2 = \mathbf{unit}$, we must have $\mathbf{v}_{env2}^* = ()$. By our assumption about VR$'$, we have that $(W', \mathbf{v}_{env1}^*, \mathbf{v}_{env2}) \in \text{VR}.R[C,C]$. Therefore, by our hypothesis that $(W, \mathbf{v}_{f1}, \mathbf{v}_{f2}) \in \mathcal{V}[\![\tau_f]\!]\rho[\beta \mapsto \text{VR}]$, we have

$$(W', \mathbf{v}_{f1}\ [\overline{\tau_1^*}]\ \mathbf{v}_{env1}^*, \overline{\mathbf{v}_1^*}, \mathbf{v}_{f2}\ [\overline{\tau_2^*}]\ \mathbf{v}_{env2}, \overline{\mathbf{v}_2^{*\prime}}) \in \mathcal{E}[\![\tau'^{\langle C \rangle}\overline{[\alpha/\lceil \alpha \rceil]}]\!]\hat{\rho}.$$

Once again, $\beta \notin \mathrm{ftv}(\tau'^{\langle \mathcal{C}\rangle}\overline{[\alpha/\lceil\alpha\rceil]})$, so

$$(W', \mathsf{v_{f1}}\ [\overline{\tau_1^*}]\ \mathsf{v_{env1}^*}, \overline{\mathsf{v_1^*}}, \mathsf{v_{f2}}\ [\overline{\tau_2^*}]\ \mathsf{v_{env2}}, \overline{\mathsf{v_2^*}'}) \in \mathcal{E}[\![\tau'^{\langle \mathcal{C}\rangle}\overline{[\alpha/\lceil\alpha\rceil]}]\!]\rho^*.$$

By Lemma 7.21,

$$\mathcal{E}[\![\tau'^{\langle \mathcal{C}\rangle}\overline{[\alpha/\lceil\alpha\rceil]}]\!]\rho^* = \mathcal{E}[\![\tau'^{\langle \mathcal{C}\rangle}]\!]\rho[\beta \mapsto \mathrm{VR}']\overline{[\alpha \mapsto \mathsf{L}\langle \mathrm{VR}^*\rangle]},$$

so we can apply the induction hypothesis to get

$$(W', \mathsf{v_{f1}}\ [\overline{\tau_1^*}]\ \mathsf{v_{env1}^*}, \overline{\mathsf{v_1^*}}, \mathcal{CM}^{\rho_2'(\tau')\overline{[\mathsf{L}\langle\tau_2^*\rangle/\alpha]}}\ ({}^{\rho_2'(\tau')\overline{[\mathsf{L}\langle\tau_2^*\rangle/\alpha]}}\mathcal{MC}\ \mathsf{v_{f2}}\ [\overline{\tau_2^*}]\ \mathsf{v_{env2}}, \overline{\mathsf{v_2^*}'})) \in \mathcal{E}[\![\tau'^{\langle \mathcal{C}\rangle}\overline{[\alpha/\lceil\alpha\rceil]}]\!]\rho^*,$$

By Lemma 7.10, given $(H_1, H_2) : W'$ it suffices to show that

$$\langle H_2 \mid \mathsf{v_{f2}'}\ [\overline{\tau_2^*}]\ (), \overline{\mathsf{v_2^*}}\rangle \longmapsto^* \langle H_2 \mid \mathcal{CM}^{\rho_2'(\tau')\overline{[\mathsf{L}\langle\tau_2^*\rangle/\alpha]}}\ ({}^{\rho_2'(\tau')\overline{[\mathsf{L}\langle\tau_2^*\rangle/\alpha]}}\mathcal{MC}\ \mathsf{v_{f2}}\ [\overline{\tau_2^*}]\ \mathsf{v_{env2}}, \overline{\mathsf{v_2^*}'})\rangle.$$

To show this, we examine the value translations to determine the structure of $\mathsf{v_{f2}'}$. We have that

$$\mathsf{v_{f2}'} = \lambda[\overline{\alpha}](\mathsf{z : unit}, \overline{\mathsf{x} : \rho_2'(\tau^{\langle \mathcal{C}\rangle})\overline{[\alpha/\lceil\alpha\rceil]}}).\mathcal{CM}^{\rho_2'(\tau')\overline{[\mathsf{L}\langle\alpha\rangle/\alpha]}}\ \mathsf{v}\ [\overline{\mathsf{L}\langle\alpha\rangle}]\ \overline{{}^{\rho_2'(\tau)\overline{[\mathsf{L}\langle\alpha\rangle/\alpha]}}\mathcal{MC}\ \mathsf{x}},$$

where

$${}^{\rho_2'(\forall[\overline{\alpha}].(\overline{\tau}) \to \tau'^{\langle \mathcal{C}\rangle})}\mathbf{MC}(\mathsf{v_2}) = \mathsf{v}.$$

In particular,

$$\mathsf{v} = \lambda[\alpha](\overline{\mathsf{x} : \tau}).{}^{\rho_2'(\tau')}\mathcal{MC}\ (\mathsf{unpack}\ \langle \beta, \mathsf{y}\rangle = \mathsf{v_2}\ \mathsf{in}\ \pi_1(\mathsf{y})\ [\overline{\lceil\alpha\rceil}]\ \pi_2(\mathsf{y}), \overline{\mathcal{CM}^{\rho_2'(\tau)}\ \mathsf{x}}).$$

Therefore,

$$\begin{aligned}
& \langle H_2 \mid \mathsf{v_{f2}'}\ [\overline{\tau_2^*}]\ (), \overline{\mathsf{v_2^*}}\rangle \\
\longmapsto\ & \langle H_2 \mid \mathcal{CM}^{\rho_2'(\tau')\overline{[\mathsf{L}\langle\tau_2^*\rangle/\alpha]}}\ \mathsf{v}\ [\overline{\mathsf{L}\langle\tau_2^*\rangle}]\ \overline{{}^{\rho_2'(\tau)\overline{[\mathsf{L}\langle\tau_2^*\rangle/\alpha]}}\mathcal{MC}\ \mathsf{v_2^*}}\rangle \\
\longmapsto^*\ & \langle H_2 \mid \mathcal{CM}^{\rho_2'(\tau')\overline{[\mathsf{L}\langle\tau_2^*\rangle/\alpha]}}\ \mathsf{v}\ [\overline{\mathsf{L}\langle\tau_2^*\rangle}]\ \overline{\mathsf{v_2^*}}\rangle \\
\longmapsto^4\ & \langle H_2 \mid \mathcal{CM}^{\rho_2'(\tau')\overline{[\mathsf{L}\langle\tau_2^*\rangle/\alpha]}}\ \mathcal{MC}\ \mathsf{v_{f2}}\ [\overline{\tau_2^*}]\ \mathsf{v_{env2}}, \overline{\mathcal{CM}^{\rho_2'(\tau)\overline{[\mathsf{L}\langle\tau_2^*\rangle/\alpha]}}\ \mathsf{v_2^*}}\rangle \\
\longmapsto^*\ & \langle H_2 \mid \mathcal{CM}^{\rho_2'(\tau')\overline{[\mathsf{L}\langle\tau_2^*\rangle/\alpha]}}\ {}^{\rho_2'(\tau')\overline{[\mathsf{L}\langle\tau_2^*\rangle/\alpha]}}\mathcal{MC}\ \mathsf{v_{f2}}\ [\overline{\tau_2^*}]\ \mathsf{v_{env2}}, \overline{\mathsf{v_2^*}'}\rangle,
\end{aligned}$$

as desired.

**Case** $\mathsf{ref}\ \tau$

$\mathsf{v_2}$ can be two possible values

- $\mathsf{v_2} = \ell$:
  $\mathbf{CM}^{\mathsf{ref}\ \tau}({}^{\mathsf{ref}\ \tau}\mathbf{MC}(\ell)) \longmapsto \mathbf{CM}^{\mathsf{ref}\ \tau}({}^{\mathsf{ref}\ \tau}\mathcal{MC}\ \ell) \longmapsto \ell$.
  Hence, $\mathsf{v_2} = \mathsf{v_2'} = \ell$
- $\mathsf{v_2} = {}^{\mathsf{ref}\ \tau}\mathcal{MC}\ \ell$:
  $\mathbf{CM}^{\mathsf{ref}\ \tau}({}^{\mathsf{ref}\ \tau}\mathbf{MC}(\mathcal{CM}^{\mathsf{ref}\ \tau}\ \ell)) \longmapsto \mathbf{CM}^{\mathsf{ref}\ \tau}(\ell) \longmapsto \mathcal{CM}^{\mathsf{ref}\ \tau}\ \ell$.
  Hence, $\mathsf{v_2} = \mathsf{v_2'} = {}^{\mathsf{ref}\ \tau}\mathcal{MC}\ \ell$

**Case** $\exists\alpha.\tau$

By Lemma 8.4, we know that $(W, \mathsf{v_1}, \mathsf{v_2'}) \in \mathrm{ValAtom}[\exists\alpha.\tau^{\langle \mathcal{C}\rangle}]\rho$.
Our hypothesis, $(W, \mathsf{v_1}, \mathsf{v_2}) \in \mathcal{V}[\![\exists\alpha.\tau^{\langle \mathcal{C}\rangle}]\!]\rho$, gives us

$$\mathsf{v_1} = \mathsf{pack}\ \langle \hat{\tau}_1, \hat{\mathsf{v}}_1\rangle\ \mathsf{as}\ \rho_1(\exists\alpha.\tau^{\langle \mathcal{C}\rangle}) \quad \text{and} \quad \mathsf{v_2} = \mathsf{pack}\ \langle \hat{\tau}_2, \hat{\mathsf{v}}_2\rangle\ \mathsf{as}\ \rho_2(\exists\alpha.\tau^{\langle \mathcal{C}\rangle})$$

and some VR such that

$$\mathrm{VR} \in \mathrm{CCValRel} \quad \mathrm{VR}.\tau_1 = \tau_1 \quad \mathrm{VR}.\tau_2 = \tau_2$$

and (applying Lemma 7.21)

$$(W, \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2) \in \mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}[\alpha/\lceil\alpha\rceil]]\!]\rho[\alpha \mapsto \text{VR}] = \mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}]\!]\rho[\alpha \mapsto \mathsf{L}\langle\text{VR}\rangle]$$

By the value translation part of the hypothesis we have

$${}^{\rho_2'(\tau)[\mathsf{L}\langle\tau_2\rangle/\alpha]}\mathbf{MC}(\hat{\mathbf{v}}_2) = \hat{\mathsf{v}} \quad \text{and} \quad \mathbf{CM}^{\rho_2'(\tau)[\mathsf{L}\langle\tau_2\rangle/\alpha]}(\hat{\mathsf{v}}) = \hat{\mathbf{v}}_2'$$

We need to show $(W, \mathbf{v}_1, \mathbf{v}_2') \in \mathcal{V}[\![\exists\alpha.\tau^{\langle\mathcal{C}\rangle}]\!]\rho$ where $\mathbf{v}_2' = \textbf{pack}\ \langle\tau_2, \hat{\mathbf{v}}_2'\rangle\ \textbf{as}\ \exists\alpha.\tau^{\langle\mathcal{C}\rangle}$ This follows by instantiating the inductive hypothesis with

$$\text{VR} \quad \text{and} \quad (W, \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2) \in \mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}]\!]\rho[\alpha \mapsto \mathsf{L}\langle\text{VR}\rangle]$$

gives us that there exists a VR′ such that

$$(W, \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2') \in \mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}]\!]\rho[\alpha \mapsto \mathsf{L}\langle\text{VR}'\rangle]$$

Using Lemma 7.21 turns this into

$$(W, \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2') \in \mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}[\alpha/\lceil\alpha\rceil]]\!]\rho[\alpha \mapsto \text{VR}']$$

Which allows us to construct the desired $(W, \mathbf{v}_1, \mathbf{v}_2') \in \mathcal{V}[\![\exists\alpha.\tau^{\langle\mathcal{C}\rangle}]\!]\rho$.

**Case** $\mu\alpha.\tau$

By Lemma 8.4, we know that $(W, \mathbf{v}_1, \mathbf{v}_2') \in \text{ValAtom}[\mu\alpha.\tau^{\langle\mathcal{C}\rangle}]\rho$.
We know that $\mathbf{v}_1 = \textbf{fold}_{\rho_1(\mu\alpha.\tau^{\langle\mathcal{C}\rangle})}\ \hat{\mathbf{v}}_1$, $\mathbf{v}_2 = \textbf{fold}_{\rho_2(\mu\alpha.\tau^{\langle\mathcal{C}\rangle})}\ \hat{\mathbf{v}}_2$, and

$$(W, \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2) \in \triangleright\mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}[\alpha/\lceil\alpha\rceil][\mu\alpha.\tau^{\langle\mathcal{C}\rangle}/\alpha]]\!]\rho = \triangleright\mathcal{V}[\![(\tau[\mu\alpha.\tau/\alpha])^{\langle\mathcal{C}\rangle}]\!]\rho.$$

By boundary value translation there exist $\hat{\mathsf{v}}$ and $\hat{\mathbf{v}}_2'$ such that

$${}^{\rho_2'(\tau[\mu\alpha.\tau/\alpha])}\mathbf{MC}(\hat{\mathbf{v}}_2) = \hat{\mathsf{v}} \quad \text{and} \quad \mathbf{CM}^{\rho_2'(\tau[\mu\alpha.\tau/\alpha])}(\hat{\mathsf{v}}) = \hat{\mathbf{v}}_2'.$$

Thus we can apply the induction hypothesis to get

$$(W, \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2') \in \triangleright\mathcal{V}[\![(\tau[\mu\alpha.\tau/\alpha])^{\langle\mathcal{C}\rangle}]\!]\rho = \triangleright\mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}[\alpha/\lceil\alpha\rceil][\mu\alpha.\tau^{\langle\mathcal{C}\rangle}/\alpha]]\!]\rho.$$

By inspection of the translations, $\mathbf{v}_2' = \textbf{fold}_{\rho_2(\mu\alpha.\tau^{\langle\mathcal{C}\rangle})}\ \hat{\mathbf{v}}_2'$, so we have the result.

**Case** $\langle\overline{\tau}\rangle$

By definition of the value translations and the induction hypothesis.

**Case** $\mathsf{L}\langle\boldsymbol{\tau}\rangle$

By inspection of the translations at type $\mathsf{L}\langle\boldsymbol{\tau}\rangle$, $\mathbf{v}_2' = \mathbf{v}_2$, so we are done.

□

**Lemma 8.6 (CM/MC Left Boundary Cancellation)**
Given $W$, $\tau$, and $\Delta$, let $\rho \in \mathcal{D}[\![\Delta, \overline{\beta}]\!]$ such that $\widetilde{\rho} \in \mathcal{D}[\![\Delta]\!]$, $\rho = \widetilde{\rho}\overline{[\beta \mapsto \text{VR}]}$, and $\rho' = \widetilde{\rho}\overline{[\beta \mapsto \text{opaqueL(VR)}]}$. Then

1. If $(W, \mathbf{e}_1, \mathbf{e}_2) \in \mathcal{E}[\![\tau^{\langle\mathcal{C}\rangle}]\!]\rho$, then $(W, \mathcal{CM}^{\rho_1'(\tau)}\ {}^{\rho_1'(\tau)}\mathcal{MC}\ \mathbf{e}_1, \mathbf{e}_2) \in \mathcal{E}[\![\tau^{\langle\mathcal{C}\rangle}]\!]\rho$.
2. If $(W, \mathbf{v}_1, \mathbf{v}_2) \in \mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}]\!]\rho$ and $\mathbf{CM}^{\rho_1'(\tau)}({}^{\rho_1'(\tau)}\mathbf{MC}(\mathbf{v}_1)) = \mathbf{v}_1'$, then

$$(W, \mathbf{v}_1', \mathbf{v}_2) \in \mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}]\!]\rho.$$

**Proof**

The same as the proof of Lemma 8.5, using symmetry of the logical relation to flip the terms. Assumes a left side version of Lemma 8.4 and a definition of opaqueL that manipulates left side terms similar to how opaqueR manipulates terms on the right of a binary relation. □

# 9 Proofs: Soundness and Completeness

**Lemma 9.1 (Bridge Lemma for M and C)**

Given $W$, $\tau$, and $\Delta$ such that $\Delta \vdash \tau$ let $\rho \in \mathcal{D}[\![\Delta]\!]$.

1. (a) If $(W, \mathbf{e_1}, \mathbf{e_2}) \in \mathcal{E}[\![\tau^{\langle \mathcal{C} \rangle}]\!]\rho$ then $(W, {}^{\rho_1(\tau)}\mathcal{MC}\, \mathbf{e_1}, {}^{\rho_2(\tau)}\mathcal{MC}\, \mathbf{e_2}) \in \mathcal{E}[\![\tau]\!]\rho$
   (b) If $(W, \mathsf{e}_1, \mathsf{e}_2) \in \mathcal{E}[\![\tau]\!]\rho$ then $(W, \mathcal{CM}^{\rho_1(\tau)}\, \mathsf{e}_1, \mathcal{CM}^{\rho_2(\tau)}\, \mathsf{e}_2) \in \mathcal{E}[\![\tau^{\langle \mathcal{C} \rangle}]\!]\rho$
2. (a) $\mathcal{MC}_{1,2}(\rho_1(\tau), \rho_2(\tau), \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}]\!]\rho) \subseteq \mathcal{V}[\![\tau]\!]\rho$
   (b) $\mathcal{CM}_{1,2}(\rho_1(\tau), \rho_2(\tau), \mathcal{V}[\![\tau]\!]\rho) \subseteq \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}]\!]\rho$

**Proof**

We can restate claim (2) as follows:

(a) If $(W, \mathbf{v_1}, \mathbf{v_2}) \in \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}]\!]\rho$ and ${}^{\rho_1(\tau)}\mathbf{MC}(\mathbf{v_1}) = \mathsf{v}_1$ and ${}^{\rho_2(\tau)}\mathbf{MC}(\mathbf{v_2}) = \mathsf{v}_2$ then $(W, \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\tau]\!]\rho$.

(b) If $(W, \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\tau]\!]\rho$ and $\mathbf{CM}^{\rho_1(\tau)}(\mathsf{v}_1) = \mathbf{v_1}$ and $\mathbf{CM}^{\rho_2(\tau)}(\mathsf{v}_2) = \mathbf{v_2}$ then $(W, \mathbf{v_1}, \mathbf{v_2}) \in \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}]\!]\rho$.

We prove all the claims simultaneously by induction on $W.k$ and the structure of $\tau$.

Here we provide a proof for Part 1a, but Part 1b follows in a similar manner.

Apply Monadic Bind (Lemma 7.13) we get to assume $W' \sqsupseteq_{\text{pub}} W$ and $(W', \mathbf{v_1}, \mathbf{v_2}) \in \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}]\!]\rho$. Since

$$(W, {}^{\rho_1(\tau)}\mathcal{MC}\, [\cdot], {}^{\rho_2(\tau)}\mathcal{MC}\, [\cdot]) \in \text{ContAtom}[\tau^{\langle \mathcal{C} \rangle}]\rho \rightsquigarrow [\tau]\rho$$

It suffices to show

$$(W', {}^{\rho_1(\tau)}\mathcal{MC}\, \mathbf{v_1}, {}^{\rho_2(\tau)}\mathcal{MC}\, \mathbf{v_2}) \in \mathcal{E}[\![\tau]\!]\rho$$

By Lemma 7.1, there exist $\mathsf{v}_1'$ and $\mathsf{v}_2'$ such that

$${}^{\rho_1(\tau)}\mathbf{MC}(\mathbf{v_1}) = \mathsf{v}_1' \qquad {}^{\rho_2(\tau)}\mathbf{MC}(\mathbf{v_2}) = \mathsf{v}_2'$$

By claim (2) we have $(W', \mathsf{v}_1', \mathsf{v}_2') \in \mathcal{V}[\![\tau]\!]\rho$ and Lemma 7.5 in turn gives us $(W', \mathsf{v}_1', \mathsf{v}_2') \in \mathcal{E}[\![\tau]\!]\rho$. We then apply anti-reduction, Lemma 7.10, leaving us to show that for some yet-to-be-determined $(H_1, H_2) : W'$

$$\langle H_1 \mid {}^{\rho_1(\tau)}\mathcal{MC}\, \mathbf{v_1} \rangle \longmapsto^* \langle H_1 \mid \mathsf{v}_1' \rangle \quad \text{and} \quad \langle H_2 \mid {}^{\rho_1(\tau)}\mathcal{MC}\, \mathbf{v_2} \rangle \longmapsto^* \langle H_2 \mid \mathsf{v}_2' \rangle$$

This follows from the operational semantics. The yet-to-be-determined $(H_1, H_2) : W'$ can be obtained by unfolding $(W', {}^{\rho_1(\tau)}\mathcal{MC}\, \mathbf{v_1}, {}^{\rho_2(\tau)}\mathcal{MC}\, \mathbf{v_2}) \in \mathcal{E}[\![\tau]\!]\rho$ down to the $\mathcal{O}$ relation, also collecting some $K_1$, $K_2$, and additional termination proof obligations. These obligations can be satisfied by instantiating $(W', \mathsf{v}_1', \mathsf{v}_2') \in \mathcal{E}[\![\tau]\!]\rho$ with $K_1, K_2$ and $H_1, H_2$.

For claim (2), we consider the possible cases of $\tau$:

**Case** $\alpha$

We have $\rho(\alpha) \in \text{MMValRel}$ from $\alpha \in \Delta$ and $\mathcal{D}[\![\Delta]\!]$.

This gives us the following two properties:

$$\mathcal{MC}_2(\rho_2(\alpha), \rho(\alpha).R[C, C]) \subseteq \rho(\alpha).R[C, M] \quad \text{and} \quad \mathcal{MC}_1(\rho_1(\alpha), \rho(\alpha).R[C, M]) \subseteq \rho(\alpha).R[M, M]$$

Hence, $\mathcal{MC}_1(\rho_1(\alpha), \mathcal{MC}_2(\rho_2(\alpha), \rho(\alpha).R[C, C])) \subseteq \mathcal{MC}_1(\rho_1(\alpha), \rho(\alpha).R[C, M]) \subseteq \rho(\alpha).R[M, M]$
Which is the original Claim 2a.

The same argument follows for Claim 2b.

**Case** $\mathsf{unit}$

Immediate.

**Case** $\mathsf{int}$

Immediate.

**Case** $\forall[\overline{\alpha}].(\overline{\tau}) \to \tau'$

For part 2a, unfolding the definition of relatedness at Mfunctions, we assume $W' \sqsupseteq W$, $\overline{\mathrm{VR} \in \mathrm{MMValRel}}$, and $\overline{(W', \hat{v}_1, \hat{v}_2) \in \mathcal{V}[\![\tau]\!]\rho\overline{[\alpha \mapsto \mathrm{VR}]}}$. For convenience, let $\overline{\hat{\tau}_1 = \mathrm{VR}.\tau_1}$, $\overline{\hat{\tau}_2 = \mathrm{VR}.\tau_2}$, and $\rho' = \rho\overline{[\alpha \mapsto \mathrm{VR}]}$. We need to show that

$$(W', \mathsf{v}_1\ [\overline{\hat{\tau}_1}]\ \overline{\hat{v}_1}, \mathsf{v}_2\ [\overline{\hat{\tau}_2}]\ \overline{\hat{v}_2}) \in \mathcal{E}[\![\tau']\!]\rho'.$$

By the lemma premise, there exists some $\mathrm{VR}^*$ such that

$$\mathbf{v_1} = \mathbf{pack}\ \langle \tau_{\mathbf{1env}}, \langle \mathbf{v_{1f}}, \mathbf{v_{1env}} \rangle\rangle\ \mathbf{as}\ \forall[\overline{\alpha}].(\overline{\tau}) \to \tau'^{\langle \mathcal{C} \rangle},$$
$$\mathbf{v_2} = \mathbf{pack}\ \langle \tau_{\mathbf{2env}}, \langle \mathbf{v_{2f}}, \mathbf{v_{2env}} \rangle\rangle\ \mathbf{as}\ \forall[\overline{\alpha}].(\overline{\tau}) \to \tau'^{\langle \mathcal{C} \rangle},$$

$\mathrm{VR}^*.\tau_1 = \tau_{\mathbf{1env}}$, $\mathrm{VR}^*.\tau_2 = \tau_{\mathbf{2env}}$, $(W, \mathbf{v_{1env}}, \mathbf{v_{2env}}) \in \mathrm{VR}^*.R[C, C]$, and

$$(W, \mathbf{v_{1f}}, \mathbf{v_{2f}}) \in \mathcal{V}[\![\forall[\overline{\alpha}].(\beta, \overline{\tau^{\langle \mathcal{C} \rangle}[\alpha/\lceil\alpha\rceil]}) \to \tau'^{\langle \mathcal{C} \rangle}[\alpha/\lceil\alpha\rceil]]\!]\rho[\beta \mapsto \mathrm{VR}^*]$$

By Lemma 7.1 there are some $\overline{\hat{\mathbf{v}}_1}$ and $\overline{\hat{\mathbf{v}}_2}$ such that $\overline{\mathbf{CM}^{\rho_1(\tau)}(\hat{v}_1) = \hat{\mathbf{v}}_1}$, $\overline{\mathbf{CM}^{\rho_2(\tau)}(\hat{v}_2) = \hat{\mathbf{v}}_2}$. This fact, paired with $\overline{(W', \hat{v}_1, \hat{v}_2) \in \mathcal{V}[\![\tau]\!]\rho\overline{[\alpha \mapsto \mathrm{VR}]}}$ and the induction hypothesis given to us from Claim 2b, via mutual induction, gives us

$$\overline{(W', \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2) \in \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}]\!]\rho'}$$

Which by Lemma 7.18 and weakening $\rho$ by $\beta$ (Lemma 7.14) becomes

$$\overline{(W', \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2) \in \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}[\alpha/\lceil\alpha\rceil]]\!]\rho\overline{[\alpha \mapsto \lfloor \mathrm{VR} \rfloor_{C,C}]}[\beta \mapsto \mathrm{VR}^*]}$$

We can use this to instantiate

$$(W, \mathbf{v_{1f}}, \mathbf{v_{2f}}) \in \mathcal{V}[\![\forall[\overline{\alpha}].(\beta, \overline{\tau^{\langle \mathcal{C} \rangle}[\alpha/\lceil\alpha\rceil]}) \to \tau'^{\langle \mathcal{C} \rangle}[\alpha/\lceil\alpha\rceil]]\!]\rho[\beta \mapsto \mathrm{VR}^*]$$

along with $W' \sqsupseteq W$, $\overline{\lfloor \mathrm{VR} \rfloor_{C,C} \in \mathrm{CCValRel}}$, and $(W, \mathbf{v_{1env}}, \mathbf{v_{2env}}) \in \mathcal{V}[\![\beta]\!]\rho\overline{[\alpha \mapsto \lfloor \mathrm{VR} \rfloor_{C,C}]}[\beta \mapsto \mathrm{VR}^*]$ to get

$$(W', \mathbf{v_{1f}}\ [\overline{\hat{\tau}_1{}^{\langle \mathcal{C} \rangle}}]\ \mathbf{v_{1env}}, \overline{\hat{\mathbf{v}}_1}, \mathbf{v_{2f}}\ [\overline{\hat{\tau}_2{}^{\langle \mathcal{C} \rangle}}]\ \mathbf{v_{2env}}, \overline{\hat{\mathbf{v}}_2}) \in \mathcal{E}[\![\tau'^{\langle \mathcal{C} \rangle}[\alpha/\lceil\alpha\rceil]]\!]\rho[\beta \mapsto \mathrm{VR}^*]\overline{[\alpha \mapsto \lfloor \mathrm{VR} \rfloor_{C,C}]}$$

Which by Lemma 7.18 and contracting $\rho$ by $\beta$, via Lemma 7.14 and the fact that $\beta \notin \mathrm{ftv}(\tau'^{\langle \mathcal{C} \rangle})$, becomes

$$(W', \mathbf{v_{1f}}\ [\overline{\hat{\tau}_1{}^{\langle \mathcal{C} \rangle}}]\ \mathbf{v_{1env}}, \overline{\hat{\mathbf{v}}_1}, \mathbf{v_{2f}}\ [\overline{\hat{\tau}_2{}^{\langle \mathcal{C} \rangle}}]\ \mathbf{v_{2env}}, \overline{\hat{\mathbf{v}}_2}) \in \mathcal{E}[\![\tau'^{\langle \mathcal{C} \rangle}]\!]\rho'$$

Since we are at a smaller type let's use the fact above to instatiate the (mutual) induction hypothesis provided by Claim 1a. The result is

$$(W', {}^{\rho'_1(\tau')}\mathcal{MC}\ \mathbf{v_{1f}}\ [\overline{\hat{\tau}_1{}^{\langle \mathcal{C} \rangle}}]\ \mathbf{v_{1env}}, \overline{\hat{\mathbf{v}}_1}, {}^{\rho'_2(\tau')}\mathcal{MC}\ \mathbf{v_{2f}}\ [\overline{\hat{\tau}_2{}^{\langle \mathcal{C} \rangle}}]\ \mathbf{v_{2env}}, \overline{\hat{\mathbf{v}}_2}) \in \mathcal{E}[\![\tau']\!]\rho'.$$

Remember we need to show that

$$(W', \mathsf{v}_1\ [\overline{\hat{\tau}_1}]\ \overline{\hat{v}_1}, \mathsf{v}_2\ [\overline{\hat{\tau}_2}]\ \overline{\hat{v}_2}) \in \mathcal{E}[\![\tau']\!]\rho'$$

Unfolding this definition, we assume $(W', K_1, K_2) \in \mathcal{K}[\![\tau', \tau']\!]\rho'$ and must show

$$(W', K_1[\mathsf{v}_1\ [\overline{\hat{\tau}_1}]\ \overline{\hat{v}_1}], K_2[\mathsf{v}_2\ [\overline{\hat{\tau}_2}]\ \overline{\hat{v}_2}]) \in \mathcal{O}$$

Unfolding this we assume $(H_1, H_2) : W'$ and must show that the expressions either terminate or are still running after $W'.k$ steps.

If we can show, for $i = 1, 2$, that

$$\langle H_i \mid \mathsf{v_i}\,[\overline{\hat{\tau}_i}]\,\overline{\hat{\mathsf{v}}_i}\rangle \longmapsto^* \langle H_i \mid {}^{\rho_i'(\tau')}\mathcal{MC}\,\mathbf{v_{if}}\,[\overline{\hat{\tau}_i{}^{\langle\mathcal{C}\rangle}}]\,\mathbf{v_{ienv}},\overline{\hat{\mathbf{v}}_i}\rangle$$

then instantiating Lemma 7.7 with this fact along with

$$(W', {}^{\rho_1'(\tau')}\mathcal{MC}\,\mathbf{v_{1f}}\,[\overline{\hat{\tau}_1{}^{\langle\mathcal{C}\rangle}}]\,\mathbf{v_{1env}},\overline{\hat{\mathbf{v}}_1}, {}^{\rho_2'(\tau')}\mathcal{MC}\,\mathbf{v_{2f}}\,[\overline{\hat{\tau}_2{}^{\langle\mathcal{C}\rangle}}]\,\mathbf{v_{2env}},\overline{\hat{\mathbf{v}}_2}) \in \mathcal{E}[\![\tau']\!]\rho'$$

gives us

$$(W', \mathsf{v_1}\,[\overline{\hat{\tau}_1}]\,\overline{\hat{\mathsf{v}}_1}, \mathsf{v_2}\,[\overline{\hat{\tau}_2}]\,\overline{\hat{\mathsf{v}}_2}) \in \mathcal{E}[\![\tau']\!]\rho'$$

Which we can in turn instantiate with the proper $K_1, K_2$ and then $H_1, H_2$ to get our desired termination or running properties.
So now we conclude by showing, for $i = 1, 2$ that

$$\langle H_i \mid \mathsf{v_i}\,[\overline{\hat{\tau}_i}]\,\overline{\hat{\mathsf{v}}_i}\rangle \longmapsto^* \langle H_i \mid {}^{\rho_i'(\tau')}\mathcal{MC}\,\mathbf{v_{if}}\,[\overline{\hat{\tau}_i{}^{\langle\mathcal{C}\rangle}}]\,\mathbf{v_{ienv}},\overline{\hat{\mathbf{v}}_i}\rangle$$

Note by the translation definitions that

$$\mathsf{v_i} = \lambda[\overline{\alpha}](\overline{\mathsf{x}\!:\!\tau}).{}^{\rho_i'(\tau')}\mathcal{MC}\,(\mathbf{unpack}\,\langle\boldsymbol{\beta},\mathbf{z}\rangle = \mathbf{v_i}\ \mathbf{in\ let}\,(\mathbf{f},\mathbf{y}) = (\boldsymbol{\pi_1}(\mathbf{z}),\boldsymbol{\pi_2}(\mathbf{z}))\ \mathbf{in\ f}\,[\overline{\lceil\alpha\rceil}]\,(\mathbf{y},\overline{\mathcal{CM}^{\rho_i'(\tau)}\,\mathsf{x}})).$$

Thus we have

$$\begin{array}{ll}
 & \langle H_i \mid \mathsf{v_i}\,[\overline{\hat{\tau}_i}]\,\overline{\hat{\mathsf{v}}_i}\rangle \\
\longmapsto & \langle H_i \mid {}^{\rho_i'(\tau')}\mathcal{MC}\,(\mathbf{unpack}\,\langle\boldsymbol{\beta},\mathbf{z}\rangle = \mathbf{v_i}\ \mathbf{in\ let}\,(\mathbf{f},\mathbf{y}) = (\boldsymbol{\pi_1}(\mathbf{z}),\boldsymbol{\pi_2}(\mathbf{z}))\ \mathbf{in\ f}\,[\hat{\tau}_i{}^{\langle\mathcal{C}\rangle}]\,(\mathbf{y},\overline{\mathcal{CM}^{\rho_i'(\tau)}\,\hat{\mathsf{v}}_i}))\rangle \\
\longmapsto^* & \langle H_i \mid {}^{\rho_i'(\tau')}\mathcal{MC}\,(\mathbf{v_{if}}\,[\overline{\hat{\tau}_i{}^{\langle\mathcal{C}\rangle}}]\,\mathbf{v_{ienv}},\overline{\mathcal{CM}^{\rho_i'(\tau)}\,\hat{\mathsf{v}}_i})\rangle \\
\longmapsto^* & \langle H_i \mid {}^{\rho_i'(\tau')}\mathcal{MC}\,(\mathbf{v_{if}}\,[\overline{\hat{\tau}_i{}^{\langle\mathcal{C}\rangle}}]\,\mathbf{v_{ienv}},\overline{\hat{\mathbf{v}}_i})\rangle,
\end{array}$$

as desired.
For part 2b, recall that

$$\forall[\overline{\alpha}].(\overline{\tau}) \to \tau'^{\langle\mathcal{C}\rangle} = \exists\boldsymbol{\beta}.\langle\boldsymbol{\tau_f},\boldsymbol{\beta}\rangle, \quad \text{where} \quad \boldsymbol{\tau_f} = \forall[\overline{\alpha}].(\boldsymbol{\beta},\overline{\tau^{\langle\mathcal{C}\rangle}[\overline{\boldsymbol{\alpha}/\lceil\alpha\rceil}]}) \to \tau'^{\langle\mathcal{C}\rangle}[\overline{\boldsymbol{\alpha}/\lceil\alpha\rceil}].$$

Throughout the rest of this case, whenever we apply Lemma 7.21 we make use of the fact that $\boldsymbol{\beta}$ does not appear in $\tau^{\langle\mathcal{C}\rangle}$ or $\tau'^{\langle\mathcal{C}\rangle}$. By inspection of the translations,

$$\mathbf{v_1} = \mathbf{pack}\,\langle\mathbf{unit},\langle\mathbf{v_1'},()\rangle\rangle\ \mathbf{as}\ \exists\boldsymbol{\beta}.\langle\rho_1(\boldsymbol{\tau_f}),\boldsymbol{\beta}\rangle \ \text{ and } \ \mathbf{v_2} = \mathbf{pack}\,\langle\mathbf{unit},\langle\mathbf{v_2'},()\rangle\rangle\ \mathbf{as}\ \exists\boldsymbol{\beta}.\langle\rho_2(\boldsymbol{\tau_f}),\boldsymbol{\beta}\rangle,$$

where

$$\mathbf{v_1'} = \boldsymbol{\lambda}[\overline{\boldsymbol{\alpha}}](\mathbf{z}:\mathbf{unit},\overline{\mathbf{x}:\rho_1(\tau)^{\langle\mathcal{C}\rangle}[\overline{\boldsymbol{\alpha}/\lceil\alpha\rceil}]}).\mathcal{CM}^{\rho_1(\tau')[\overline{\mathsf{L}\langle\boldsymbol{\alpha}\rangle/\alpha}]}\,\mathsf{v_1}\,[\overline{\mathsf{L}\langle\boldsymbol{\alpha}\rangle}]\,\overline{{}^{\rho_1(\tau)[\overline{\mathsf{L}\langle\boldsymbol{\alpha}\rangle/\alpha}]}\mathcal{MC}\,\mathbf{x}}$$

and

$$\mathbf{v_2'} = \boldsymbol{\lambda}[\overline{\boldsymbol{\alpha}}](\mathbf{z}:\mathbf{unit},\overline{\mathbf{x}:\rho_2(\tau)^{\langle\mathcal{C}\rangle}[\overline{\boldsymbol{\alpha}/\lceil\alpha\rceil}]}).\mathcal{CM}^{\rho_2(\tau')[\overline{\mathsf{L}\langle\boldsymbol{\alpha}\rangle/\alpha}]}\,\mathsf{v_2}\,[\overline{\mathsf{L}\langle\boldsymbol{\alpha}\rangle}]\,\overline{{}^{\rho_2(\tau)[\overline{\mathsf{L}\langle\boldsymbol{\alpha}\rangle/\alpha}]}\mathcal{MC}\,\mathbf{x}}.$$

Let

$$\text{VR} = (\mathbf{unit},\mathbf{unit},[\{(W,(),()) \mid W \in \text{World}\}].$$

Clearly, VR $\in$ CCValRel. It suffices to prove that

$$(W,\mathbf{v_1'},\mathbf{v_2'}) \in \mathcal{V}[\![\boldsymbol{\tau_f}]\!]\rho[\boldsymbol{\beta} \mapsto \text{VR}].$$

To do this, assume $W' \sqsupseteq W$, $\overline{\text{VR}' \in \text{CCValRel}}$, $\rho' = \rho[\boldsymbol{\beta} \mapsto \text{VR}][\overline{\boldsymbol{\alpha} \mapsto \text{VR}'}]$, $(W',(),()) \in \mathcal{V}[\![\boldsymbol{\beta}]\!]\rho'$, and

$$\overline{(W',\hat{\mathbf{v}}_1,\hat{\mathbf{v}}_2) \in \mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}[\overline{\boldsymbol{\alpha}/\lceil\alpha\rceil}]]\!]\rho'}$$

For convenience, also let $\overline{\hat{\tau}_1 = \mathrm{VR}'.\tau_1}$ and $\overline{\hat{\tau}_2 = \mathrm{VR}'.\tau_2}$. We need to show that

$$(W', \mathsf{v}_1' \; \overline{[\hat{\tau}_1]} \; (), \overline{\hat{\mathsf{v}}_1}, \mathsf{v}_2' \; \overline{[\hat{\tau}_2]} \; (), \overline{\hat{\mathsf{v}}_2}) \in \mathcal{E}[\![\tau'^{\langle \mathcal{C} \rangle} \overline{[\alpha / \lceil \alpha \rceil]}]\!]\rho'$$

By Lemma 7.21 we have that $\mathcal{E}[\![\tau'^{\langle \mathcal{C} \rangle} \overline{[\alpha / \lceil \alpha \rceil]}]\!]\rho' = \mathcal{E}[\![\tau'^{\langle \mathcal{C} \rangle}]\!]\rho[\beta \mapsto \mathrm{VR}]\overline{[\alpha \mapsto \mathsf{L}\langle \mathrm{VR}' \rangle]}$.
Taking the assumed

$$\overline{(W', \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2) \in \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle} \overline{[\alpha / \lceil \alpha \rceil]}]\!]\rho'}$$

By Lemma 7.21 we can lump $\mathrm{VR}'$ and by Lemma 7.14 we can contract away the $\beta$

$$\mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle} \overline{[\alpha / \lceil \alpha \rceil]}]\!]\rho' = \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}]\!]\rho[\beta \mapsto \mathrm{VR}]\overline{[\alpha \mapsto \mathsf{L}\langle \mathrm{VR}' \rangle]} = \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}]\!]\rho\overline{[\alpha \mapsto \mathsf{L}\langle \mathrm{VR}' \rangle]}$$

Let $\hat{\rho} = \rho\overline{[\alpha \mapsto \mathsf{L}\langle \mathrm{VR}' \rangle]}$. By Lemma 7.1, there are some $\overline{\hat{\mathsf{v}}_1}$ and $\overline{\hat{\mathsf{v}}_2}$ such that

$$\overline{{}^{\hat{\rho}_1(\tau)}\mathbf{MC}(\hat{\mathbf{v}}_1) = \hat{\mathsf{v}}_1} \quad \text{and} \quad \overline{{}^{\hat{\rho}_2(\tau)}\mathbf{MC}(\hat{\mathbf{v}}_2) = \hat{\mathsf{v}}_2}.$$

The induction hypothesis given to us from Claim 2a, via mutual induction, provides us with

$$\overline{(W', \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2) \in \mathcal{V}[\![\tau]\!]\hat{\rho}}$$

Which we can use, along with $W' \sqsupseteq W$ and $\overline{\mathsf{L}\langle \mathrm{VR}' \rangle \in \mathrm{MMValRel}}$ (holding by Lemma 7.20), to instantiate the premise $(W, \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\forall[\overline{\alpha}].(\overline{\tau}) \to \tau']\!]\rho$. This gives us

$$(W', \mathsf{v}_1 \; \overline{[\mathsf{L}\langle \hat{\tau}_1 \rangle]} \; \overline{\hat{\mathsf{v}}_1}, \mathsf{v}_2 \; \overline{[\mathsf{L}\langle \hat{\tau}_2 \rangle]} \; \overline{\hat{\mathsf{v}}_2}) \in \mathcal{E}[\![\tau']\!]\hat{\rho}$$

From the above fact and the induction hypothesis from Claim 1b,

$$(W', \mathcal{CM}^{\hat{\rho}_1(\tau')} \; \mathsf{v}_1 \; \overline{[\mathsf{L}\langle \hat{\tau}_1 \rangle]} \; \overline{\hat{\mathsf{v}}_1}, \mathcal{CM}^{\hat{\rho}_2(\tau')} \; \mathsf{v}_2 \; \overline{[\mathsf{L}\langle \hat{\tau}_2 \rangle]} \; \overline{\hat{\mathsf{v}}_2}) \in \mathcal{E}[\![\tau'^{\langle \mathcal{C} \rangle}]\!]\hat{\rho}$$

Remember we need to show that

$$(W', \mathsf{v}_1' \; \overline{[\hat{\tau}_1]} \; \overline{\hat{\mathsf{v}}_1}, \mathsf{v}_2' \; \overline{[\hat{\tau}_2]} \; \overline{\hat{\mathsf{v}}_2}) \in \mathcal{E}[\![\tau'^{\langle \mathcal{C} \rangle} \overline{[\alpha / \lceil \alpha \rceil]}]\!]\rho[\beta \mapsto \mathrm{VR}]\overline{[\alpha \mapsto \mathrm{VR}']}$$

or by Lemmas 7.21 and 7.14

$$(W', \mathsf{v}_1' \; \overline{[\hat{\tau}_1]} \; \overline{\hat{\mathsf{v}}_1}, \mathsf{v}_2' \; \overline{[\hat{\tau}_2]} \; \overline{\hat{\mathsf{v}}_2}) \in \mathcal{E}[\![\tau'^{\langle \mathcal{C} \rangle}]\!]\rho\overline{[\alpha \mapsto \mathsf{L}\langle \mathrm{VR}' \rangle]}$$

Unfolding this definition, we assume $(W', K_1, K_2) \in \mathcal{K}[\![\tau'^{\langle \mathcal{C} \rangle}, \tau'^{\langle \mathcal{C} \rangle}]\!]\hat{\rho}$ and must show

$$(W', K_1[\mathsf{v}_1' \; \overline{[\hat{\tau}_1]} \; \overline{\hat{\mathsf{v}}_1}], K_2[\mathsf{v}_2' \; \overline{[\hat{\tau}_2]} \; \overline{\hat{\mathsf{v}}_2}]) \in \mathcal{O}$$

Unfolding this we assume $(H_1, H_2) : W'$ and must show that the expressions either terminate or are still running after $W'.k$ steps.
If we can show, for $i = 1, 2$, that

$$\langle H_i \mid \mathsf{v}_i' \; \overline{[\hat{\tau}_i]} \; \overline{\hat{\mathsf{v}}_i} \rangle \longmapsto^* \langle H_i \mid \mathcal{CM}^{\hat{\rho}_1(\tau')} \; \mathsf{v}_i \; \overline{[\mathsf{L}\langle \hat{\tau}_i \rangle]} \; \overline{\hat{\mathsf{v}}_i} \rangle$$

then instantiating Lemma 7.10 with this fact along with

$$(W', \mathcal{CM}^{\hat{\rho}_1(\tau')} \; \mathsf{v}_1 \; \overline{[\mathsf{L}\langle \hat{\tau}_1 \rangle]} \; \overline{\hat{\mathsf{v}}_1}, \mathcal{CM}^{\hat{\rho}_2(\tau')} \; \mathsf{v}_2 \; \overline{[\mathsf{L}\langle \hat{\tau}_2 \rangle]} \; \overline{\hat{\mathsf{v}}_2}) \in \mathcal{E}[\![\tau'^{\langle \mathcal{C} \rangle}]\!]\hat{\rho}$$

gives us

$$(W', \mathsf{v}_1' \; \overline{[\hat{\tau}_1]} \; \overline{\hat{\mathsf{v}}_1}, \mathsf{v}_2' \; \overline{[\hat{\tau}_2]} \; \overline{\hat{\mathsf{v}}_2}) \in \mathcal{E}[\![\tau'^{\langle \mathcal{C} \rangle}]\!]\hat{\rho}$$

Which we can in turn instantiate with the proper $K_1, K_2$ and then $H_1, H_2$ to get our desired termination or running properties.
So now we conclude by showing, for $i = 1, 2$ that

$$\langle H_i \mid \mathsf{v}_i' \; \overline{[\hat{\tau}_i]} \; \overline{\hat{\mathsf{v}}_i} \rangle \longmapsto^* \langle H_i \mid \mathcal{CM}^{\hat{\rho}_1(\tau')} \; \mathsf{v}_i \; \overline{[\mathsf{L}\langle \hat{\tau}_i \rangle]} \; \overline{\hat{\mathsf{v}}_i} \rangle$$

By inspection of the operational semantics, we have

$$\begin{aligned}
& \langle H_i \mid \mathsf{v}'_\mathsf{i}\ [\overline{\hat{\tau}_\mathsf{i}}]\ \overline{\hat{\mathsf{v}}_\mathsf{i}} \rangle \\
\longmapsto\ & \langle H_i \mid \mathcal{CM}^{\hat{\rho}_i(\tau')}\ \mathsf{v}_\mathsf{i}\ [\overline{\mathsf{L}\langle \hat{\tau}_\mathsf{i} \rangle}]\ \overline{{}^{\hat{\rho}_i(\tau)}\mathcal{MC}\ \hat{\mathsf{v}}_\mathsf{i}} \rangle \\
\longmapsto^*\ & \langle H_i \mid \mathcal{CM}^{\hat{\rho}_i(\tau')}\ \mathsf{v}_\mathsf{i}\ [\overline{\mathsf{L}\langle \hat{\tau}_\mathsf{i} \rangle}]\ \overline{\hat{\mathsf{v}}_\mathsf{i}} \rangle,
\end{aligned}$$

as desired.

**Case** $\mathsf{ref}\ \tau$ For part 2a, we have four cases:

- $\mathsf{v_1} = \boldsymbol{\ell_1}$ and $\mathsf{v_2} = \boldsymbol{\ell_2}$:

$$^{\rho_1(\mathsf{ref}\ \tau)}\mathbf{MC}(\boldsymbol{\ell_1}) = {}^{\rho_1(\mathsf{ref}\ \tau)}\mathcal{MC}\ \boldsymbol{\ell_1} \quad \text{and} \quad ^{\rho_2(\mathsf{ref}\ \tau)}\mathbf{MC}(\boldsymbol{\ell_2}) = {}^{\rho_2(\mathsf{ref}\ \tau)}\mathcal{MC}\ \boldsymbol{\ell_2}$$

  We're given $(W, \boldsymbol{\ell_1}, \boldsymbol{\ell_2}) \in \mathcal{V}[\![\mathsf{ref}\ \tau^{\langle \mathcal{C} \rangle}]\!]\rho$ and need to show:

$$(W, {}^{\rho_1(\mathsf{ref}\ \tau)}\mathcal{MC}\ \boldsymbol{\ell_1}, {}^{\rho_2(\mathsf{ref}\ \tau)}\mathcal{MC}\ \boldsymbol{\ell_2}) \in \mathcal{V}[\![\mathsf{ref}\ \tau]\!]\rho$$

  Take $i$ from the given and assume $W' \sqsupseteq W$. Now show:

$$(\mathrm{loc}({}^{\rho_1(\mathsf{ref}\ \tau)}\mathcal{MC}\ \boldsymbol{\ell_1}), \mathrm{loc}({}^{\rho_2(\mathsf{ref}\ \tau)}\mathcal{MC}\ \boldsymbol{\ell_2})) \in W'(i).\mathrm{bij}(W'(i).s)$$

  Which follows by instantiating the given $(W, \boldsymbol{\ell_1}, \boldsymbol{\ell_2}) \in \mathcal{V}[\![\mathsf{ref}\ \tau^{\langle \mathcal{C} \rangle}]\!]\rho$ with $W'$ and the definition of loc. Note this instantiation also gives us $\varphi_H$ and

$$W'(i).\mathrm{HR}(W'(i).s) = \varphi_H \otimes \varphi_{\boldsymbol{\ell}}$$

  Where $\varphi_{\boldsymbol{\ell}} = \{ (\widetilde{W}, H_1, H_2) \in \mathrm{HeapAtom} \mid \forall v'_1, v'_2.$
  $\mathrm{lookup}^{\tau^{\langle \mathcal{C} \rangle}}(\boldsymbol{\ell_1}, H_1) \hookrightarrow (v'_1, \tau_1)\ \wedge \mathrm{lookup}^{\tau^{\langle \mathcal{C} \rangle}}(\boldsymbol{\ell_2}, H_2) \hookrightarrow (v'_2, \tau_2) \implies$
  $(\widetilde{W}, v'_1, v'_2) \in \mathcal{V}[\![\tau_1, \tau_2]\!]\rho \}$

  Taking $\varphi_H$ from the given we now must show

$$W'(i).\mathrm{HR}(W'(i).s) = \varphi_H \otimes \hat{\varphi}_{\ell}$$

  Where $\hat{\varphi}_{\ell} = \{ (\widetilde{W}, H_1, H_2) \in \mathrm{HeapAtom} \mid \forall v'_1, v'_2.$
  $\mathrm{lookup}^{\tau}({}^{\rho_1(\mathsf{ref}\ \tau)}\mathcal{MC}\ \boldsymbol{\ell_1}, H_1) \hookrightarrow (v'_1, \tau_1)\ \wedge \mathrm{lookup}^{\tau}({}^{\rho_2(\mathsf{ref}\ \tau)}\mathcal{MC}\ \boldsymbol{\ell_2}, H_2) \hookrightarrow (v'_2, \tau_2) \implies$
  $(\widetilde{W}, v'_1, v'_2) \in \mathcal{V}[\![\tau_1, \tau_2]\!]\rho \}$

  By the definition of lookup it is easy to show that $\varphi_{\boldsymbol{\ell}} = \hat{\varphi}_{\ell}$.

- $\mathsf{v_1} = \mathcal{CM}^{\rho_1(\mathsf{ref}\ \tau)}\ \ell_1$ and $\mathsf{v_2} = \mathcal{CM}^{\rho_2(\mathsf{ref}\ \tau)}\ \ell_2$:

$$^{\rho_1(\mathsf{ref}\ \tau)}\mathbf{MC}(\mathcal{CM}^{\rho_1(\mathsf{ref}\ \tau)}\ \ell_1) = \ell_1 \quad \text{and} \quad ^{\rho_2(\mathsf{ref}\ \tau)}\mathbf{MC}(\mathcal{CM}^{\rho_2(\mathsf{ref}\ \tau)}\ \ell_2) = \ell_2$$

  We're given $(W, \mathcal{CM}^{\rho_1(\mathsf{ref}\ \tau)}\ \ell_1, \mathcal{CM}^{\rho_2(\mathsf{ref}\ \tau)}\ \ell_2) \in \mathcal{V}[\![\mathsf{ref}\ \tau^{\langle \mathcal{C} \rangle}]\!]\rho$ and need to show:

$$(W, \ell_1, \ell_2) \in \mathcal{V}[\![\mathsf{ref}\ \tau]\!]\rho$$

  Follows by a similar argument to the case above.

- $\mathsf{v_1} = \boldsymbol{\ell_1}$ and $\mathsf{v_2} = \mathcal{CM}^{\rho_2(\mathsf{ref}\ \tau)}\ \ell_2$:

$$^{\rho_1(\mathsf{ref}\ \tau)}\mathbf{MC}(\boldsymbol{\ell_1}) = {}^{\rho_1(\mathsf{ref}\ \tau)}\mathcal{MC}\ \boldsymbol{\ell_1} \quad \text{and} \quad ^{\rho_2(\mathsf{ref}\ \tau)}\mathbf{MC}(\mathcal{CM}^{\rho_2(\mathsf{ref}\ \tau)}\ \ell_2) = \ell_2$$

  We're given $(W, \boldsymbol{\ell_1}, \mathcal{CM}^{\rho_2(\mathsf{ref}\ \tau)}\ \ell_2) \in \mathcal{V}[\![\mathsf{ref}\ \tau^{\langle \mathcal{C} \rangle}]\!]\rho$ and need to show:

$$(W, {}^{\rho_1(\mathsf{ref}\ \tau)}\mathcal{MC}\ \boldsymbol{\ell_1}, \ell_2) \in \mathcal{V}[\![\mathsf{ref}\ \tau]\!]\rho$$

  Take $i$ from the given and assume $W' \sqsupseteq W$. Now show:

$$(\mathrm{loc}({}^{\rho_1(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell_1}), \mathrm{loc}(\ell_2)) \in W'(i).\mathrm{bij}(W'(i).s)$$

Which follows by instantiating the given $(W, \boldsymbol{\ell_1}, \mathcal{CM}^{\rho_2(\mathsf{ref}\,\tau)}\,\ell_2) \in \mathcal{V}[\![\mathsf{ref}\,\tau^{\langle\mathcal{C}\rangle}]\!]\rho$ with $W'$ and the definition of loc. Note this instantiation also gives us $\varphi_H$ and

$$W'(i).\mathrm{HR}(W'(i).s) = \varphi_H \otimes \varphi_{\boldsymbol{\ell}}$$

Where $\varphi_{\boldsymbol{\ell}} = \{\, (\widetilde{W}, H_1, H_2) \in \mathrm{HeapAtom} \mid \forall v_1', v_2'.$
$\mathrm{lookup}^{\tau^{\langle\mathcal{C}\rangle}}(\boldsymbol{\ell_1}, H_1) \hookrightarrow (v_1', \tau_1) \;\wedge \mathrm{lookup}^{\tau^{\langle\mathcal{C}\rangle}}(\mathcal{CM}^{\rho_2(\mathsf{ref}\,\tau)}\,\ell_2, H_2) \hookrightarrow (v_2', \tau_2) \implies$
$(\widetilde{W}, v_1', v_2') \in \mathcal{V}[\![\tau_1, \tau_2]\!]\rho \,\}$

Taking $\varphi_H$ from the given we now must show

$$W'(i).\mathrm{HR}(W'(i).s) = \varphi_H \otimes \hat{\varphi}_{\ell}$$

Where $\hat{\varphi}_{\ell} = \{\, (\widetilde{W}, H_1, H_2) \in \mathrm{HeapAtom} \mid \forall v_1', v_2'.$
$\mathrm{lookup}^{\tau}({}^{\rho_1(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell_1}, H_1) \hookrightarrow (v_1', \tau_1) \;\wedge \mathrm{lookup}^{\tau}(\ell_2, H_2) \hookrightarrow (v_2', \tau_2) \implies$
$(\widetilde{W}, v_1', v_2') \in \mathcal{V}[\![\tau_1, \tau_2]\!]\rho \,\}$

By the definition of lookup it is easy to show that $\varphi_{\boldsymbol{\ell}} = \hat{\varphi}_{\ell}$.

- $\mathbf{v_1} = \mathcal{CM}^{\rho_1(\mathsf{ref}\,\tau)}\,\ell_1$ and $\mathbf{v_2} = \boldsymbol{\ell_2}$:

$${}^{\rho_1(\mathsf{ref}\,\tau)}\mathbf{MC}(\mathcal{CM}^{\rho_1(\mathsf{ref}\,\tau)}\,\ell_1) = \ell_1 \quad \text{and} \quad {}^{\rho_2(\mathsf{ref}\,\tau)}\mathbf{MC}(\boldsymbol{\ell_2}) = {}^{\rho_2(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell_2}$$

We're given $(W, \mathcal{CM}^{\rho_1(\mathsf{ref}\,\tau)}\,\ell_1, \boldsymbol{\ell_2}) \in \mathcal{V}[\![\mathsf{ref}\,\tau^{\langle\mathcal{C}\rangle}]\!]\rho$ and need to show:

$$(W, \ell_1, {}^{\rho_2(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell_2}) \in \mathcal{V}[\![\mathsf{ref}\,\tau]\!]\rho$$

Follows by a similar argument to the case above.

Part 2b follows in a similar manner.

**Case** $\exists\alpha.\tau$

For part 2a, we have $\mathbf{v_1} = \mathbf{pack}\,\langle\hat{\boldsymbol{\tau}}_1, \hat{\mathbf{v}}_1\rangle\,\mathbf{as}\,\rho_1(\exists\alpha.\tau^{\langle\mathcal{C}\rangle})$, $\mathbf{v_2} = \mathbf{pack}\,\langle\hat{\boldsymbol{\tau}}_2, \hat{\mathbf{v}}_2\rangle\,\mathbf{as}\,\rho_2(\exists\alpha.\tau^{\langle\mathcal{C}\rangle})$, and that there is some $\mathrm{VR} \in \mathrm{CCValRel}$ such that $\mathrm{VR}.\boldsymbol{\tau_1} = \hat{\boldsymbol{\tau}}_1$, $\mathrm{VR}.\boldsymbol{\tau_2} = \hat{\boldsymbol{\tau}}_2$, and

$$(W, \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2) \in \mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}[\boldsymbol{\alpha}/\lceil\alpha\rceil]]\!]\rho[\boldsymbol{\alpha} \mapsto \mathrm{VR}]$$

By Lemma 7.21

$$(W, \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2) \in \mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}]\!]\rho[\alpha \mapsto \mathsf{L}\langle\mathrm{VR}\rangle]$$

By inspection of the translations ${}^{\rho_1(\exists\alpha.\tau)}\mathbf{MC}(\mathbf{v_1}) = \mathsf{v}_1$ and ${}^{\rho_2(\exists\alpha.\tau)}\mathbf{MC}(\mathbf{v_2}) = \mathsf{v}_2$ we have that

$$\mathsf{v}_1 = \mathsf{pack}\,\langle\mathsf{L}\langle\hat{\boldsymbol{\tau}}_1\rangle, \hat{\mathsf{v}}_1\rangle\,\mathsf{as}\,\rho_1(\exists\alpha.\tau) \quad \text{and} \quad \mathsf{v}_2 = \mathsf{pack}\,\langle\mathsf{L}\langle\hat{\boldsymbol{\tau}}_2\rangle, \hat{\mathsf{v}}_2\rangle\,\mathsf{as}\,\rho_2(\exists\alpha.\tau)$$

where

$${}^{\rho_1(\tau)[\mathsf{L}\langle\hat{\boldsymbol{\tau}}_1\rangle/\alpha]}\mathbf{MC}(\hat{\mathbf{v}}_1) = \hat{\mathsf{v}}_1 \quad \text{and} \quad {}^{\rho_2(\tau)[\mathsf{L}\langle\hat{\boldsymbol{\tau}}_2\rangle/\alpha]}\mathbf{MC}(\hat{\mathbf{v}}_2) = \hat{\mathsf{v}}_2$$

By the induction hypothesis,

$$(W, \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2) \in \mathcal{V}[\![\tau]\!]\rho[\alpha \mapsto \mathsf{L}\langle\mathrm{VR}\rangle].$$

By Lemma 7.20 we can use $\mathsf{L}\langle\mathrm{VR}\rangle$ to instantiate the definition of $\mathcal{V}[\![\exists\alpha.\tau]\!]\rho$ and reach the result.
Part 2b is similar: we have $\mathsf{v}_1 = \mathsf{pack}\,\langle\hat{\tau}_1, \hat{\mathsf{v}}_1\rangle\,\mathsf{as}\,\rho_1(\exists\alpha.\tau)$, $\mathsf{v}_2 = \mathsf{pack}\,\langle\hat{\tau}_2, \hat{\mathsf{v}}_2\rangle\,\mathsf{as}\,\rho_2(\exists\alpha.\tau)$, and that there is some $\mathrm{VR} \in \mathrm{MMValRel}$ such that $\mathrm{VR}.\tau_1 = \hat{\tau}_1$, $\mathrm{VR}.\tau_2 = \hat{\tau}_2$, and

$$(W, \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2) \in \mathcal{V}[\![\tau]\!]\rho[\alpha \mapsto \mathrm{VR}].$$

By inspection of the translations,

$$\mathbf{v_1} = \mathbf{pack}\ \langle \hat{\tau}_1{}^{\langle \mathcal{C} \rangle}, \hat{\mathbf{v}}_1 \rangle\ \mathbf{as}\ \rho_1(\exists\alpha.\tau^{\langle \mathcal{C} \rangle}) \quad \text{and} \quad \mathbf{v_2} = \mathbf{pack}\ \langle \hat{\tau}_2{}^{\langle \mathcal{C} \rangle}, \hat{\mathbf{v}}_2 \rangle\ \mathbf{as}\ \rho_2(\exists\alpha.\tau^{\langle \mathcal{C} \rangle}),$$

where

$$\mathbf{CM}^{\tau[\hat{\tau}_1/\alpha]}(\hat{\mathsf{v}}_1) = \hat{\mathbf{v}}_1 \quad \text{and} \quad \mathbf{CM}^{\tau[\hat{\tau}_2/\alpha]}(\hat{\mathsf{v}}_2) = \hat{\mathbf{v}}_2.$$

By the induction hypothesis and Lemma 7.18,

$$(W, \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2) \in \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}]\!]\rho[\alpha \mapsto \mathrm{VR}] = \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}[\boldsymbol{\alpha}/\lceil\alpha\rceil]]\!]\rho[\boldsymbol{\alpha} \mapsto \lfloor \mathrm{VR} \rfloor_{C,C}].$$

By Lemma 7.17 we instantiate $\mathcal{V}[\![\exists\alpha.\tau^{\langle \mathcal{C} \rangle}]\!]\rho$ with $\lfloor \mathrm{VR} \rfloor_{C,C}$ to complete the proof.

**Case** $\mu\alpha.\tau$

For part 2a, we have $\mathbf{v_1} = \mathbf{fold}_{\rho_1(\mu\alpha.\tau^{\langle \mathcal{C} \rangle})}\ \hat{\mathbf{v}}_1$, $\mathbf{v_2} = \mathbf{fold}_{\rho_2(\mu\alpha.\tau^{\langle \mathcal{C} \rangle})}\ \hat{\mathbf{v}}_2$, and

$$(W, \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2) \in \triangleright\mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}[\boldsymbol{\alpha}/\lceil\alpha\rceil][(\mu\alpha.\tau)^{\langle \mathcal{C} \rangle}/\boldsymbol{\alpha}]]\!]\rho = \triangleright\mathcal{V}[\![\tau[\mu\alpha.\tau/\alpha]^{\langle \mathcal{C} \rangle}]\!]\rho$$

By inspection of the translations,

$$\mathsf{v}_1 = \mathsf{fold}_{\rho_1(\mu\alpha.\tau)}\ \hat{\mathsf{v}}_1 \quad \text{and} \quad \mathsf{v}_2 = \mathsf{fold}_{\rho_2(\mu\alpha.\tau)}\ \hat{\mathsf{v}}_2,$$

where

$$^{\rho_1(\tau[\mu\alpha.\tau/\alpha])}\mathbf{MC}(\hat{\mathbf{v}}_1) = \hat{\mathsf{v}}_1 \quad \text{and} \quad ^{\rho_2(\tau[\mu\alpha.\tau/\alpha])}\mathbf{MC}(\hat{\mathbf{v}}_2) = \hat{\mathsf{v}}_2.$$

By the induction hypothesis,

$$(W, \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2) \in \triangleright\mathcal{V}[\![\tau[\mu\alpha.\tau/\alpha]]\!]\rho,$$

which is sufficient to prove $(W, \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\mu\alpha.\tau]\!]\rho$.

Part 2b is similar: we have $\mathsf{v}_1 = \mathsf{fold}_{\rho_1(\mu\alpha.\tau)}\ \hat{\mathsf{v}}_1$, $\mathsf{v}_2' = \mathsf{fold}_{\rho_2(\mu\alpha.\tau)}\ \hat{\mathsf{v}}_2$, and

$$(W, \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2) \in \triangleright\mathcal{V}[\![\tau[\mu\alpha.\tau/\alpha]]\!]\rho.$$

By inspection of the translations,

$$\mathbf{v_1} = \mathbf{fold}_{\rho_1(\mu\alpha.\tau^{\langle \mathcal{C} \rangle})}\ \hat{\mathbf{v}}_1 \quad \text{and} \quad \mathbf{v_2} = \mathbf{fold}_{\rho_2(\mu\alpha.\tau^{\langle \mathcal{C} \rangle})}\ \hat{\mathbf{v}}_2,$$

where

$$\mathbf{CM}^{\rho_1(\tau[\mu\alpha.\tau/\alpha])}(\hat{\mathsf{v}}_1) = \hat{\mathbf{v}}_1 \quad \text{and} \quad \mathbf{CM}^{\rho_2(\tau[\mu\alpha.\tau/\alpha])}(\hat{\mathsf{v}}_2) = \hat{\mathbf{v}}_2.$$

By the induction hypothesis,

$$(W, \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2) \in \triangleright\mathcal{V}[\![\tau[\mu\alpha.\tau/\alpha]^{\langle \mathcal{C} \rangle}]\!]\rho = \triangleright\mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}[\boldsymbol{\alpha}/\lceil\alpha\rceil][\boldsymbol{\mu\alpha.\tau}/\boldsymbol{\alpha}]]\!]\rho.$$

**Case** $\langle \overline{\tau} \rangle$

By definition of the value translations and the induction hypothesis.

**Case** $\mathsf{L}\langle \boldsymbol{\tau} \rangle$

Follows immediately from the definitions of $\mathcal{V}[\![\mathsf{L}\langle \boldsymbol{\tau} \rangle]\!]\rho$ and the translation rules for lumps.

□

## 9.1 Substitution

**Lemma 9.2 (Admissibility of $\mathcal{V}$)**
Let $\rho \in \mathcal{D}[\![\Delta]\!]$.

1. If $\Delta \vdash \tau$, then $(\rho_1(\tau), \rho_2(\tau), R) \in$ MMValRel.
   where $R = \begin{bmatrix} \mathcal{V}[\![\tau, \tau]\!]\rho & \mathcal{V}[\![\tau, \tau^{\langle \mathcal{C} \rangle}]\!]\rho \\ \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}, \tau]\!]\rho & \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}, \tau^{\langle \mathcal{C} \rangle}]\!]\rho \end{bmatrix}$
2. If $\Delta \vdash \boldsymbol{\tau}$, then $(\rho_1(\boldsymbol{\tau}), \rho_2(\boldsymbol{\tau}), [\mathcal{V}[\![\boldsymbol{\tau}]\!]\rho]) \in$ CCValRel.

**Proof**

For claim 1 we first show that the entries of $R$ are in ValRel by monotonicity (Lemma 7.6) Second we must show that relations are closed under $\mathcal{MC}_i(\,)$ and $\mathcal{CM}_i(\,)$ relation boundary translations This follows from the definitions of the off diagonal V relations, $\mathcal{V}[\![\tau, \tau^{\langle \mathcal{C} \rangle}]\!]\rho$ and $\mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}, \tau]\!]\rho$, as well as the Boundary Cancellation and Bridge Lemmas.

Claim 2 follows from monotonicity (Lemma 7.6) □

**Lemma 9.3 (Syntactic type substitution is equivalent to semantic type substitution in C)**
Given $\rho \in \mathcal{D}[\![\Delta]\!]$, $\Delta \vdash \tau$, and $\alpha \notin \Delta$ where

$$\text{if } \tau = \tau \quad \text{let} \quad \text{VR} = (\rho_1(\tau), \rho_2(\tau), \begin{bmatrix} \mathcal{V}[\![\tau, \tau]\!]\rho & \mathcal{V}[\![\tau, \tau^{\langle \mathcal{C} \rangle}]\!]\rho \\ \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}, \tau]\!]\rho & \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}, \tau^{\langle \mathcal{C} \rangle}]\!]\rho \end{bmatrix})$$
$$\text{if } \tau = \boldsymbol{\tau} \quad \text{let} \quad \text{VR} = (\rho_1(\boldsymbol{\tau}), \rho_2(\boldsymbol{\tau}), [\mathcal{V}[\![\boldsymbol{\tau}]\!]\rho])$$

Then $\forall \hat{\boldsymbol{\tau}} \,.\, \Delta, \alpha \vdash \hat{\boldsymbol{\tau}}$

1. $\mathcal{V}[\![\hat{\boldsymbol{\tau}}]\!]\rho[\alpha \mapsto \text{VR}] = \mathcal{V}[\![\hat{\boldsymbol{\tau}}[\tau/\alpha]]\!]\rho$.
2. $\mathcal{E}[\![\hat{\boldsymbol{\tau}}]\!]\rho[\alpha \mapsto \text{VR}] = \mathcal{E}[\![\hat{\boldsymbol{\tau}}[\tau/\alpha]]\!]\rho$
3. $\mathcal{K}[\![\hat{\boldsymbol{\tau}}]\!]\rho[\alpha \mapsto \text{VR}] = \mathcal{K}[\![\hat{\boldsymbol{\tau}}[\tau/\alpha]]\!]\rho$

**Proof**

Note that by Lemma 9.2, $\rho[\alpha \mapsto \text{VR}] \in \mathcal{D}[\![\Delta, \alpha]\!]$. Follows the induction structure of Lemma 7.14. The base cases (for type variables and suspensions) follow from the definition of VR. □

**Lemma 9.4 (Syntactic type substitution is equivalent to semantic type substitution in M)**
Given $\rho \in \mathcal{D}[\![\Delta]\!]$, $\Delta \vdash \tau$, and $\alpha \notin \Delta$ where

$$\text{if } \tau = \tau \quad \text{let} \quad \text{VR} = (\rho_1(\tau), \rho_2(\tau) \begin{bmatrix} \mathcal{V}[\![\tau, \tau]\!]\rho & \mathcal{V}[\![\tau, \tau^{\langle \mathcal{C} \rangle}]\!]\rho \\ \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}, \tau]\!]\rho & \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}, \tau^{\langle \mathcal{C} \rangle}]\!]\rho \end{bmatrix})$$
$$\text{if } \tau = \boldsymbol{\tau} \quad \text{let} \quad \text{VR} = (\rho_1(\boldsymbol{\tau}), \rho_2(\boldsymbol{\tau}), [\mathcal{V}[\![\boldsymbol{\tau}]\!]\rho])$$

Then $\forall \hat{\tau} \,.\, \Delta, \alpha \vdash \hat{\tau}$

1. $\mathcal{V}[\![\hat{\tau}]\!]\rho[\alpha \mapsto \text{VR}] = \mathcal{V}[\![\hat{\tau}[\tau/\alpha]]\!]\rho$.
2. $\mathcal{E}[\![\hat{\tau}]\!]\rho[\alpha \mapsto \text{VR}] = \mathcal{E}[\![\hat{\tau}[\tau/\alpha]]\!]\rho$
3. $\mathcal{K}[\![\hat{\tau}]\!]\rho[\alpha \mapsto \text{VR}] = \mathcal{K}[\![\hat{\tau}[\tau/\alpha]]\!]\rho$.

**Proof**

Note that by Lemma 9.2, $\rho[\alpha \mapsto \text{VR}] \in \mathcal{D}[\![\Delta, \alpha]\!]$. Follows the induction structure of Lemma 7.15. The base case (for type variables) follows from the definition of VR. □

## 9.2 Compatibility Lemmas

Because of the recursive dependence between the relations $\mathcal{E}[\![\tau]\!]\rho$ and $\mathcal{V}[\![\tau]\!]\rho$ to prove the fundamental property we will need to define relations for open values:

**Definition 9.5 (Logical Relation for Values)**

$$\begin{aligned}\Psi;\Delta;\Gamma \vdash v_1 \approx_{\mathrm{v}} v_2 : \tau \stackrel{\mathrm{def}}{=} \;& \Psi;\Delta;\Gamma \vdash v_1 : \tau \;\wedge\; \Psi;\Delta;\Gamma \vdash v_2 : \tau \;\wedge \\ & \forall W, \rho, \gamma.\; W \in \mathcal{H}[\![\Psi]\!] \;\wedge\; \rho \in \mathcal{D}[\![\Delta]\!] \;\wedge\; (W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho \\ & \quad \Longrightarrow\; (W, \rho_1(\gamma_1(v_1)), \rho_2(\gamma_2(v_2))) \in \mathcal{V}[\![\tau]\!]\rho\end{aligned}$$

The compatibility lemmas for value forms will have two similar statements: one for the logical relation for terms, and one for the logical relation for values. We will only address the former in the proofs, because the latter can always be shown with a very similar proof.

**Lemma 9.6 (M Variable)**
If $\mathsf{x} : \tau \in \Gamma$, then $\Psi;\Delta;\Gamma \vdash \mathsf{x} \approx^{log}_{M+C} \mathsf{x} : \tau$.

**Proof**

First note that $\Psi;\Delta;\Gamma \vdash \mathsf{x} : \tau$.

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. By definition of $\mathcal{G}[\![\Gamma]\!]\rho$,

$$(W, \rho_1(\gamma_1(\mathsf{x})), \rho_2(\gamma_2(\mathsf{x}))) = (W, \gamma_1(\mathsf{x}), \gamma_2(\mathsf{x})) \in \mathcal{V}[\![\tau]\!]\rho.$$

Then by Lemma 7.5, $(W, \rho_1(\gamma_1(\mathsf{x})), \rho_2(\gamma_2(\mathsf{x}))) \in \mathcal{E}[\![\tau]\!]\rho$, as desired. □

**Lemma 9.7 (M Unit)**
$\Psi;\Delta;\Gamma \vdash () \approx^{log}_{M+C} () : \mathsf{unit}$

**Proof**

First note that $\Psi;\Delta;\Gamma \vdash () : \mathsf{unit}$.

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. By definition,

$$(W, \rho_1(\gamma_1(())), \rho_2(\gamma_2(()))) = (W, (), ()) \in \mathcal{V}[\![\mathsf{unit}]\!]\rho.$$

By Lemma 7.5, $(W, \rho_1(\gamma_1(())), \rho_2(\gamma_2(()))) \in \mathcal{E}[\![\mathsf{unit}]\!]\rho$, as desired. □

**Lemma 9.8 (M Int)**
$\Psi;\Delta;\Gamma \vdash \mathsf{n} \approx^{log}_{M+C} \mathsf{n} : \mathsf{int}$

**Proof**

First note that $\Psi;\Delta;\Gamma \vdash \mathsf{n} : \mathsf{int}$.

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. By definition,

$$(W, \rho_1(\gamma_1(\mathsf{n})), \rho_2(\gamma_2(\mathsf{n}))) = (W, \mathsf{n}, \mathsf{n}) \in \mathcal{V}[\![\mathsf{int}]\!]\rho.$$

By Lemma 7.5, $(W, \rho_1(\gamma_1(\mathsf{n})), \rho_2(\gamma_2(\mathsf{n}))) \in \mathcal{E}[\![\mathsf{int}]\!]\rho$, as desired. □

**Lemma 9.9 (M Primitive)**
If $\Psi;\Delta;\Gamma \vdash \mathsf{v}_1 \approx^{log}_{M+C} \mathsf{v}_2 : \mathsf{int}$ and $\Psi;\Delta;\Gamma \vdash \mathsf{v}_1' \approx^{log}_{M+C} \mathsf{v}_2' : \mathsf{int}$, then $\Psi;\Delta;\Gamma \vdash \mathsf{v}_1\ \mathsf{p}\ \mathsf{v}_1' \approx^{log}_{M+C} \mathsf{v}_2\ \mathsf{p}\ \mathsf{v}_2' : \mathsf{int}$.

**Proof**

First note that $\Psi; \Delta; \Gamma \vdash \mathsf{v_1\ p\ v_1'} : \mathsf{int}$ and $\Psi; \Delta; \Gamma \vdash \mathsf{v_2\ p\ v_2'} : \mathsf{int}$.
Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$.
We need to show that

$$(W, \rho_1(\gamma_1(\mathsf{v_1\ p\ v_1'})), \rho_2(\gamma_2(\mathsf{v_2\ p\ v_2'}))) = (W, \rho_1(\gamma_1(\mathsf{v_1}))\ \mathsf{p}\ \rho_1(\gamma_1(\mathsf{v_1'})), \rho_2(\gamma_2(\mathsf{v_2}))\ \mathsf{p}\ \rho_2(\gamma_2(\mathsf{v_2'}))) \in \mathcal{E}[\![\mathsf{int}]\!]\rho.$$

By assumption, $(W, \rho_1(\gamma_1(\mathsf{v_1})), \rho_2(\gamma_2(\mathsf{v_2}))) \in \mathcal{E}[\![\mathsf{int}]\!]\rho$ and $(W, \rho_1(\gamma_1(\mathsf{v_1'})), \rho_2(\gamma_2(\mathsf{v_2'}))) \in \mathcal{E}[\![\mathsf{int}]\!]\rho$.
Let $W' \sqsupseteq_{\text{pub}} W$ and $(W', \mathsf{v_1}, \mathsf{v_2}) \in \mathcal{V}[\![\mathsf{int}]\!]\rho$. By Lemma 7.13, it suffices to show that

$$(W', \mathsf{v_1\ p}\ \rho_1(\gamma_1(\mathsf{v_1'})), \mathsf{v_2\ p}\ \rho_2(\gamma_2(\mathsf{v_2'}))) \in \mathcal{E}[\![\mathsf{int}]\!]\rho.$$

Let $W'' \sqsupseteq_{\text{pub}} W'$ and $(W', \mathsf{v_1'}, \mathsf{v_2'}) \in \mathcal{V}[\![\mathsf{int}]\!]\rho$. By another application of Lemma 7.13, it suffices to show that

$$(W'', \mathsf{v_1\ p\ v_1'}, \mathsf{v_2\ p\ v_2'}) \in \mathcal{E}[\![\mathsf{int}]\!]\rho.$$

But by definition of $\mathcal{V}[\![\mathsf{int}]\!]\rho$, $\mathsf{v_1} = \mathsf{v_2} = \mathsf{m}$ and $\mathsf{v_1'} = \mathsf{v_2'} = \mathsf{n}$. For any $(H_1, H_2) : W''$,

$$\langle H_i \mid \mathsf{m\ p\ n} \rangle \longmapsto \langle H_i \mid \mathsf{n'} \rangle.$$

By definition, $(W'', \mathsf{n'}, \mathsf{n'}) \in \mathcal{V}[\![\mathsf{int}]\!]\rho$. So by Lemma 7.5 and Lemma 7.10, we have the result. □

**Lemma 9.10 (M If0)**
If $\Psi; \Delta; \Gamma \vdash \mathsf{v_1} \approx^{log}_{M+C} \mathsf{v_2} : \mathsf{int}$, $\Psi; \Delta; \Gamma \vdash \mathsf{e_1} \approx^{log}_{M+C} \mathsf{e_2} : \tau$, and $\Psi; \Delta; \Gamma \vdash \mathsf{e_1'} \approx^{log}_{M+C} \mathsf{e_2'} : \tau$, then

$$\Psi; \Delta; \Gamma \vdash \mathsf{if0\, v_1\ e_1\ e_1'} \approx^{log}_{M+C} \mathsf{if0\, v_2\ e_2\ e_2'} : \tau.$$

**Proof**

First note that $\Psi; \Delta; \Gamma \vdash \mathsf{if0\, v_1\ e_1\ e_1'} : \tau$ and $\Psi; \Delta; \Gamma \vdash \mathsf{if0\, v_2\ e_2\ e_2'} : \tau$.
Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$.
We need to show that

$$\begin{aligned}&(W, \rho_1(\gamma_1(\mathsf{if0\, v_1\ e_1\ e_1'})), \rho_2(\gamma_2(\mathsf{if0\, v_2\ e_2\ e_2'})))\\ &\quad = (W, \mathsf{if0}\, \rho_1(\gamma_1(\mathsf{v_1}))\ \rho_1(\gamma_1(\mathsf{e_1}))\ \rho_1(\gamma_1(\mathsf{e_1'})), \mathsf{if0}\, \rho_2(\gamma_2(\mathsf{v_2}))\ \rho_2(\gamma_2(\mathsf{e_2}))\ \rho_2(\gamma_2(\mathsf{e_2'}))) \in \mathcal{E}[\![\tau]\!]\rho.\end{aligned}$$

By assumption, $(W, \rho_1(\gamma_1(\mathsf{v_1})), \rho_2(\gamma_2(\mathsf{v_2}))) \in \mathcal{E}[\![\mathsf{int}]\!]\rho$,

$$(W, \rho_1(\gamma_1(\mathsf{e_1})), \rho_2(\gamma_2(\mathsf{e_2}))) \in \mathcal{E}[\![\tau]\!]\rho, \quad \text{and} \quad (W, \rho_1(\gamma_1(\mathsf{e_1'})), \rho_2(\gamma_2(\mathsf{e_2'}))) \in \mathcal{E}[\![\tau]\!]\rho.$$

Let $W' \sqsupseteq_{\text{pub}} W$ and $(W', \mathsf{v_1}, \mathsf{v_2}) \in \mathcal{V}[\![\mathsf{int}]\!]\rho$. By Lemma 7.13, it suffices to show that

$$(W', \mathsf{if0\, v_1}\ \rho_1(\gamma_1(\mathsf{e_1}))\ \rho_1(\gamma_1(\mathsf{e_1'})), \mathsf{if0\, v_2}\ \rho_2(\gamma_2(\mathsf{e_2}))\ \rho_2(\gamma_2(\mathsf{e_2'}))) \in \mathcal{E}[\![\tau]\!]\rho.$$

By definition of $\mathcal{V}[\![\mathsf{int}]\!]\rho$, $\mathsf{v_1} = \mathsf{v_2} = \mathsf{n}$. Depending on whether $\mathsf{n} = \mathsf{0}$, for any $(H_1, H_2) : W'$, either

$$\langle H_i \mid \mathsf{if0\, v_i}\ \rho_i(\gamma_i(\mathsf{e_i}))\ \rho_i(\gamma_i(\mathsf{e_i'})) \rangle \longmapsto \langle H_i \mid \rho_i(\gamma_i(\mathsf{e_i})) \rangle$$

or

$$\langle H_i \mid \mathsf{if0\, v_i}\ \rho_i(\gamma_i(\mathsf{e_i}))\ \rho_i(\gamma_i(\mathsf{e_i'})) \rangle \longmapsto \langle H_i \mid \rho_i(\gamma_i(\mathsf{e_i'})) \rangle.$$

Thus, by Lemma 7.10, it suffices to show that

$$(W, \rho_1(\gamma_1(\mathsf{e_1})), \rho_2(\gamma_2(\mathsf{e_2}))) \in \mathcal{E}[\![\tau]\!]\rho \quad \text{and} \quad (W, \rho_1(\gamma_1(\mathsf{e_1'})), \rho_2(\gamma_2(\mathsf{e_2'}))) \in \mathcal{E}[\![\tau]\!]\rho.$$

But we already have these facts, so we are done. □

**Lemma 9.11 (M Function)**
If $\Psi;(\Delta,\overline{\alpha});(\Gamma,\overline{x\!:\!\tau}) \vdash e_1 \approx^{log}_{M+C} e_2 : \tau'$, then $\Psi;\Delta;\Gamma \vdash \lambda[\overline{\alpha}](\overline{x\!:\!\tau}).e_1 \approx^{log}_{M+C} \lambda[\overline{\alpha}](\overline{x\!:\!\tau}).e_2 : \forall[\overline{\alpha}].(\overline{\tau}) \to \tau'$

**Proof**

First note that $\Psi;\Delta;\Gamma \vdash \lambda[\overline{\alpha}](\overline{x\!:\!\tau}).e_1 : \forall[\overline{\alpha}].(\overline{\tau}) \to \tau'$ and $\Psi;\Delta;\Gamma \vdash \lambda[\overline{\alpha}](\overline{x\!:\!\tau}).e_2 : \forall[\overline{\alpha}].(\overline{\tau}) \to \tau'$.
Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W,\gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$.
We need to show that

$$(W, \rho_1(\gamma_1(\lambda[\overline{\alpha}](\overline{x\!:\!\tau}).e_1)), \rho_2(\gamma_2(\lambda[\overline{\alpha}](\overline{x\!:\!\tau}).e_2))) \\ = (W, \lambda[\overline{\alpha}](\overline{x\!:\!\tau}).\rho_1(\gamma_1(e_1)), \lambda[\overline{\alpha}](\overline{x\!:\!\tau}).\rho_2(\gamma_2(e_2))) \in \mathcal{E}[\![\forall[\overline{\alpha}].(\overline{\tau}) \to \tau']\!]\rho.$$

By Lemma 7.5, it suffices to show that

$$(W, \lambda[\overline{\alpha}](\overline{x\!:\!\tau}).\rho_1(\gamma_1(e_1)), \lambda[\overline{\alpha}](\overline{x\!:\!\tau}).\rho_2(\gamma_2(e_2))) \in \mathcal{V}[\![\forall[\overline{\alpha}].(\overline{\tau}) \to \tau']\!]\rho.$$

Let $W' \sqsupseteq W$, $\overline{\mathrm{VR} \in \mathrm{MMValRel}}$, and $\overline{(W', v_1, v_2) \in \mathcal{V}[\![\tau]\!]\rho\overline{[\alpha \mapsto \mathrm{VR}]}}$. For convenience, also let $\overline{\tau_1 = \mathrm{VR}.\tau_1}$ and $\overline{\tau_2 = \mathrm{VR}.\tau_2}$ We need to show that

$$(W, \lambda[\overline{\alpha}](\overline{x\!:\!\tau}).\rho_1(\gamma_1(e_1))\ [\overline{\tau_1}]\ \overline{v_1}, \lambda[\overline{\alpha}](\overline{x\!:\!\tau}).\rho_2(\gamma_2(e_2))\ [\overline{\tau_2}]\ \overline{v_2}) \in \mathcal{E}[\![\tau']\!]\rho\overline{[\alpha \mapsto \mathrm{VR}]}.$$

By Lemma 7.10, it suffices to show that

$$(W, \rho_1(\gamma_1(e_1))\overline{[\tau_1/\alpha][v_1/x]}, \rho_2(\gamma_2(e_2))\overline{[\tau_2/\alpha][v_2/x]}) \in \mathcal{E}[\![\tau']\!]\rho\overline{[\alpha \mapsto \mathrm{VR}]}.$$

By definition, $W' \in \mathcal{H}[\![\Psi]\!]$ and $\rho\overline{[\alpha \mapsto \mathrm{VR}]} \in \mathcal{D}[\![\Delta, \overline{\alpha}]\!]$. By definition and by monotonicity,

$$(W', \gamma\overline{[x \mapsto (v_1, v_2)]}) \in \mathcal{G}[\![\Gamma, \overline{x\!:\!\tau}]\!]\rho\overline{[\alpha \mapsto \mathrm{VR}]}.$$

Applying our assumption gives the result. □

**Lemma 9.12 (M Application)**
If $\Psi;\Delta;\Gamma \vdash v_1 \approx^{log}_{M+C} v_2 : \forall[\overline{\alpha}].(\overline{\tau}) \to \tau'$, $\overline{\Delta \vdash \hat{\tau}}$, and $\overline{\Psi;\Delta;\Gamma \vdash v_1' \approx^{log}_{M+C} v_2' : \tau\overline{[\hat{\tau}/\alpha]}}$, then

$$\Psi;\Delta;\Gamma \vdash v_1\ [\overline{\hat{\tau}}]\ \overline{v_1'} \approx^{log}_{M+C} v_2\ [\overline{\hat{\tau}}]\ \overline{v_2'} : \tau'\overline{[\hat{\tau}/\alpha]}.$$

**Proof**

First note that $\Psi;\Delta;\Gamma \vdash v_1\ [\overline{\hat{\tau}}]\ \overline{v_1'} : \tau'\overline{[\hat{\tau}/\alpha]}$ and $\Psi;\Delta;\Gamma \vdash v_2\ [\overline{\hat{\tau}}]\ \overline{v_2'} : \tau'\overline{[\hat{\tau}/\alpha]}$.
Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W,\gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$(W, \rho_1(\gamma_1(v_1\ [\overline{\hat{\tau}}]\ \overline{v_1'})), \rho_2(\gamma_2(v_2\ [\overline{\hat{\tau}}]\ \overline{v_2'}))) \\ = (W, \rho_1(\gamma_1(v_1))\ [\overline{\rho_1(\hat{\tau})}]\ \overline{\rho_1(\gamma_1(v_1'))}, \rho_2(\gamma_2(v_2))\ [\overline{\rho_2(\hat{\tau})}]\ \overline{\rho_2(\gamma_2(v_2'))}) \in \mathcal{E}[\![\tau'\overline{[\hat{\tau}/\alpha]}]\!]\rho.$$

Let $\overline{\mathrm{VR} = (\rho_1(\hat{\tau}), \rho_2(\hat{\tau}), R)}$ Where $\overline{R = \begin{bmatrix} \mathcal{V}[\![\hat{\tau},\hat{\tau}]\!]\rho & \mathcal{V}[\![\hat{\tau},\hat{\tau}^{\langle\mathcal{C}\rangle}]\!]\rho \\ \mathcal{V}[\![\hat{\tau}^{\langle\mathcal{C}\rangle},\hat{\tau}]\!]\rho & \mathcal{V}[\![\hat{\tau}^{\langle\mathcal{C}\rangle},\hat{\tau}^{\langle\mathcal{C}\rangle}]\!]\rho \end{bmatrix}}$
By Lemma 9.4,

$$\mathcal{E}[\![\tau'\overline{[\hat{\tau}/\alpha]}]\!]\rho = \mathcal{E}[\![\tau']\!]\rho\overline{[\alpha \mapsto \mathrm{VR}]} \quad \text{and} \quad \overline{\mathcal{V}[\![\tau\overline{[\hat{\tau}/\alpha]}]\!]\rho = \mathcal{V}[\![\tau]\!]\rho\overline{[\alpha \mapsto \mathrm{VR}]}}.$$

We will use these equalities throughout the proof.
Let $W_0 \sqsupseteq_{\mathrm{pub}} W$ and $(W_0, v_1, v_2) \in \mathcal{V}[\![\forall[\overline{\alpha}].(\overline{\tau}) \to \tau']\!]\rho$. By Lemma 7.13, it suffices to show that

$$(W_0, v_1\ [\overline{\rho_1(\hat{\tau})}]\ \overline{\rho_1(\gamma_1(v_1'))}, v_2\ [\overline{\rho_2(\hat{\tau})}]\ \overline{\rho_2(\gamma_2(v_2'))}) \in \mathcal{E}[\![\tau'\overline{[\hat{\tau}/\alpha]}]\!]\rho.$$

Let $W_i \sqsupseteq_{\text{pub}} W_{i-1}$ and $\overline{(W_i, \mathsf{v}'_{1i}, \mathsf{v}'_{2i}) \in \mathcal{V}[\![\tau[\overline{\hat{\tau}/\alpha}]]\!]\rho}$. By further applications of 7.13, it suffices to show that

$$(W_n, \mathsf{v}_1\ [\overline{\rho_1(\hat{\tau})}]\ \overline{\mathsf{v}'_1}, \mathsf{v}_2\ [\overline{\rho_2(\hat{\tau})}]\ \overline{\mathsf{v}'_2}) \in \mathcal{E}[\![\tau'[\overline{\hat{\tau}/\alpha}]]\!]\rho.$$

Since $W_n \sqsupseteq W_0$, $\overline{\mathrm{VR} \in \mathrm{MMValRel}}$, and $\overline{(W_n, \mathsf{v}'_1, \mathsf{v}'_2) \in \mathcal{V}[\![\tau]\!]\rho\overline{[\alpha \mapsto \mathrm{VR}]}}$, we can instantiate our assumption that

$$(W_0, \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\forall[\overline{\alpha}].(\overline{\tau}) \to \tau']\!]\rho$$

to get exactly the needed result. □

**Lemma 9.13 (M Pack)**
If $\Psi; \Delta; \Gamma \vdash \mathsf{v}_1 \approx^{log}_{M+C} \mathsf{v}_2 : \tau[\tau'/\alpha]$, then $\Psi; \Delta; \Gamma \vdash \mathsf{pack}\ \langle \tau', \mathsf{v}_1 \rangle\ \mathsf{as}\ \exists\alpha.\tau \approx^{log}_{M+C} \mathsf{pack}\ \langle \tau', \mathsf{v}_2 \rangle\ \mathsf{as}\ \exists\alpha.\tau : \exists\alpha.\tau$

**Proof**

Note that $\Psi; \Delta; \Gamma \vdash \mathsf{pack}\ \langle \tau', \mathsf{v}_1 \rangle\ \mathsf{as}\ \exists\alpha.\tau : \exists\alpha.\tau$ and $\Psi; \Delta; \Gamma \vdash \mathsf{pack}\ \langle \tau', \mathsf{v}_2 \rangle\ \mathsf{as}\ \exists\alpha.\tau : \exists\alpha.\tau$.

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$\begin{aligned}&(W, \rho_1(\gamma_1(\mathsf{pack}\ \langle \tau', \mathsf{v}_1 \rangle\ \mathsf{as}\ \exists\alpha.\tau)), \rho_2(\gamma_2(\mathsf{pack}\ \langle \tau', \mathsf{v}_2 \rangle\ \mathsf{as}\ \exists\alpha.\tau)))\\ &\qquad = (W, \mathsf{pack}\ \langle \rho_1(\tau'), \rho_1(\gamma_1(\mathsf{v}_1)) \rangle\ \mathsf{as}\ \exists\alpha.\tau, \mathsf{pack}\ \langle \rho_2(\tau'), \rho_2(\gamma_2(\mathsf{v}_2)) \rangle\ \mathsf{as}\ \exists\alpha.\tau) \in \mathcal{E}[\![\exists\alpha.\tau]\!]\rho.\end{aligned}$$

Let $\mathrm{VR} = (\rho_1(\tau'), \rho_2(\tau'), [\mathcal{V}[\![\tau']\!]\rho])$
By our assumption and Lemma 9.4,

$$(W, \rho_1(\gamma_1(\mathsf{v}_1)), \rho_2(\gamma_2(\mathsf{v}_2))) \in \mathcal{E}[\![\tau[\tau'/\alpha]]\!]\rho = \mathcal{E}[\![\tau]\!]\rho[\alpha \mapsto \mathrm{VR}].$$

Additionally, $(W, \mathsf{pack}\ \langle \rho_1(\tau'), [\cdot] \rangle\ \mathsf{as}\ \exists\alpha.\tau, \mathsf{pack}\ \langle \rho_2(\tau'), [\cdot] \rangle\ \mathsf{as}\ \exists\alpha.\tau) \in \mathrm{ContAtom}[\tau[\tau'/\alpha]]\rho \rightsquigarrow [\exists\alpha.\tau]\rho$ holds from the premises.
We use these facts to apply Lemma 7.13 and assume $W' \sqsupseteq_{\text{pub}} W$ and $(W', \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2) \in \mathcal{V}[\![\tau]\!]\rho[\alpha \mapsto \mathrm{VR}]$.
It then suffices to show that

$$(W, \mathsf{pack}\ \langle \rho_1(\tau'), \hat{\mathsf{v}}_1 \rangle\ \mathsf{as}\ \exists\alpha.\tau, \mathsf{pack}\ \langle \rho_2(\tau'), \hat{\mathsf{v}}_2 \rangle\ \mathsf{as}\ \exists\alpha.\tau) \in \mathcal{E}[\![\exists\alpha.\tau]\!]\rho.$$

By Lemma 7.5, it suffices to show that

$$(W, \mathsf{pack}\ \langle \rho_1(\tau'), \hat{\mathsf{v}}_1 \rangle\ \mathsf{as}\ \exists\alpha.\tau, \mathsf{pack}\ \langle \rho_2(\tau'), \hat{\mathsf{v}}_2 \rangle\ \mathsf{as}\ \exists\alpha.\tau) \in \mathcal{V}[\![\exists\alpha.\tau]\!]\rho.$$

But $(W', \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2) \in \mathcal{V}[\![\tau]\!]\rho[\alpha \mapsto \mathrm{VR}]$ is sufficient to give this. □

**Lemma 9.14 (M Unpack)**
If $\Psi; \Delta; \Gamma \vdash \mathsf{v}_1 \approx^{log}_{M+C} \mathsf{v}_2 : \exists\alpha.\tau$ and $\Psi; (\Delta, \alpha); (\Gamma, \mathsf{x} : \tau) \vdash \mathsf{e}_1 \approx^{log}_{M+C} \mathsf{e}_2 : \tau'$, then

$$\Psi; \Delta; \Gamma \vdash \mathsf{unpack}\ \langle \alpha, \mathsf{x} \rangle = \mathsf{v}_1\ \mathsf{in}\ \mathsf{e}_1 \approx^{log}_{M+C} \mathsf{unpack}\ \langle \alpha, \mathsf{x} \rangle = \mathsf{v}_2\ \mathsf{in}\ \mathsf{e}_2 : \tau'.$$

**Proof**

Note that $\Psi; \Delta; \Gamma \vdash \mathsf{unpack}\ \langle \alpha, \mathsf{x} \rangle = \mathsf{v}_1\ \mathsf{in}\ \mathsf{e}_1 : \tau'$ and $\Psi; \Delta; \Gamma \vdash \mathsf{unpack}\ \langle \alpha, \mathsf{x} \rangle = \mathsf{v}_2\ \mathsf{in}\ \mathsf{e}_2 : \tau'$.

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$\begin{aligned}&(W, \rho_1(\gamma_1(\mathsf{unpack}\ \langle \alpha, \mathsf{x} \rangle = \mathsf{v}_1\ \mathsf{in}\ \mathsf{e}_1)), \rho_2(\gamma_2(\mathsf{unpack}\ \langle \alpha, \mathsf{x} \rangle = \mathsf{v}_2\ \mathsf{in}\ \mathsf{e}_2)))\\ &\quad = (W, \mathsf{unpack}\ \langle \alpha, \mathsf{x} \rangle = \rho_1(\gamma_1(\mathsf{v}_1))\ \mathsf{in}\ \rho_1(\gamma_1(\mathsf{e}_1)), \mathsf{unpack}\ \langle \alpha, \mathsf{x} \rangle = \rho_2(\gamma_2(\mathsf{v}_2))\ \mathsf{in}\ \rho_2(\gamma_2(\mathsf{e}_2))) \in \mathcal{E}[\![\tau']\!]\rho.\end{aligned}$$

Note that $(W, \rho_1(\gamma_1(\mathsf{v}_1)), \rho_2(\gamma_2(\mathsf{v}_2))) \in \mathcal{E}[\![\exists\alpha.\tau]\!]\rho$ and

$$(W, \mathsf{unpack}\ \langle \alpha, \mathsf{x} \rangle = [\cdot]\ \mathsf{in}\ \mathsf{e}_1, \mathsf{unpack}\ \langle \alpha, \mathsf{x} \rangle = [\cdot]\ \mathsf{in}\ \mathsf{e}_2) \in \mathrm{ContAtom}[\exists\alpha.\tau]\rho \rightsquigarrow [\tau']\rho$$

holds from the premises. We use these facts to apply Lemma 7.13 and assume $W' \sqsupseteq_{\text{pub}} W$ and $(W', \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2) \in \mathcal{V}[\![\exists\alpha.\tau]\!]\rho$. It then suffices to show that

$$(W', \mathsf{unpack}\ \langle\alpha, \mathsf{x}\rangle = \hat{\mathsf{v}}_1\ \mathsf{in}\ \rho_1(\gamma_1(\mathsf{e}_1)), \mathsf{unpack}\ \langle\alpha, \mathsf{x}\rangle = \hat{\mathsf{v}}_2\ \mathsf{in}\ \rho_2(\gamma_2(\mathsf{e}_2))) \in \mathcal{E}[\![\tau']\!]\rho.$$

By definition of $\mathcal{V}[\![\exists\alpha.\tau]\!]\rho$, $\hat{\mathsf{v}}_1 = \mathsf{pack}\ \langle\tau_1, \hat{\mathsf{v}}'_1\rangle\ \mathsf{as}\ \rho_1(\exists\alpha.\tau)$ and $\hat{\mathsf{v}}_2 = \mathsf{pack}\ \langle\tau_2, \hat{\mathsf{v}}'_2\rangle\ \mathsf{as}\ \rho_2(\exists\alpha.\tau)$, where there is some $\text{VR} \in \text{MMValRel}$ such that $\text{VR}.\tau_1 = \tau_1$, $\text{VR}.\tau_2 = \tau_2$, and $(W', \hat{\mathsf{v}}'_1, \hat{\mathsf{v}}'_2) \in \mathcal{V}[\![\tau]\!]\rho[\alpha \mapsto \text{VR}]$. By the operational semantics and by Lemma 7.10, it suffices to show that

$$(W', \rho_1(\gamma_1(\mathsf{e}_1))[\tau_1/\alpha][\hat{\mathsf{v}}'_1/\mathsf{x}], \rho_2(\gamma_2(\mathsf{e}_2))[\tau_2/\alpha][\hat{\mathsf{v}}'_2/\mathsf{x}]) \in \mathcal{E}[\![\tau']\!]\rho.$$

By our hypothesis, this follows from $W' \in \mathcal{H}[\![\Psi]\!]$, $\rho[\alpha \mapsto \text{VR}] \in \mathcal{D}[\![\Delta, \alpha]\!]$, and

$$(W', \gamma[\mathsf{x} \mapsto (\hat{\mathsf{v}}'_1, \hat{\mathsf{v}}'_2)]) \in \mathcal{G}[\![\Gamma, \mathsf{x}\!:\!\tau]\!]\rho[\alpha \mapsto \text{VR}].$$

The first two of these conditions hold immediately, and the last holds by monotonicity and since $(W', \hat{\mathsf{v}}'_1, \hat{\mathsf{v}}'_2) \in \mathcal{V}[\![\tau]\!]\rho[\alpha \mapsto \text{VR}]$. □

**Lemma 9.15 (M Fold)**
If $\Psi; \Delta; \Gamma \vdash \mathsf{v}_1 \approx^{log}_{M+C} \mathsf{v}_2 : \tau[\mu\alpha.\tau/\alpha]$, then $\Psi; \Delta; \Gamma \vdash \mathsf{fold}_{\mu\alpha.\tau}\ \mathsf{v}_1 \approx^{log}_{M+C} \mathsf{fold}_{\mu\alpha.\tau}\ \mathsf{v}_2 : \mu\alpha.\tau$

**Proof**

Note that $\Psi; \Delta; \Gamma \vdash \mathsf{fold}_{\mu\alpha.\tau}\ \mathsf{v}_1 : \mu\alpha.\tau$ and $\Psi; \Delta; \Gamma \vdash \mathsf{fold}_{\mu\alpha.\tau}\ \mathsf{v}_2 : \mu\alpha.\tau$.

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$\begin{aligned}&(W, \rho_1(\gamma_1(\mathsf{fold}_{\mu\alpha.\tau}\ \mathsf{v}_1)), \rho_2(\gamma_2(\mathsf{fold}_{\mu\alpha.\tau}\ \mathsf{v}_2)))\\ &\qquad = (W, \mathsf{fold}_{\rho_1(\mu\alpha.\tau)}\ \rho_1(\gamma_1(\mathsf{v}_1)), \mathsf{fold}_{\rho_2(\mu\alpha.\tau)}\ \rho_2(\gamma_2(\mathsf{v}_2))) \in \mathcal{E}[\![\mu\alpha.\tau]\!]\rho.\end{aligned}$$

By our assumption,

$$(W, \rho_1(\gamma_1(\mathsf{v}_1)), \rho_2(\gamma_2(\mathsf{v}_2))) \in \mathcal{E}[\![\tau[\mu\alpha.\tau/\alpha]]\!]\rho.$$

Let $W' \sqsupseteq_{\text{pub}} W$ and $(W', \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\tau[\mu\alpha.\tau/\alpha]]\!]\rho$. By Lemma 7.13, it suffices to show that

$$(W', \mathsf{fold}_{\rho_1(\mu\alpha.\tau)}\ \mathsf{v}_1, \mathsf{fold}_{\rho_2(\mu\alpha.\tau)}\ \mathsf{v}_2) \in \mathcal{E}[\![\mu\alpha.\tau]\!]\rho.$$

By Lemma 7.5, it suffices to show that

$$(W', \mathsf{fold}_{\rho_1(\mu\alpha.\tau)}\ \mathsf{v}_1, \mathsf{fold}_{\rho_2(\mu\alpha.\tau)}\ \mathsf{v}_2) \in \mathcal{V}[\![\mu\alpha.\tau]\!]\rho.$$

It suffices to show $(W', \mathsf{v}_1, \mathsf{v}_2) \in \triangleright\mathcal{V}[\![\tau[\mu\alpha.\tau/\alpha]]\!]\rho$, which follows by monotonicity. □

**Lemma 9.16 (M Unfold)**
If $\Psi; \Delta; \Gamma \vdash \mathsf{v}_1 \approx^{log}_{M+C} \mathsf{v}_2 : \mu\alpha.\tau$, then $\Psi; \Delta; \Gamma \vdash \mathsf{unfold}\ \mathsf{v}_1 \approx^{log}_{M+C} \mathsf{unfold}\ \mathsf{v}_2 : \tau[\mu\alpha.\tau/\alpha]$.

**Proof**

Note that $\Psi; \Delta; \Gamma \vdash \mathsf{unfold}\ \mathsf{v}_1 : \tau[\mu\alpha.\tau/\alpha]$ and $\Psi; \Delta; \Gamma \vdash \mathsf{unfold}\ \mathsf{v}_2 : \tau[\mu\alpha.\tau/\alpha]$.

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$(W, \rho_1(\gamma_1(\mathsf{unfold}\ \mathsf{v}_1)), \rho_2(\gamma_2(\mathsf{unfold}\ \mathsf{v}_2))) = (W, \mathsf{unfold}\ \rho_1(\gamma_1(\mathsf{v}_1)), \mathsf{unfold}\ \rho_2(\gamma_2(\mathsf{v}_2))) \in \mathcal{E}[\![\tau[\mu\alpha.\tau/\alpha]]\!]\rho.$$

By our assumption, $(W, \rho_1(\gamma_1(\mathsf{v}_1)), \rho_2(\gamma_2(\mathsf{v}_2))) \in \mathcal{E}[\![\mu\alpha.\tau]\!]\rho$. Let $W' \sqsupseteq_{\text{pub}} W$ and $(W', \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\mu\alpha.\tau]\!]\rho$. By Lemma 7.13, it suffices to show that

$$(W', \mathsf{unfold}\ \mathsf{v}_1, \mathsf{unfold}\ \mathsf{v}_2) \in \mathcal{E}[\![\tau[\mu\alpha.\tau/\alpha]]\!]\rho.$$

By definition of $\mathcal{V}[\![\mu\alpha.\tau]\!]\rho$, $\mathsf{v}_1 = \mathsf{fold}_{\rho_1(\mu\alpha.\tau)}\ \hat{\mathsf{v}}_1$ and $\mathsf{v}_2 = \mathsf{fold}_{\rho_2(\mu\alpha.\tau)}\ \hat{\mathsf{v}}_2$, where

$$(W', \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2) \in \triangleright\mathcal{V}[\![\tau[\mu\alpha.\tau/\alpha]]\!]\rho.$$

By the operational semantics and by Lemma 7.10, it suffices to show that

$$(W', \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2) \in \mathcal{E}[\![\tau[\mu\alpha.\tau/\alpha]]\!]\rho.$$

But this follows from Lemma 7.5. □

**Lemma 9.17 (M Reference)**
If $\Psi(\ell) = \tau$ then $\Psi;\Delta;\Gamma \vdash \ell \approx^{log}_{M+C} \ell : \mathsf{ref}\ \tau$.

**Proof**

First note that $\Psi;\Delta;\Gamma \vdash \ell : \mathsf{ref}\ \tau$.

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$.
By definition of $\mathcal{H}[\![\Psi]\!]$, $(W, \ell, \ell) \in \mathcal{V}[\![\mathsf{ref}\ \tau]\!]\rho$
Then by Lemma 7.5, $(W, \rho_1(\gamma_1(\ell)), \rho_2(\gamma_2(\ell))) \in \mathcal{E}[\![\mathsf{ref}\ \tau]\!]\rho$, as desired. □

**Lemma 9.18 (M New)**
If $\Psi;\Delta;\Gamma \vdash \mathsf{v}_1 \approx^{log}_{M+C} \mathsf{v}_2 : \tau$, then $\Psi;\Delta;\Gamma \vdash \mathsf{new}\ \mathsf{v}_1 \approx^{log}_{M+C} \mathsf{new}\ \mathsf{v}_2 : \mathsf{ref}\ \tau$.

**Proof**

First note that $\Psi;\Delta;\Gamma \vdash \mathsf{new}\ \mathsf{v}_1 : \mathsf{ref}\ \tau$ and $\Psi;\Delta;\Gamma \vdash \mathsf{new}\ \mathsf{v}_2 : \mathsf{ref}\ \tau$.

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$(W, \rho_1(\gamma_1(\mathsf{new}\ \mathsf{v}_1)), \rho_2(\gamma_2(\mathsf{new}\ \mathsf{v}_2))) = (W, \mathsf{new}\ \rho_1(\gamma_1(\mathsf{v}_1)), \mathsf{new}\ \rho_2(\gamma_2(\mathsf{v}_2))) \in \mathcal{E}[\![\mathsf{ref}\ \tau]\!]\rho$$

By our assumption, we have $(W, \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{E}[\![\tau]\!]\rho$. Let $W' \sqsupseteq_{\mathrm{pub}} W$ and $(W', \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2) \in \mathcal{V}[\![\tau]\!]\rho$. By Lemma 7.13, it suffices to show that

$$(W', \mathsf{new}\ \hat{\mathsf{v}}_1, \mathsf{new}\ \hat{\mathsf{v}}_2) \in \mathcal{E}[\![\mathsf{ref}\ \tau]\!]\rho$$

To show this we first pick $\ell_1 \notin W'.\Psi_1$ and $\ell_2 \notin W'.\Psi_2$ and build $W''$:

$$\begin{aligned}
W'' &= (W'.k, (W'.\Psi_1, \ell_1 : \tau), (W'.\Psi_2, \ell_2 : \tau), (W'.\Theta, \theta)) \\
\theta &= (\bullet, \{\bullet\}, \emptyset, \emptyset, \lambda s.\varphi_H, \lambda s.\{(\ell_1, \ell_2)\}]) \\
\varphi_H &= \{ (\widetilde{W}, \{\ell_1 \mapsto \mathsf{v}_1'\}, \{\ell_2 \mapsto \mathsf{v}_2'\}) \in \mathrm{HeapAtom} \mid (\widetilde{W}, \mathsf{v}_1', \mathsf{v}_2') \in \mathcal{V}[\![\tau]\!]\rho \}
\end{aligned}$$

We then instantiate Lemma 7.9 with:
$W'' \sqsupseteq W'$ and a proof of

$$\forall (H_1, H_2) : W'.\ \exists (H_1', H_2') : W''.\ \langle H_1 \mid \mathsf{new}\ \hat{\mathsf{v}}_1 \rangle \longmapsto^* \langle H_1' \mid \ell_1 \rangle \ \wedge\ \langle H_2 \mid \mathsf{new}\ \hat{\mathsf{v}}_2 \rangle \longmapsto^* \langle H_2' \mid \ell_2 \rangle$$

Which follows by assuming $H_1, H_2$, picking $H_1' = H_1[\ell_1 \mapsto \hat{\mathsf{v}}_1]$ and $H_2' = H_2[\ell_2 \mapsto \hat{\mathsf{v}}_2]$, showing $(H_1', H_2') : W''$, and invoking the operational semantics.
Lastly, we must show that $(W'', \ell_1, \ell_2) \in \mathcal{V}[\![\mathsf{ref}\ \tau]\!]\rho$, which follows from our construction of $W''$

□

**Lemma 9.19 (M Assignment)**
If $\Psi;\Delta;\Gamma \vdash \mathsf{v}_1 \approx^{log}_{M+C} \mathsf{v}_2 : \mathsf{ref}\ \tau$ and $\Psi;\Delta;\Gamma \vdash \mathsf{v}_1' \approx^{log}_{M+C} \mathsf{v}_2' : \tau$,
then $\Psi;\Delta;\Gamma \vdash \mathsf{v}_1 := \mathsf{v}_1' \approx^{log}_{M+C} \mathsf{v}_2 := \mathsf{v}_2' : \mathsf{unit}$.

**Proof**

First note that $\Psi; \Delta; \Gamma \vdash \mathsf{v}_1 := \mathsf{v}_1' : \mathsf{unit}$ and $\Psi; \Delta; \Gamma \vdash \mathsf{v}_2 := \mathsf{v}_2' : \mathsf{unit}$.

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$(W, \rho_1(\gamma_1(\mathsf{v}_1 := \mathsf{v}_1')), \rho_2(\gamma_2(\mathsf{v}_2 := \mathsf{v}_2'))) = (W, \rho_1(\gamma_1(\mathsf{v}_1)) := \rho_1(\gamma_1(\mathsf{v}_1')), \rho_2(\gamma_2(\mathsf{v}_2)) := \rho_2(\gamma_2(\mathsf{v}_2'))) \in \mathcal{E}[\![\mathsf{unit}]\!]\rho$$

From the second premise, $(W, \rho_1(\gamma_1(\mathsf{v}_1')), \rho_2(\gamma_2(\mathsf{v}_2'))) \in \mathcal{E}[\![\tau]\!]\rho$. Let $W' \sqsupseteq_{\mathrm{pub}} W$ and $(W', \hat{\mathsf{v}}_1', \hat{\mathsf{v}}_2') \in \mathcal{V}[\![\tau]\!]\rho$. By Lemma 7.13, it suffices to show that

$$(W', \rho_1(\gamma_1(\mathsf{v}_1)) := \hat{\mathsf{v}}_1', \rho_1(\gamma_1(\mathsf{v}_2)) := \hat{\mathsf{v}}_2') \in \mathcal{E}[\![\mathsf{unit}]\!]\rho$$

Instantiating the second premise with $W'$, noting that by monotonicity we have $W' \in \mathcal{H}[\![\Psi]\!]$ and $(W', \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$, we get

$$(W', \rho_1(\gamma_1(\mathsf{v}_1)), \rho_2(\gamma_2(\mathsf{v}_2))) \in \mathcal{E}[\![\mathsf{ref}\,\tau]\!]\rho$$

Applying Lemma 7.13 again, we assume $W'' \sqsupseteq_{\mathrm{pub}} W'$ and $(W'', \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2) \in \mathcal{V}[\![\mathsf{ref}\,\tau]\!]\rho$, and need to show that

$$(W'', \hat{\mathsf{v}}_1 := \hat{\mathsf{v}}_1', \hat{\mathsf{v}}_2 := \hat{\mathsf{v}}_2') \in \mathcal{E}[\![\mathsf{unit}]\!]\rho$$

Next we consider the 4 possible cases for $\hat{\mathsf{v}}_1$ and $\hat{\mathsf{v}}_2$.

For each case, we proceed by applying Lemma 7.9. For each case start by unfolding the $\mathcal{E}$ relation down to the $\mathcal{O}$ relation: we get to assume $(W'', K_1, K_2) \in \mathcal{K}[\![\mathsf{ref}\,\tau]\!]\rho$ and $(H_1, H_2) : W''$ and must show that the assignment expressions wrapped in the continuations terminate or are still running after $W''.k$ steps. We proceed by applying Lemma **??** with $W'' \sqsupseteq W'', W''.k \leq W''.k + 1, (H_1, H_2) : W''$.

- $\hat{\mathsf{v}}_1 = \ell_1$ and $\hat{\mathsf{v}}_2 = \ell_2$:
  Let $H_1' = H_1[\ell_1 \mapsto \hat{\mathsf{v}}_1']$ and $H_2' = H_2[\ell_2 \mapsto \hat{\mathsf{v}}_2']$.
  By $(W'', \hat{\mathsf{v}}_1', \hat{\mathsf{v}}_2') \in \mathcal{V}[\![\tau]\!]\rho$ it follows that $(H_1', H_2') : W''$.
  By the operational semantics

  $$\langle H_1 \mid \ell_1 := \hat{\mathsf{v}}_1' \rangle \longmapsto^1 \langle H_1[\ell_1 \mapsto \hat{\mathsf{v}}_1'] \mid () \rangle \;\wedge\; \langle H_2 \mid \ell_2 := \hat{\mathsf{v}}_2' \rangle \longmapsto^1 \langle H_2[\ell_2 \mapsto \hat{\mathsf{v}}_2'] \mid () \rangle$$

  We now must show $\langle H_1' \mid K_1[\ell_1] \rangle \Downarrow, \langle H_2' \mid K_2[\ell_2] \rangle \Downarrow$ or that they keep running after $W''.k$ steps. This is obtained by turning the premise $(W'', \ell_1, \ell_2) \in \mathcal{V}[\![\mathsf{ref}\,\tau]\!]\rho$ into $(W'', \ell_1, \ell_2) \in \mathcal{E}[\![\mathsf{ref}\,\tau]\!]\rho$ by Lemma 7.5 and then instantiating the result with the $K_1, K_2, H_1', H_2'$.
- $\hat{\mathsf{v}}_1 = \ell_1$ and $\hat{\mathsf{v}}_2 = {}^{\rho_2(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell_2}$:
  Let $H_1' = H_1[\ell_1 \mapsto \hat{\mathsf{v}}_1']$ and $H_2' = H_2[\boldsymbol{\ell_2} \mapsto \boldsymbol{\hat{\mathsf{v}}_2''}]$, where $\boldsymbol{\hat{\mathsf{v}}_2''} = \mathbf{CM}^{\rho_2(\tau)}(\hat{\mathsf{v}}_2')$
  In order to show $(H_1', H_2') : W''$ we must prove $(W'', \hat{\mathsf{v}}_1', \boldsymbol{\hat{\mathsf{v}}_2''}) \in \mathcal{V}[\![\tau, \tau^{\langle \mathcal{C} \rangle}]\!]\rho$:
  - $(W'', \hat{\mathsf{v}}_1', {}^{\rho_2(\tau)}\mathbf{MC}(\boldsymbol{\hat{\mathsf{v}}_2''})) \in \mathcal{V}[\![\tau]\!]\rho$: follows by the Boundary Cancellation Lemma (8.2)
  - $(W'', \mathbf{CM}^{\rho_1(\tau)}(\hat{\mathsf{v}}_1'), \boldsymbol{\hat{\mathsf{v}}_2''}) \in \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}]\!]\rho$: follows by the Bridge Lemma (9.1).

  Now, by the operational semantics

  $$\langle H_1 \mid \ell_1 := \hat{\mathsf{v}}_1' \rangle \longmapsto^1 \langle H_1[\ell_1 \mapsto \hat{\mathsf{v}}_1'] \mid \ell_1 \rangle$$

  $$\langle H_2 \mid {}^{\rho_2(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell_2} := \hat{\mathsf{v}}_2' \rangle \longmapsto^1 \langle H_2 \mid {}^{\rho_2(\mathsf{ref}\,\tau)}\mathcal{MC}\,(\boldsymbol{\ell_2} := \mathcal{CM}^{\rho_2(\tau)}\,\hat{\mathsf{v}}_2')\rangle \longmapsto^2 \langle H_2[\boldsymbol{\ell_2} \mapsto \boldsymbol{\hat{\mathsf{v}}_2''}] \mid {}^{\rho_2(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell_2} \rangle$$

  We now must show $\langle H_1' \mid K_1[\ell_1] \rangle \Downarrow, \langle H_2' \mid K_2[{}^{\rho_2(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell_2}] \rangle \Downarrow$ or that they keep running after $W''.k$ steps. This is wrapped up in a similar fashion to the case above.
- $\hat{\mathsf{v}}_1 = {}^{\rho_1(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell_1}$ and $\hat{\mathsf{v}}_2 = \ell_2$:
  Let $H_1' = H_1[\boldsymbol{\ell_1} \mapsto \boldsymbol{\hat{\mathsf{v}}_1''}]$ and $H_2' = H_2[\ell_2 \mapsto \hat{\mathsf{v}}_2']$, where $\boldsymbol{\hat{\mathsf{v}}_1''} = \mathbf{CM}^{\rho_1(\tau)}(\hat{\mathsf{v}}_1')$
  In order to show $(H_1', H_2') : W''$ we must prove $(W'', \boldsymbol{\hat{\mathsf{v}}_1''}, \hat{\mathsf{v}}_2') \in \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}, \tau]\!]\rho$:
  - $(W'', {}^{\rho_1(\tau)}\mathbf{MC}(\boldsymbol{\hat{\mathsf{v}}_1''}), \hat{\mathsf{v}}_2') \in \mathcal{V}[\![\tau]\!]\rho$: follows by the Left Boundary Cancellation Lemma (8.3).
  - $(W'', \boldsymbol{\hat{\mathsf{v}}_1''}, \mathbf{CM}^{\rho_2(\tau)}(\hat{\mathsf{v}}_2')) \in \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}]\!]\rho$: follows by the Bridge Lemma (9.1).

Now, by the operational semantics

$$\langle H_1 \mid {}^{\rho_1(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell_1} := \hat{\mathsf{v}}_1'\rangle \longmapsto^1 \langle H_1 \mid {}^{\rho_1(\mathsf{ref}\,\tau)}\mathcal{MC}\,(\boldsymbol{\ell_1} := \mathcal{CM}^{\rho_1(\tau)}\,\hat{\mathsf{v}}_1')\rangle \longmapsto^2 \langle H_1[\boldsymbol{\ell_1} \mapsto \hat{\mathbf{v}}_1''] \mid {}^{\rho_1(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell_1}\rangle$$

$$\langle H_2 \mid \ell_2 := \hat{\mathsf{v}}_2'\rangle \longmapsto^1 \langle H_2[\ell_2 \mapsto \hat{\mathsf{v}}_2'] \mid \ell_2\rangle$$

We now must show $\langle H_1' \mid K_1[{}^{\rho_1(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell_1}]\rangle \Downarrow, \langle H_2' \mid K_2[\ell_2]\rangle \Downarrow$ or that they keep running after $W''.k$ steps. This is wrapped up in a similar fashion to the case above.

- $\hat{\mathsf{v}}_1 = {}^{\rho_1(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell_1}$ and $\hat{\mathsf{v}}_2 = {}^{\rho_2(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell_2}$:
  Now assuming $(H_1, H_2) : W''$, let $H_1' = H_1[\boldsymbol{\ell_1} \mapsto \hat{\mathbf{v}}_1'']$ and $H_2' = H_2[\boldsymbol{\ell_2} \mapsto \hat{\mathbf{v}}_2'']$
  where $\hat{\mathbf{v}}_1'' = \mathbf{CM}^{\rho_1(\tau)}(\hat{\mathsf{v}}_1')$ and $\hat{\mathbf{v}}_2'' = \mathbf{CM}^{\rho_2(\tau)}(\hat{\mathsf{v}}_2')$
  By applying the Bridge Lemma (9.1) to $(W'', \hat{\mathsf{v}}_1', \hat{\mathsf{v}}_2') \in \mathcal{V}[\![\tau]\!]\rho$ we get that $(H_1', H_2') : W''$.
  By the operational semantics

$$\langle H_1 \mid {}^{\rho_1(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell_1} := \hat{\mathsf{v}}_1'\rangle \longmapsto^1 \langle H_1 \mid {}^{\rho_1(\mathsf{ref}\,\tau)}\mathcal{MC}\,(\boldsymbol{\ell_1} := \mathcal{CM}^{\rho_1(\tau)}\,\hat{\mathsf{v}}_1')\rangle \longmapsto^2 \langle H_1[\boldsymbol{\ell_1} \mapsto \hat{\mathbf{v}}_1''] \mid {}^{\rho_1(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell_1}\rangle$$

$$\langle H_2 \mid {}^{\rho_2(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell_2} := \hat{\mathsf{v}}_2'\rangle \longmapsto^1 \langle H_2 \mid {}^{\rho_2(\mathsf{ref}\,\tau)}\mathcal{MC}\,(\boldsymbol{\ell_2} := \mathcal{CM}^{\rho_2(\tau)}\,\hat{\mathsf{v}}_2')\rangle \longmapsto^2 \langle H_2[\boldsymbol{\ell_2} \mapsto \hat{\mathbf{v}}_2''] \mid {}^{\rho_2(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell_2}\rangle$$

  We now must show $\langle H_1' \mid K_1[{}^{\rho_2(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell_1}]\rangle \Downarrow, \langle H_2' \mid K_2[{}^{\rho_2(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell_2}]\rangle \Downarrow$ or that they keep running after $W''.k$ steps. This is wrapped up in a similar fashion to the case above.

□

**Lemma 9.20 (M Dereference)**
If $\Psi; \Delta; \Gamma \vdash \mathsf{v}_1 \approx^{log}_{M+C} \mathsf{v}_2 : \mathsf{ref}\,\tau$, then $\Psi; \Delta; \Gamma \vdash !\mathsf{v}_1 \approx^{log}_{M+C} !\mathsf{v}_2 : \tau$.

**Proof**

First note that $\Psi; \Delta; \Gamma \vdash !\mathsf{v}_1 : \tau$ and $\Psi; \Delta; \Gamma \vdash !\mathsf{v}_2 : \tau$.

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$(W, \rho_1(\gamma_1(!\mathsf{v}_1)), \rho_2(\gamma_2(!\mathsf{v}_2))) = (W, !\rho_1(\gamma_1(\mathsf{v}_1)), !\rho_2(\gamma_2(\mathsf{v}_2))) \in \mathcal{E}[\![\tau]\!]\rho$$

We begin by showing $(W, \rho_1(\gamma_1(\mathsf{v}_1)), \rho_2(\gamma_2(\mathsf{v}_2))) \in \mathcal{E}[\![\mathsf{ref}\,\tau]\!]\rho$, $(W, ![\cdot], ![\cdot]) \in \text{ContAtom}[\mathsf{ref}\,\tau]\rho \rightsquigarrow [\tau]\rho$.
Using these facts, we applying Lemma 7.13.
Assuming $W' \sqsupseteq_{\text{pub}} W$ and $(W', \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2) \in \mathcal{V}[\![\mathsf{ref}\,\tau]\!]\rho$ it suffices to show that

$$(W', !\hat{\mathsf{v}}_1, !\hat{\mathsf{v}}_2) \in \mathcal{E}[\![\tau]\!]\rho$$

We consider the 4 cases of $\hat{\mathsf{v}}_1$ and $\hat{\mathsf{v}}_2$:
For each case start by unfolding down past the $\mathcal{O}$ relation to assume $(W', K_1, K_2) \in \mathcal{K}[\![\tau]\!]\rho$ and $(H_1, H_2) : W'$. We must show $\langle H_1 \mid K_1[\mathsf{v}_1]\rangle \Downarrow \wedge \langle H_2 \mid K_2[\mathsf{v}_2]\rangle \Downarrow$ or $\text{running}(W'k, \langle H_1 \mid K_1[\mathsf{v}_1]\rangle) \wedge \text{running}(W'.k, \langle H_2 \mid K_2[\mathsf{v}_2]\rangle)$ and continue by applying Lemma **??** and instantiating it with $W' \sqsupseteq W'$, $H_1' = H_1$, and $H_2' = H_2$.

- $\hat{\mathsf{v}}_1 = \ell_1$ and $\hat{\mathsf{v}}_2 = \ell_2$:
  The operational semantics give us

$$\langle H_1 \mid !\ell_1\rangle \longmapsto^1 \langle H_1 \mid \mathsf{v}_1'\rangle \;\wedge\; \langle H_2 \mid !\ell_2\rangle \longmapsto^1 \langle H_2 \mid \mathsf{v}_2'\rangle$$

  where $H_1(\ell_1) = \mathsf{v}_1'$ and $H_2(\ell_2) = \mathsf{v}_2'$
  We need to show $\langle H_1 \mid K_1[\mathsf{v}_1']\rangle \Downarrow, \langle H_2 \mid K_2[\mathsf{v}_2']\rangle \Downarrow$ or that they keep running after $W'.k$ steps. This follows by instantiating $(W', \ell_1, \ell_2) \in \mathcal{V}[\![\mathsf{ref}\,\tau]\!]\rho$ with $W'$ to get $(W', \mathsf{v}_1', \mathsf{v}_2') \in \mathcal{V}[\![\tau]\!]\rho$. By Lemma 7.5 this becomes $(W', \mathsf{v}_1', \mathsf{v}_2') \in \mathcal{E}[\![\tau]\!]\rho$, which we can instantiate with $K_1, K_2, H_1, H_2$.

- $\hat{v}_1 = \ell_1$ and $\hat{v}_2 = {}^{\rho_2(\text{ref }\tau)}\mathcal{MC}\,\boldsymbol{\ell_2}$:
  The operational semantics give us

  $$\langle H_1 \mid !\ell_1\rangle \longmapsto^1 \langle H_1 \mid v_1'\rangle \;\wedge\; \langle H_2 \mid !^{\rho_2(\text{ref }\tau)}\mathcal{MC}\,\boldsymbol{\ell_2}\rangle \longmapsto^1 \langle H_2 \mid {}^{\rho_2(\tau)}\mathcal{MC}\,\boldsymbol{!\ell_2}\rangle \longmapsto^2 \langle H_2 \mid v_2''\rangle$$

  where $H_1(\ell_1) = v_1'$, $H_2(\boldsymbol{\ell_2}) = \mathbf{v_2'}$ and $v_2'' = {}^{\rho_2(\tau)}\mathbf{MC}(\mathbf{v_2'})$
  We need to show $\langle H_1 \mid K_1[v_1']\rangle \Downarrow, \langle H_2 \mid K_2[v_2'']\rangle \Downarrow$ or that they keep running after $W'.k$ steps. This follows by instantiating $(W', \ell_1, {}^{\rho_2(\text{ref }\tau)}\mathcal{MC}\,\boldsymbol{\ell_2}) \in \mathcal{V}[\![\text{ref }\tau]\!]\rho$ with $W'$ to give us $(W', v_1', \mathbf{v_2'}) \in \mathcal{V}[\![\tau, \tau^{\langle\mathcal{C}\rangle}]\!]\rho$. By $v_2'' = {}^{\rho_2(\tau)}\mathbf{MC}(\mathbf{v_2'})$ and the definition of $\mathcal{V}[\![\tau, \tau^{\langle\mathcal{C}\rangle}]\!]\rho$, we have $(W', v_1', v_2'') \in \mathcal{V}[\![\tau]\!]\rho$, Lemma 7.5 then gives us $(W', v_1', v_2'') \in \mathcal{E}[\![\tau]\!]\rho$ which we can instantiate with the proper continuations and heaps for the desired result.
- $\hat{v}_1 = {}^{\rho_1(\text{ref }\tau)}\mathcal{MC}\,\boldsymbol{\ell_1}$ and $\hat{v}_2 = \ell_2$:
  The operational semantics give us

  $$\langle H_1 \mid !^{\rho_1(\text{ref }\tau)}\mathcal{MC}\,\boldsymbol{\ell_1}\rangle \longmapsto^1 \langle H_1 \mid {}^{\rho_1(\tau)}\mathcal{MC}\,\boldsymbol{!\ell_1}\rangle \longmapsto^2 \langle H_1 \mid v_1''\rangle \;\wedge\; \langle H_2 \mid !\ell_2\rangle \longmapsto^1 \langle H_2 \mid v_2'\rangle$$

  where $H_1(\boldsymbol{\ell_1}) = \mathbf{v_1'}$, $H_2(\ell_2) = v_2'$ and $v_1'' = {}^{\rho_1(\tau)}\mathbf{MC}(\mathbf{v_1'})$
  We need to show $\langle H_1 \mid K_1[v_1'']\rangle \Downarrow, \langle H_2 \mid K_2[v_2']\rangle \Downarrow$ or that they keep running after $W'.k$ steps. This follows by instantiating $(W', {}^{\rho_1(\text{ref }\tau)}\mathcal{MC}\,\boldsymbol{\ell_1}, \ell_2) \in \mathcal{V}[\![\text{ref }\tau]\!]\rho$ with $W'$, giving us $(W', \mathbf{v_1'}, v_2') \in \mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}, \tau]\!]\rho$.
  From $v_1'' = {}^{\rho_1(\tau)}\mathbf{MC}(\mathbf{v_1'})$ and the definition of $\mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}, \tau]\!]\rho$ we have $(W', v_1'', v_2') \in \mathcal{V}[\![\tau]\!]\rho$. Lemma 7.5 turns this into $(W', v_1'', v_2') \in \mathcal{E}[\![\tau]\!]\rho$. We wrap up in a similar fashion to the case above.
- $\hat{v}_1 = {}^{\rho_1(\text{ref }\tau)}\mathcal{MC}\,\boldsymbol{\ell_1}$ and $\hat{v}_2 = {}^{\rho_2(\text{ref }\tau)}\mathcal{MC}\,\boldsymbol{\ell_2}$:
  The operational semantics give us

  $$\langle H_1 \mid !^{\rho_1(\text{ref }\tau)}\mathcal{MC}\,\boldsymbol{\ell_1}\rangle \longmapsto^1 \langle H_1 \mid {}^{\rho_1(\tau)}\mathcal{MC}\,\boldsymbol{!\ell_1}\rangle \longmapsto^2 \langle H_1 \mid v_1''\rangle$$

  $$\langle H_2 \mid !^{\rho_2(\rho_2(\text{ref }\tau))}\mathcal{MC}\,\boldsymbol{\ell_2}\rangle \longmapsto^1 \langle H_2 \mid {}^{\rho_2(\tau)}\mathcal{MC}\,\boldsymbol{!\ell_2}\rangle \longmapsto^2 \langle H_2 \mid v_2''\rangle$$

  where $H_1(\boldsymbol{\ell_1}) = \mathbf{v_1'}$, $H_2(\boldsymbol{\ell_2}) = \mathbf{v_2'}$, $v_1'' = {}^{\rho_1(\tau)}\mathbf{MC}(\mathbf{v_1'})$, and $v_2'' = {}^{\rho_2(\tau)}\mathbf{MC}(\mathbf{v_2'})$
  We need to show $\langle H_1 \mid K_1[v_1'']\rangle \Downarrow, \langle H_2 \mid K_2[v_2'']\rangle \Downarrow$ or that they keep running after $W'.k$ steps. Instantiating $(W', {}^{\rho_1(\text{ref }\tau)}\mathcal{MC}\,\boldsymbol{\ell_1}, {}^{\rho_2(\text{ref }\tau)}\mathcal{MC}\,\boldsymbol{\ell_2}) \in \mathcal{V}[\![\text{ref }\tau]\!]\rho$ with $W'$ gives us $(W', \mathbf{v_1'}, \mathbf{v_2'}) \in \mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}]\!]\rho$. Applying the Bridge Lemma (9.1) provides us with $(W', v_1'', v_2'') \in \mathcal{V}[\![\tau]\!]\rho$. We wrap up with Lemma 7.5 and plugging in the correct continuations and heaps.

□

**Lemma 9.21 (M Tuple)**
If $\Psi; \Delta; \Gamma \vdash v_1 \approx^{log}_{M+C} v_2 : \tau$, then $\Psi; \Delta; \Gamma \vdash \langle\overline{v_1}\rangle \approx^{log}_{M+C} \langle\overline{v_2}\rangle : \langle\overline{\tau}\rangle$

**Proof**

Note that $\Psi; \Delta; \Gamma \vdash \langle\overline{v_1}\rangle : \langle\overline{\tau}\rangle$ and $\Psi; \Delta; \Gamma \vdash \langle\overline{v_2}\rangle : \langle\overline{\tau}\rangle$.
Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$(W, \rho_1(\gamma_1(\langle\overline{v_1}\rangle)), \rho_2(\gamma_2(\langle\overline{v_2}\rangle))) = (W, \langle\overline{\rho_1(\gamma_1(v_1))}\rangle, \langle\overline{\rho_2(\gamma_2(v_2))}\rangle) \in \mathcal{E}[\![\langle\overline{\tau}\rangle]\!]\rho.$$

By our assumption,

$$\overline{(W, \rho_1(\gamma_1(v_1)), \rho_2(\gamma_2(v_2))) \in \mathcal{E}[\![\tau]\!]\rho}.$$

Let $W_0 = W$, $W_i \sqsupseteq_{\text{pub}} W_{i-1}$, and $\overline{(W_i, v_1, v_2) \in \mathcal{V}[\![\tau]\!]\rho}$. By repeated use of Lemma 7.13, it suffices to show that

$$(W_n, \langle\overline{v_1}\rangle, \langle\overline{v_2}\rangle) \in \mathcal{E}[\![\langle\overline{\tau}\rangle]\!]\rho.$$

By Lemma 7.5, it suffices to show that

$$(W_n, \langle\overline{v_1}\rangle, \langle\overline{v_2}\rangle) \in \mathcal{V}[\![\langle\overline{\tau}\rangle]\!]\rho.$$

But we have this by its definition and by monotonicity. □

**Lemma 9.22 (M Projection)**
If $\Psi; \Delta; \Gamma \vdash \mathsf{v_1} \approx^{log}_{M+C} \mathsf{v_2} : \langle\overline{\tau}\rangle$, then $\Psi; \Delta; \Gamma \vdash \pi_\mathsf{i}(\mathsf{v_1}) \approx^{log}_{M+C} \pi_\mathsf{i}(\mathsf{v_2}) : \tau_\mathsf{i}$.

**Proof**

Note that $\Psi; \Delta; \Gamma \vdash \pi_\mathsf{i}(\mathsf{v_1}) : \tau_\mathsf{i}$ and $\Psi; \Delta; \Gamma \vdash \pi_\mathsf{i}(\mathsf{v_2}) : \tau_\mathsf{i}$.

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$(W, \rho_1(\gamma_1(\pi_\mathsf{i}(\mathsf{v_1}))), \rho_2(\gamma_2(\pi_\mathsf{i}(\mathsf{v_2})))) = (W, \pi_\mathsf{i}(\rho_1(\gamma_1(\mathsf{v_1}))), \pi_\mathsf{i}(\rho_2(\gamma_2(\mathsf{v_2})))) \in \mathcal{E}[\![\tau_\mathsf{i}]\!]\rho.$$

By our assumption, $(W, \rho_1(\gamma_1(\mathsf{v_1})), \rho_2(\gamma_2(\mathsf{v_2}))) \in \mathcal{E}[\![\langle\overline{\tau}\rangle]\!]\rho$. Let $W' \sqsupseteq_{\text{pub}} W$ and $(W', \mathsf{v_1}, \mathsf{v_2}) \in \mathcal{V}[\![\langle\overline{\tau}\rangle]\!]\rho$. By Lemma 7.13, it suffices to show that

$$(W', \pi_\mathsf{i}(\mathsf{v_1}), \pi_\mathsf{i}(\mathsf{v_2})) \in \mathcal{E}[\![\tau_\mathsf{i}]\!]\rho.$$

By definition of $\mathcal{V}[\![\langle\overline{\tau}\rangle]\!]\rho$, $\mathsf{v_1} = \langle\overline{\hat{\mathsf{v}}_1}\rangle$ and $\mathsf{v_2} = \langle\overline{\hat{\mathsf{v}}_2}\rangle$, where $\overline{(W', \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2) \in \mathcal{V}[\![\tau]\!]\rho}$. By the operational semantics and by Lemma 7.10, it suffices to show that

$$(W', \hat{\mathsf{v}}_{1\mathsf{i}}, \hat{\mathsf{v}}_{2\mathsf{i}}) \in \mathcal{E}[\![\tau_\mathsf{i}]\!]\rho.$$

But this follows from Lemma 7.5. □

**Lemma 9.23 (M Let)**
If $\Psi; \Delta; \Gamma \vdash \mathsf{e_1} \approx^{log}_{M+C} \mathsf{e_1'} : \tau_1$ and $\Psi; \Delta; \Gamma, \mathsf{x} : \tau_1 \vdash \mathsf{e_2} \approx^{log}_{M+C} \mathsf{e_2'} : \tau_2$,
then $\Psi; \Delta; \Gamma \vdash \mathsf{let\,x = e_1 \ in \ e_2} \approx^{log}_{M+C} \mathsf{let\,x = e_1' \ in \ e_2'} : \tau_2$.

**Proof**

Note that $\Psi; \Delta; \Gamma \vdash \mathsf{let\,x = e_1 \ in \ e_2} : \tau_2$ and $\Psi; \Delta; \Gamma \vdash \mathsf{let\,x = e_1' \ in \ e_2'} : \tau_2$

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$\begin{aligned}&(W, \rho_1(\gamma_1(\mathsf{let\,x = e_1 \ in \ e_2})), \rho_2(\gamma_2(\mathsf{let\,x = e_1' \ in \ e_2'})))\\ &\quad = (W, \mathsf{let\,x} = \rho_1(\gamma_1(\mathsf{e_1})) \ \mathsf{in} \ \rho_2(\gamma_2(\mathsf{e_2})), \mathsf{let\,x} = \rho_1(\gamma_1(\mathsf{e_1'})) \ \mathsf{in} \ \rho_2(\gamma_2(\mathsf{e_2'}))) \in \mathcal{E}[\![\tau_2]\!]\rho.\end{aligned}$$

We start by showing $(W, \rho_1(\gamma_1(\mathsf{e_1})), \rho_2(\gamma_2(\mathsf{e_1'}))) \in \mathcal{E}[\![\tau_1]\!]\rho$ and
$(W, \mathsf{let\,x} = [\cdot] \ \mathsf{in} \ \rho_1(\gamma_1(\mathsf{e_2})), \mathsf{let\,x} = [\cdot] \ \mathsf{in} \ \rho_2(\gamma_2(\mathsf{e_2'}))) \in \mathrm{ContAtom}[\tau_1]\rho \rightsquigarrow [\tau_2]\rho$.
Using these facts, we applying Lemma 7.13.
Assuming $W' \sqsupseteq_{\text{pub}} W$ and $(W', \mathsf{v_1}, \mathsf{v_1'}) \in \mathcal{V}[\![\tau_1]\!]\rho$ it suffices to show that

$$(W, \mathsf{let\,x = v_1 \ in} \ \rho_1(\gamma_1(\mathsf{e_2})), \mathsf{let\,x = v_1' \ in} \ \rho_2(\gamma_2(\mathsf{e_2'}))) \in \mathcal{E}[\![\tau_2]\!]\rho$$

By Lemma 7.10 it suffices to show

$$(W, \rho_1(\gamma_1(\mathsf{e_2}))[\mathsf{v_1/x}], \rho_2(\gamma_2(\mathsf{e_2'}))[\mathsf{v_1'/x}]) \in \mathcal{E}[\![\tau_2]\!]\rho$$

We now instantiate our second premise with $W' \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and

$$(W', \gamma\overline{[\mathsf{x} \mapsto (\mathsf{v_1}, \mathsf{v_2})]}) \in \mathcal{G}[\![\Gamma, \overline{\mathsf{x} : \tau}]\!]\rho\overline{[\alpha \mapsto \mathrm{VR}]}$$

which hold by monotonicity, to achieve this.

□

**Lemma 9.24 (MC Boundary)**
If $\Psi; \Delta; \Gamma \vdash \mathbf{e_1} \approx^{log}_{M+C} \mathbf{e_2} : \tau^{\langle\mathcal{C}\rangle}$, then $\Psi; \Delta; \Gamma \vdash {}^{\tau}\mathcal{MC}\,\mathbf{e_1} \approx^{log}_{M+C} {}^{\tau}\mathcal{MC}\,\mathbf{e_2} : \tau$.

**Proof**

Note that $\Psi;\Delta;\Gamma \vdash {}^{\tau}\mathcal{MC}\,\mathbf{e_2} : \tau$ and $\Psi;\Delta;\Gamma \vdash {}^{\tau}\mathcal{MC}\,\mathbf{e_2} : \tau$.

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W,\gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. By our assumption,

$$(W, \rho_1(\gamma_1(\mathbf{e_1})), \rho_2(\gamma_2(\mathbf{e_2}))) \in \mathcal{E}[\![\tau^{\langle \mathbf{C} \rangle}]\!]\rho.$$

By the bridge lemma,

$$(W, {}^{\rho_1(\tau)}\mathcal{MC}\,\rho_1(\gamma_1(\mathbf{e_1})), {}^{\rho_2(\tau)}\mathcal{MC}\,\rho_2(\gamma_2(\mathbf{e_2}))) = (W, \rho_1(\gamma_1({}^{\tau}\mathcal{MC}\,\mathbf{e_1})), \rho_2(\gamma_2({}^{\tau}\mathcal{MC}\,\mathbf{e_2}))) \in \mathcal{E}[\![\tau]\!]\rho,$$

as desired. □

**Lemma 9.25 (C Function)**
If $\Psi; (\overline{\alpha}); (\overline{\mathbf{x} : \boldsymbol{\tau}}) \vdash \mathbf{e_1} \approx^{log}_{M+C} \mathbf{e_2} : \boldsymbol{\tau}'$, then $\Psi;\Delta;\Gamma \vdash \boldsymbol{\lambda}[\overline{\alpha}](\overline{\mathbf{x} : \boldsymbol{\tau}}).\mathbf{e_1} \approx^{log}_{M+C} \boldsymbol{\lambda}[\overline{\alpha}](\overline{\mathbf{x} : \boldsymbol{\tau}}).\mathbf{e_2} : \forall[\overline{\alpha}].(\overline{\boldsymbol{\tau}}) \to \boldsymbol{\tau}'$

**Proof**

First note that $\Psi;\Delta;\Gamma \vdash \boldsymbol{\lambda}[\overline{\alpha}](\overline{\mathbf{x} : \boldsymbol{\tau}}).\mathbf{e_1} : \forall[\overline{\alpha}].(\overline{\boldsymbol{\tau}}) \to \boldsymbol{\tau}'$ and $\Psi;\Delta;\Gamma \vdash \boldsymbol{\lambda}[\overline{\alpha}](\overline{\mathbf{x} : \boldsymbol{\tau}}).\mathbf{e_2} : \forall[\overline{\alpha}].(\overline{\boldsymbol{\tau}}) \to \boldsymbol{\tau}'$.

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W,\gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$.

We need to show that

$$(W, \rho_1(\gamma_1(\boldsymbol{\lambda}[\overline{\alpha}](\overline{\mathbf{x} : \boldsymbol{\tau}}).\mathbf{e_1})), \rho_2(\gamma_2(\boldsymbol{\lambda}[\overline{\alpha}](\overline{\mathbf{x} : \boldsymbol{\tau}}).\mathbf{e_2}))) \in \mathcal{E}[\![\forall[\overline{\alpha}].(\overline{\boldsymbol{\tau}}) \to \boldsymbol{\tau}']\!]\rho$$

The bodies of the functions are closed, due to being the target of closure conversion, so we really must show

$$(W, \boldsymbol{\lambda}[\overline{\alpha}](\overline{\mathbf{x} : \boldsymbol{\tau}}).\mathbf{e_1}, \boldsymbol{\lambda}[\overline{\alpha}](\overline{\mathbf{x} : \boldsymbol{\tau}}).\mathbf{e_2}) \in \mathcal{E}[\![\forall[\overline{\alpha}].(\overline{\boldsymbol{\tau}}) \to \boldsymbol{\tau}']\!]\rho$$

By Lemma 7.5, it suffices to show that

$$(W, \boldsymbol{\lambda}[\overline{\alpha}](\overline{\mathbf{x} : \boldsymbol{\tau}}).\mathbf{e_1}, \boldsymbol{\lambda}[\overline{\alpha}](\overline{\mathbf{x} : \boldsymbol{\tau}}).\mathbf{e_2}) \in \mathcal{V}[\![\forall[\overline{\alpha}].(\overline{\boldsymbol{\tau}}) \to \boldsymbol{\tau}']\!]\rho.$$

Let $W' \sqsupseteq W$, $\overline{\mathrm{VR} \in \mathrm{CCValRel}}$, and $\overline{(W', \mathbf{v_1}, \mathbf{v_2}) \in \mathcal{V}[\![\boldsymbol{\tau}]\!]\rho\overline{[\alpha \mapsto \mathrm{VR}]}}$. For convenience, let $\overline{\boldsymbol{\tau_1} = \mathrm{VR}.\boldsymbol{\tau_1}}$ and $\overline{\boldsymbol{\tau_2} = \mathrm{VR}.\boldsymbol{\tau_2}}$. We need to show that

$$(W, (\boldsymbol{\lambda}[\overline{\alpha}](\overline{\mathbf{x} : \boldsymbol{\tau}}).\mathbf{e_1})\ [\overline{\boldsymbol{\tau_1}}]\ \overline{\mathbf{v_1}}, (\boldsymbol{\lambda}[\overline{\alpha}](\overline{\mathbf{x} : \boldsymbol{\tau}}).\mathbf{e_2})\ [\overline{\boldsymbol{\tau_2}}]\ \overline{\mathbf{v_2}}) \in \mathcal{E}[\![\boldsymbol{\tau}']\!]\rho\overline{[\alpha \mapsto \mathrm{VR}]}.$$

By Lemma 7.10, it suffices to show that

$$(W, \mathbf{e_1}\overline{[\boldsymbol{\tau_1}/\alpha][\mathbf{v_1}/\mathbf{x}]}, \mathbf{e_2}\overline{[\boldsymbol{\tau_2}/\alpha][\mathbf{v_2}/\mathbf{x}]}) \in \mathcal{E}[\![\boldsymbol{\tau}']\!]\rho\overline{[\alpha \mapsto \mathrm{VR}]}.$$

By definition, $W' \in \mathcal{H}[\![\Psi]\!]$, $\cdot\overline{[\alpha \mapsto \mathrm{VR}]} \in \mathcal{D}[\![\overline{\alpha}]\!]$, and $(W', \cdot\overline{[\mathbf{x} \mapsto (\mathbf{v_1}, \mathbf{v_2})]}) \in \mathcal{G}[\![\overline{\mathbf{x} : \boldsymbol{\tau}}]\!]\rho\overline{[\alpha \mapsto \mathrm{VR}]}$. Applying our assumption gives the result. □

**Lemma 9.26 (C Application)**
If $\Psi;\Delta;\Gamma \vdash \mathbf{v_1} \approx^{log}_{M+C} \mathbf{v_2} : \forall[].(\overline{\boldsymbol{\tau}}) \to \boldsymbol{\tau}'$ and $\overline{\Psi;\Delta;\Gamma \vdash \mathbf{v_1'} \approx^{log}_{M+C} \mathbf{v_2'} : \boldsymbol{\tau}}$, then $\Psi;\Delta;\Gamma \vdash \mathbf{v_1}\ []\ \overline{\mathbf{v_1'}} \approx^{log}_{M+C} \mathbf{v_2}\ []\ \overline{\mathbf{v_2'}} : \boldsymbol{\tau}'$.

**Proof**

First note that $\Psi;\Delta;\Gamma \vdash \mathbf{v_1}\ []\ \overline{\mathbf{v_1'}} : \boldsymbol{\tau}'$ and $\Psi;\Delta;\Gamma \vdash \mathbf{v_2}\ []\ \overline{\mathbf{v_2'}} : \boldsymbol{\tau}'$.

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W,\gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$(W, \rho_1(\gamma_1(\mathbf{v_1}\ []\ \overline{\mathbf{v_1'}})), \rho_2(\gamma_2(\mathbf{v_2}\ []\ \overline{\mathbf{v_2'}}))) = (W, \rho_1(\gamma_1(\mathbf{v_1}))\ []\ \overline{\rho_1(\gamma_1(\mathbf{v_1'}))}, \rho_2(\gamma_2(\mathbf{v_2}))\ []\ \overline{\rho_2(\gamma_2(\mathbf{v_2'}))}) \in \mathcal{E}[\![\boldsymbol{\tau}']\!]\rho.$$

Let $W_0 \sqsupseteq_{\mathrm{pub}} W$ and $(W_0, \mathbf{v_1}, \mathbf{v_2}) \in \mathcal{V}[\![\forall[].(\overline{\boldsymbol{\tau}}) \to \boldsymbol{\tau}']\!]\rho$. By Lemma 7.13, it suffices to show that

$$(W_0, \mathbf{v_1}\ []\ \overline{\rho_1(\gamma_1(\mathbf{v_1'}))}, \mathbf{v_2}\ []\ \overline{\rho_2(\gamma_2(\mathbf{v_2'}))}) \in \mathcal{E}[\![\boldsymbol{\tau}']\!]\rho.$$

Let $W_i \sqsupseteq_{\mathrm{pub}} W_{i-1}$ and $\overline{(W_i, \mathbf{v_{1i}'}, \mathbf{v_{2i}'}) \in \mathcal{V}[\![\boldsymbol{\tau}]\!]\rho}$. By further applications of 7.13, it suffices to show that

$$(W_n, \mathbf{v_1}\ []\ \overline{\mathbf{v_1'}}, \mathbf{v_2}\ []\ \overline{\mathbf{v_2'}}) \in \mathcal{E}[\![\boldsymbol{\tau}']\!]\rho.$$

Since $W_n \sqsupseteq W_0$ and $\overline{(W_n, \mathbf{v_1'}, \mathbf{v_2'}) \in \mathcal{V}[\![\boldsymbol{\tau}]\!]\rho}$, we can instantiate our assumption that

$$(W_0, \mathbf{v_1}, \mathbf{v_2}) \in \mathcal{V}[\![\forall[].(\overline{\boldsymbol{\tau}}) \to \boldsymbol{\tau}']\!]\rho$$

to get exactly the needed result. □

**Lemma 9.27 (C Type Application)**
If $\Psi;\Delta;\Gamma \vdash \mathsf{v_1} \approx^{log}_{M+C} \mathsf{v_2} : \forall[\beta, \overline{\alpha}].(\overline{\tau}) \rightarrow \tau'$ and $\Delta \vdash \hat{\tau}$, then

$$\Psi;\Delta;\Gamma \vdash \mathsf{v_1}[\hat{\tau}] \approx^{log}_{M+C} \mathsf{v_2}[\hat{\tau}] : \forall[\overline{\alpha}].(\overline{\tau[\hat{\tau}/\beta]}) \rightarrow \tau'[\hat{\tau}/\beta].$$

**Proof**

First note that $\Psi;\Delta;\Gamma \vdash \mathsf{v_i}[\hat{\tau}] : \forall[\overline{\alpha}].(\overline{\tau[\hat{\tau}/\beta]}) \rightarrow \tau'[\hat{\tau}/\beta]$ for $i \in \{1,2\}$.
Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W,\gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$(W, \rho_1(\gamma_1(\mathsf{v_1}[\hat{\tau}])), \rho_2(\gamma_2(\mathsf{v_2}[\hat{\tau}]))) = (W, \rho_1(\gamma_1(\mathsf{v_1}))[\rho_1(\hat{\tau})], \rho_2(\gamma_2(\mathsf{v_2}))[\rho_2(\hat{\tau})]) \in \mathcal{E}[\![\forall[\overline{\alpha}].(\overline{\tau[\hat{\tau}/\beta]}) \rightarrow \tau'[\hat{\tau}/\beta]]\!]\rho.$$

By our assumption, $(W, \rho_1(\gamma_1(\mathsf{v_1})), \rho_2(\gamma_2(\mathsf{v_2}))) \in \mathcal{E}[\![\forall[\beta,\overline{\alpha}].(\overline{\tau}) \rightarrow \tau']\!]\rho$. Let $W' \sqsupseteq_{\text{pub}} W$ and

$$(W', \mathsf{v_1}, \mathsf{v_2}) \in \mathcal{V}[\![\forall[\beta,\overline{\alpha}].(\overline{\tau}) \rightarrow \tau']\!]\rho.$$

By Lemma 7.13, it suffices to show that

$$(W', \mathsf{v_1}[\rho_1(\hat{\tau})], \mathsf{v_2}[\rho_2(\hat{\tau})]) \in \mathcal{E}[\![\forall[\overline{\alpha}].(\overline{\tau[\hat{\tau}/\beta]}) \rightarrow \tau'[\hat{\tau}/\beta]]\!]\rho.$$

Let $\text{VR} = (\rho_1(\hat{\tau}), \rho_2(\hat{\tau}), [\mathcal{V}[\![\hat{\tau}]\!]\rho])$. By Lemma 7.5 and Lemma 11.9, it suffices to show that

$$(W', \mathsf{v_1}[\rho_1(\hat{\tau})], \mathsf{v_2}[\rho_2(\hat{\tau})]) \in \mathcal{V}[\![\forall[\overline{\alpha}].(\overline{\tau[\hat{\tau}/\beta]}) \rightarrow \tau'[\hat{\tau}/\beta]]\!]\rho = \mathcal{V}[\![\forall[\overline{\alpha}].(\overline{\tau}) \rightarrow \tau']\!]\rho[\beta \mapsto \text{VR}].$$

We can reach this easily from our hypothesis that $(W', \mathsf{v_1}, \mathsf{v_2}) \in \mathcal{V}[\![\forall[\beta,\overline{\alpha}].(\overline{\tau}) \rightarrow \tau']\!]\rho$. □

**Lemma 9.28 (CM Boundary)**
If $\Psi;\Delta;\Gamma \vdash \mathsf{e_1} \approx^{log}_{M+C} \mathsf{e_2} : \tau$, then $\Psi;\Delta;\Gamma \vdash \mathcal{CM}^{\tau}\ \mathsf{e_1} \approx^{log}_{M+C} \mathcal{CM}^{\tau}\ \mathsf{e_2} : \tau^{\langle C \rangle}$.

**Proof**

Note that $\Psi;\Delta;\Gamma \vdash \mathcal{CM}^{\tau}\ \mathsf{e_1} : \tau^{\langle C \rangle}$ and $\Psi;\Delta;\Gamma \vdash \mathcal{CM}^{\tau}\ \mathsf{e_2} : \tau^{\langle C \rangle}$.
Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W,\gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. By our assumption,

$$(W, \rho_1(\gamma_1(\mathsf{e_1})), \rho_2(\gamma_2(\mathsf{e_2}))) \in \mathcal{E}[\![\tau]\!]\rho.$$

By the bridge lemma,

$$(W, \mathcal{CM}^{\rho_1(\tau)}\ \rho_1(\gamma_1(\mathsf{e_1})), \mathcal{CM}^{\rho_2(\tau)}\ \rho_2(\gamma_2(\mathsf{e_2}))) = (W, \rho_1(\gamma_1(\mathcal{CM}^{\tau}\ \mathsf{e_1})), \rho_2(\gamma_2(\mathcal{CM}^{\tau}\ \mathsf{e_2}))) \in \mathcal{E}[\![\tau^{\langle C \rangle}]\!]\rho,$$

as desired. □

**Lemma 9.29 (C Pack)**
If $\Psi;\Delta;\Gamma \vdash \mathsf{v_1} \approx^{log}_{M+C} \mathsf{v_2} : \tau[\tau'/\alpha]$, then $\Psi;\Delta;\Gamma \vdash \mathsf{pack}\ \langle \tau', \mathsf{v_1} \rangle\ \mathsf{as}\ \exists\alpha.\tau \approx^{log}_{M+C} \mathsf{pack}\ \langle \tau', \mathsf{v_2} \rangle\ \mathsf{as}\ \exists\alpha.\tau : \exists\alpha.\tau$

**Proof**

Note that $\Psi;\Delta;\Gamma \vdash \mathsf{pack}\ \langle \tau', \mathsf{v_1} \rangle\ \mathsf{as}\ \exists\alpha.\tau : \exists\alpha.\tau$ and $\Psi;\Delta;\Gamma \vdash \mathsf{pack}\ \langle \tau', \mathsf{v_2} \rangle\ \mathsf{as}\ \exists\alpha.\tau : \exists\alpha.\tau$.
Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W,\gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$(W, \rho_1(\gamma_1(\mathsf{pack}\ \langle \tau', \mathsf{v_1} \rangle\ \mathsf{as}\ \exists\alpha.\tau)), \rho_2(\gamma_2(\mathsf{pack}\ \langle \tau', \mathsf{v_2} \rangle\ \mathsf{as}\ \exists\alpha.\tau))) = (W, \mathsf{pack}\ \langle \rho_1(\tau'), \rho_1(\gamma_1(\mathsf{v_1})) \rangle\ \mathsf{as}\ \exists\alpha.\tau, \mathsf{pack}\ \langle \rho_2(\tau'), \rho_2(\gamma_2(\mathsf{v_2})) \rangle\ \mathsf{as}\ \exists\alpha.\tau) \in \mathcal{E}[\![\exists\alpha.\tau]\!]\rho.$$

Let $\text{VR} = (\rho_1(\tau'), \rho_2(\tau'), [\mathcal{V}[\![\tau']\!]\rho])$
By our assumption and Lemma 11.9,

$$(W, \rho_1(\gamma_1(\mathsf{v_1})), \rho_2(\gamma_2(\mathsf{v_2}))) \in \mathcal{E}[\![\tau[\tau'/\alpha]]\!]\rho = \mathcal{E}[\![\tau]\!]\rho[\alpha \mapsto \text{VR}].$$

Additionally, $(W, \mathbf{pack}\ \langle \rho_1(\tau'), [\cdot]\rangle\ \mathbf{as}\ \exists\alpha.\tau, \mathbf{pack}\ \langle \rho_2(\tau'), [\cdot]\rangle\ \mathbf{as}\ \exists\alpha.\tau) \in \text{ContAtom}[\tau[\tau'/\alpha]]\rho \rightsquigarrow [\exists\alpha.\tau]\rho$ holds from the premises.
We use these facts to apply Lemma 7.13 and assume $W' \sqsupseteq_{\text{pub}} W$ and $(W', \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2) \in \mathcal{V}[\![\tau]\!]\rho[\alpha \mapsto \text{VR}]$. It then suffices to show that

$$(W, \mathbf{pack}\ \langle \rho_1(\tau'), \hat{\mathbf{v}}_1\rangle\ \mathbf{as}\ \exists\alpha.\tau, \mathbf{pack}\ \langle \rho_2(\tau'), \hat{\mathbf{v}}_2\rangle\ \mathbf{as}\ \exists\alpha.\tau) \in \mathcal{E}[\![\exists\alpha.\tau]\!]\rho.$$

By Lemma 7.5, it suffices to show that

$$(W, \mathbf{pack}\ \langle \rho_1(\tau'), \hat{\mathbf{v}}_1\rangle\ \mathbf{as}\ \exists\alpha.\tau, \mathbf{pack}\ \langle \rho_2(\tau'), \hat{\mathbf{v}}_2\rangle\ \mathbf{as}\ \exists\alpha.\tau) \in \mathcal{V}[\![\exists\alpha.\tau]\!]\rho.$$

But $(W', \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2) \in \mathcal{V}[\![\tau]\!]\rho[\alpha \mapsto \text{VR}]$ is sufficient to give this. □

**Lemma 9.30 (C Unpack)**
If $\Psi; \Delta; \Gamma \vdash \mathbf{v}_1 \approx^{log}_{M+C} \mathbf{v}_2 : \exists\alpha.\tau$ and $\Psi; (\Delta, \alpha); (\Gamma, \mathbf{x} : \tau) \vdash \mathbf{e}_1 \approx^{log}_{M+C} \mathbf{e}_2 : \tau'$, then

$$\Psi; \Delta; \Gamma \vdash \mathbf{unpack}\ \langle \alpha, \mathbf{x}\rangle = \mathbf{v}_1\ \mathbf{in}\ \mathbf{e}_1 \approx^{log}_{M+C} \mathbf{unpack}\ \langle \alpha, \mathbf{x}\rangle = \mathbf{v}_2\ \mathbf{in}\ \mathbf{e}_2 : \tau'.$$

**Proof**

Note that $\Psi; \Delta; \Gamma \vdash \mathbf{unpack}\ \langle \alpha, \mathbf{x}\rangle = \mathbf{v}_1\ \mathbf{in}\ \mathbf{e}_1 : \tau'$ and $\Psi; \Delta; \Gamma \vdash \mathbf{unpack}\ \langle \alpha, \mathbf{x}\rangle = \mathbf{v}_2\ \mathbf{in}\ \mathbf{e}_2 : \tau'$.
Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$\begin{aligned}&(W, \rho_1(\gamma_1(\mathbf{unpack}\ \langle \alpha, \mathbf{x}\rangle = \mathbf{v}_1\ \mathbf{in}\ \mathbf{e}_1)), \rho_2(\gamma_2(\mathbf{unpack}\ \langle \alpha, \mathbf{x}\rangle = \mathbf{v}_2\ \mathbf{in}\ \mathbf{e}_2)))\\ =\ &(W, \mathbf{unpack}\ \langle \alpha, \mathbf{x}\rangle = \rho_1(\gamma_1(\mathbf{v}_1))\ \mathbf{in}\ \rho_1(\gamma_1(\mathbf{e}_1)), \mathbf{unpack}\ \langle \alpha, \mathbf{x}\rangle = \rho_2(\gamma_2(\mathbf{v}_2))\ \mathbf{in}\ \rho_2(\gamma_2(\mathbf{e}_2))) \in \mathcal{E}[\![\tau']\!]\rho.\end{aligned}$$

Note that $(W, \rho_1(\gamma_1(\mathbf{v}_1)), \rho_2(\gamma_2(\mathbf{v}_2))) \in \mathcal{E}[\![\exists\alpha.\tau]\!]\rho$ and

$$(W, \mathbf{unpack}\ \langle \alpha, \mathbf{x}\rangle = [\cdot]\ \mathbf{in}\ \mathbf{e}_1, \mathbf{unpack}\ \langle \alpha, \mathbf{x}\rangle = [\cdot]\ \mathbf{in}\ \mathbf{e}_2) \in \text{ContAtom}[\exists\alpha.\tau]\rho \rightsquigarrow [\tau']\rho$$

holds from the premises. We use these facts to apply Lemma 7.13 and assume $W' \sqsupseteq_{\text{pub}} W$ and $(W', \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2) \in \mathcal{V}[\![\exists\alpha.\tau]\!]\rho$. It then suffices to show that

$$(W', \mathbf{unpack}\ \langle \alpha, \mathbf{x}\rangle = \hat{\mathbf{v}}_1\ \mathbf{in}\ \rho_1(\gamma_1(\mathbf{e}_1)), \mathbf{unpack}\ \langle \alpha, \mathbf{x}\rangle = \hat{\mathbf{v}}_2\ \mathbf{in}\ \rho_2(\gamma_2(\mathbf{e}_2))) \in \mathcal{E}[\![\tau']\!]\rho.$$

By definition of $\mathcal{V}[\![\exists\alpha.\tau]\!]\rho$, $\hat{\mathbf{v}}_1 = \mathbf{pack}\ \langle \tau_1, \hat{\mathbf{v}}'_1\rangle\ \mathbf{as}\ \rho_1(\exists\alpha.\tau)$ and $\hat{\mathbf{v}}_2 = \mathbf{pack}\ \langle \tau_2, \hat{\mathbf{v}}'_2\rangle\ \mathbf{as}\ \rho_2(\exists\alpha.\tau)$, where there is some $\text{VR} \in \text{CCValRel}$ such that $\text{VR}.\tau_1 = \tau_1$, $\text{VR}.\tau_2 = \tau_2$, and $(W', \hat{\mathbf{v}}'_1, \hat{\mathbf{v}}'_2) \in \mathcal{V}[\![\tau]\!]\rho[\alpha \mapsto \text{VR}]$. By the operational semantics and by Lemma 7.10, it suffices to show that

$$(W', \rho_1(\gamma_1(\mathbf{e}_1))[\tau_1/\alpha][\hat{\mathbf{v}}'_1/\mathbf{x}], \rho_2(\gamma_2(\mathbf{e}_2))[\tau_2/\alpha][\hat{\mathbf{v}}'_2/\mathbf{x}]) \in \mathcal{E}[\![\tau']\!]\rho.$$

By our hypothesis, this follows from $W' \in \mathcal{H}[\![\Psi]\!]$, $\rho[\alpha \mapsto \text{VR}] \in \mathcal{D}[\![\Delta, \alpha]\!]$, and

$$(W', \gamma[\mathbf{x} \mapsto (\hat{\mathbf{v}}'_1, \hat{\mathbf{v}}'_2)]) \in \mathcal{G}[\![\Gamma, \mathbf{x} : \tau]\!]\rho[\alpha \mapsto \text{VR}].$$

The first two of these conditions hold immediately, and the last holds by monotonicity and since $(W', \hat{\mathbf{v}}'_1, \hat{\mathbf{v}}'_2) \in \mathcal{V}[\![\tau]\!]\rho[\alpha \mapsto \text{VR}]$. □

**Lemma 9.31 (C Assignment)**
If $\Psi; \Delta; \Gamma \vdash \mathbf{v}_1 \approx^{log}_{M+C} \mathbf{v}_2 : \mathbf{ref}\ \tau$ and $\Psi; \Delta; \Gamma \vdash \mathbf{v}'_1 \approx^{log}_{M+C} \mathbf{v}'_2 : \tau$,
then $\Psi; \Delta; \Gamma \vdash \mathbf{v}_1 := \mathbf{v}'_1 \approx^{log}_{M+C} \mathbf{v}_2 := \mathbf{v}'_2 : \mathbf{unit}$.

**Proof**

First note that $\Psi;\Delta;\Gamma \vdash \mathbf{v_1} := \mathbf{v'_1} : \mathbf{unit}$ and $\Psi;\Delta;\Gamma \vdash \mathbf{v_2} := \mathbf{v'_2} : \mathbf{unit}$.
Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W,\gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$(W, \rho_1(\gamma_1(\mathbf{v_1} := \mathbf{v'_1})), \rho_2(\gamma_2(\mathbf{v_2} := \mathbf{v'_2}))) = (W, \rho_1(\gamma_1(\mathbf{v_1})) := \rho_1(\gamma_1(\mathbf{v'_1})), \rho_2(\gamma_2(\mathbf{v_2})) := \rho_2(\gamma_2(\mathbf{v'_2}))) \in \mathcal{E}[\![\mathbf{unit}]\!]\rho$$

We begin by showing $(W, \rho_1(\gamma_1(\mathbf{v'_1})), \rho_2(\gamma_2(\mathbf{v'_2}))) \in \mathcal{E}[\![\boldsymbol{\tau}]\!]\rho$ and $(W, \rho_1(\gamma_1(\mathbf{v_1})) := [\cdot], \rho_1(\gamma_1(\mathbf{v_2})) := [\cdot]) \in \mathrm{ContAtom}[\boldsymbol{\tau}]\rho \rightsquigarrow [\mathbf{unit}]\rho$, which follow from the premises.
Using these facts, we apply Lemma 7.13, where assuming $W' \sqsupseteq_{\mathrm{pub}} W$ and $(W', \hat{\mathbf{v}}'_1, \hat{\mathbf{v}}'_2) \in \mathcal{V}[\![\boldsymbol{\tau}]\!]\rho$, it suffices to show that

$$(W', \rho_1(\gamma_1(\mathbf{v_1})) := \hat{\mathbf{v}}'_1, \rho_1(\gamma_1(\mathbf{v_2})) := \hat{\mathbf{v}}'_2) \in \mathcal{E}[\![\mathbf{unit}]\!]\rho$$

We continue by showing $(W', \rho_1(\gamma_1(\mathbf{v_1})), \rho_2(\gamma_2(\mathbf{v_2}))) \in \mathcal{E}[\![\mathbf{ref}\ \boldsymbol{\tau}]\!]\rho$ and $(W', [\cdot] := \mathbf{v'_1}, [\cdot] := \mathbf{v'_2}) \in \mathrm{ContAtom}[\mathbf{ref}\ \boldsymbol{\tau}]\rho \rightsquigarrow [\mathbf{unit}]\rho$, which follow from monotonicity (7.6) and the premises.
We use these facts and again apply Lemma 7.13. Assuming $W'' \sqsupseteq_{\mathrm{pub}} W'$ and $(W'', \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2) \in \mathcal{V}[\![\mathbf{ref}\ \boldsymbol{\tau}]\!]\rho$, we must show that

$$(W'', \hat{\mathbf{v}}_1 := \hat{\mathbf{v}}'_1, \hat{\mathbf{v}}_2 := \hat{\mathbf{v}}'_2) \in \mathcal{E}[\![\mathbf{unit}]\!]\rho$$

We assume $(W'', K_1, K_2) \in \mathcal{K}[\![\mathbf{ref}\ \boldsymbol{\tau}]\!]\rho$ and must show

$$(W'', K_1[\hat{\mathbf{v}}_1 := \hat{\mathbf{v}}'_1], K_2[\hat{\mathbf{v}}_2 := \hat{\mathbf{v}}'_2]) \in \mathcal{O}$$

Unfolding this definition we assume $(H_1, H_2) : W''$ and must show co-termination or lack of it for the expressions in question. We proceed by considering the 4 possible cases of $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$:
For each case start by applying Lemma **??** with $W'' \sqsupseteq W'', W''.k \leq W''.k+1, (H_1,H_2) : W''$

- $\hat{\mathbf{v}}_1 = \boldsymbol{\ell}_1$ and $\hat{\mathbf{v}}_2 = \boldsymbol{\ell}_2$:
  Let $H'_1 = H_1[\boldsymbol{\ell}_1 \mapsto \hat{\mathbf{v}}'_1]$ and $H'_2 = H_2[\boldsymbol{\ell}_2 \mapsto \hat{\mathbf{v}}'_2]$.
  By $(W'', \hat{\mathbf{v}}'_1, \hat{\mathbf{v}}'_2) \in \mathcal{V}[\![\boldsymbol{\tau}]\!]\rho$ it follows that $(H'_1, H'_2) : W''$.
  By the operational semantics

  $$\langle H_1 \mid \boldsymbol{\ell}_1 := \hat{\mathbf{v}}'_1 \rangle \longmapsto^1 \langle H_1[\boldsymbol{\ell}_1 \mapsto \hat{\mathbf{v}}'_1] \mid \boldsymbol{\ell}_1 \rangle \ \wedge\ \langle H_2 \mid \boldsymbol{\ell}_2 := \hat{\mathbf{v}}'_2 \rangle \longmapsto^1 \langle H_2[\boldsymbol{\ell}_2 \mapsto \hat{\mathbf{v}}'_2] \mid \boldsymbol{\ell}_2 \rangle$$

  We now must show $\langle H'_1 \mid K_1[\boldsymbol{\ell}_1] \rangle \Downarrow, \langle H'_2 \mid K_2[\boldsymbol{\ell}_2] \rangle \Downarrow$ or that they keep running after $W''.k$ steps. This is obtained by turning the premise $(W'', \boldsymbol{\ell}_1, \boldsymbol{\ell}_2) \in \mathcal{V}[\![\mathbf{ref}\ \boldsymbol{\tau}]\!]\rho$ into $(W'', \boldsymbol{\ell}_1, \boldsymbol{\ell}_2) \in \mathcal{E}[\![\mathbf{ref}\ \boldsymbol{\tau}]\!]\rho$ by Lemma 7.5 and then instantiating the result with the $K_1, K_2, H'_1, H'_2$.
- $\hat{\mathbf{v}}_1 = \boldsymbol{\ell}_1$ and $\hat{\mathbf{v}}_2 = \mathcal{CM}^{\rho_2(\mathsf{ref}\,\tau)}\, \ell_2$:
  Let $H'_1 = H_1[\boldsymbol{\ell}_1 \mapsto \hat{\mathbf{v}}'_1]$ and $H'_2 = H_2[\ell_2 \mapsto \hat{\mathrm{v}}''_2]$, where $\hat{\mathrm{v}}''_2 = {}^{\rho_2(\tau)}\mathbf{MC}(\hat{\mathbf{v}}'_2)$
  Since $\Psi;\Delta;\Gamma \vdash \mathcal{CM}^{\rho_2(\tau)}\, \ell_2 : \boldsymbol{\tau}$ tells us that $\exists \tau\ .\ \tau^{\langle \mathcal{C} \rangle} = \boldsymbol{\tau}$, let us use $\tau^{\langle \mathcal{C} \rangle}$ in place of $\boldsymbol{\tau}$.
  In order to show $(H'_1, H'_2) : W''$ we must prove $(W'', \hat{\mathbf{v}}'_1, \hat{\mathrm{v}}''_2) \in \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}, \tau]\!]\rho$:
  - $(W'', {}^{\rho_1(\tau)}\mathbf{MC}(\hat{\mathbf{v}}'_1), \hat{\mathrm{v}}''_2) \in \mathcal{V}[\![\tau]\!]\rho$: follows by the Bridge Lemma (9.1).
  - $(W'', \hat{\mathbf{v}}'_1, \mathbf{CM}^{\rho_2(\tau)}(\hat{\mathrm{v}}''_2)) \in \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}]\!]\rho$: follows by the Boundary Cancellation Lemma (8.5)

  Now, by the operational semantics

  $$\langle H_1 \mid \boldsymbol{\ell}_1 := \hat{\mathbf{v}}'_1 \rangle \longmapsto^1 \langle H_1[\boldsymbol{\ell}_1 \mapsto \hat{\mathbf{v}}'_1] \mid \boldsymbol{\ell}_1 \rangle$$

  $$\langle H_2 \mid \mathcal{CM}^{\rho_2(\mathsf{ref}\,\tau)}\, \ell_2 := \hat{\mathbf{v}}'_2 \rangle \longmapsto^1 \langle H_2 \mid \mathcal{CM}^{\rho_2(\mathsf{ref}\,\tau)}\, (\ell_2 := {}^{\rho_2(\tau)}\mathcal{MC}\, \hat{\mathbf{v}}'_2) \rangle \longmapsto^2 \langle H_2[\ell_2 \mapsto \hat{\mathrm{v}}''_2] \mid \mathcal{CM}^{\rho_2(\mathsf{ref}\,\tau)}\, \ell_2 \rangle$$

  We now must show $\langle H'_1 \mid K_1[\boldsymbol{\ell}_1] \rangle \Downarrow, \langle H'_2 \mid K_2[\mathcal{CM}^{\rho_2(\mathsf{ref}\,\tau)}\, \ell_2] \rangle \Downarrow$ or that they keep running after $W''.k$ steps. This is wrapped up in a similar fashion to the case above.
- $\hat{\mathbf{v}}_1 = \mathcal{CM}^{\rho_1(\mathsf{ref}\,\tau)}\, \ell_1$ and $\hat{\mathbf{v}}_2 = \boldsymbol{\ell}_2$:
  Let $H'_1 = H_1[\ell_1 \mapsto \hat{\mathrm{v}}''_1]$ and $H'_2 = H_2[\boldsymbol{\ell}_2 \mapsto \hat{\mathbf{v}}'_2]$, where $\hat{\mathrm{v}}''_1 = {}^{\rho_1(\tau)}\mathbf{MC}(\hat{\mathbf{v}}'_1)$
  Since $\Psi;\Delta;\Gamma \vdash \mathcal{CM}^{\rho_1(\tau)}\, \ell_1 : \boldsymbol{\tau}$ tells us that $\exists \tau\ .\ \tau^{\langle \mathcal{C} \rangle} = \boldsymbol{\tau}$, let us use $\tau^{\langle \mathcal{C} \rangle}$ in place of $\boldsymbol{\tau}$.
  In order to show $(H'_1, H'_2) : W''$ we must prove $(W'', \hat{\mathrm{v}}''_1, \hat{\mathbf{v}}'_2) \in \mathcal{V}[\![\tau, \tau^{\langle \mathcal{C} \rangle}]\!]\rho$:
  - $(W'', \hat{\mathrm{v}}''_1, {}^{\rho_2(\tau)}\mathbf{MC}(\hat{\mathbf{v}}'_2)) \in \mathcal{V}[\![\tau]\!]\rho$: follows by the Bridge Lemma (9.1).
  - $(W'', \mathcal{CM}^{\rho_1(\tau)}\, \hat{\mathrm{v}}''_1, \hat{\mathbf{v}}'_2) \in \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}]\!]\rho$: follows by the Left Boundary Cancellation Lemma (8.6).

Now, by the operational semantics

$$\langle H_1 \mid \mathcal{CM}^{\rho_1(\mathsf{ref}\,\tau)}\,\ell_1 := \hat{\mathbf{v}}_1'\rangle \longmapsto^1 \langle H_1 \mid \mathcal{CM}^{\rho_1(\mathsf{ref}\,\tau)}\,(\ell_1 := {}^{\rho_1(\tau)}\mathcal{MC}\,\hat{\mathbf{v}}_1')\rangle \longmapsto^2 \langle H_1[\ell_1 \mapsto \hat{\mathsf{v}}_1''] \mid \mathcal{CM}^{\rho_1(\mathsf{ref}\,\tau)}\,\ell_1\rangle$$

$$\langle H_2 \mid \ell_2 := \hat{\mathbf{v}}_2'\rangle \longmapsto^1 \langle H_2[\ell_2 \mapsto \hat{\mathbf{v}}_2'] \mid \ell_2\rangle$$

We now must show $\langle H_1' \mid K_1[\mathcal{CM}^{\rho_1(\mathsf{ref}\,\tau)}\,\ell_1]\rangle \Downarrow, \langle H_2' \mid K_2[\ell_2]\rangle \Downarrow$ or that they keep running after $W''.k$ steps. This is wrapped up in a similar fashion to the case above.

- $\hat{\mathbf{v}}_1 = \mathcal{CM}^{\rho_1(\mathsf{ref}\,\tau)}\,\ell_1$ and $\hat{\mathbf{v}}_2 = \mathcal{CM}^{\rho_2(\mathsf{ref}\,\tau)}\,\ell_2$:
  Let $H_1' = H_1[\ell_1 \mapsto \hat{\mathsf{v}}_1'']$ and $H_2' = H_2[\ell_2 \mapsto \hat{\mathsf{v}}_2'']$
  where $\hat{\mathsf{v}}_1'' = {}^{\rho_1(\tau)}\mathbf{MC}(\hat{\mathbf{v}}_1')$ and $\hat{\mathsf{v}}_2'' = {}^{\rho_2(\tau)}\mathbf{MC}(\hat{\mathbf{v}}_2')$
  Since $\Psi; \Delta; \Gamma \vdash \mathcal{CM}^{\rho_1(\tau)}\,\ell_1 : \boldsymbol{\tau}$ tells us that $\exists \tau\ .\ \tau^{\langle \mathcal{C}\rangle} = \boldsymbol{\tau}$, let us use $\tau^{\langle \mathcal{C}\rangle}$ in place of $\boldsymbol{\tau}$.
  By applying the Bridge Lemma (9.1) to $(W'', \hat{\mathbf{v}}_1', \hat{\mathbf{v}}_2') \in \mathcal{V}[\![\tau^{\langle \mathcal{C}\rangle}]\!]\rho$ we get that $(H_1', H_2') : W''$.
  By the operational semantics

$$\langle H_1 \mid \mathcal{CM}^{\rho_1(\mathsf{ref}\,\tau)}\,\ell_1 := \hat{\mathbf{v}}_1'\rangle \longmapsto^1 \langle H_1 \mid \mathcal{CM}^{\rho_1(\mathsf{ref}\,\tau)}\,(\ell_1 := {}^{\rho_1(\tau)}\mathcal{MC}\,\hat{\mathbf{v}}_1')\rangle \longmapsto^2 \langle H_1[\ell_1 \mapsto \hat{\mathsf{v}}_1''] \mid \mathcal{CM}^{\rho_1(\mathsf{ref}\,\tau)}\,\ell_1\rangle$$

$$\langle H_2 \mid \mathcal{CM}^{\rho_2(\mathsf{ref}\,\tau)}\,\ell_2 := \hat{\mathbf{v}}_2'\rangle \longmapsto^1 \langle H_2 \mid \mathcal{CM}^{\rho_2(\mathsf{ref}\,\tau)}\,(\ell_2 := {}^{\rho_2(\tau)}\mathcal{MC}\,\hat{\mathbf{v}}_2')\rangle \longmapsto^2 \langle H_2[\ell_2 \mapsto \hat{\mathsf{v}}_2''] \mid \mathcal{CM}^{\rho_2(\mathsf{ref}\,\tau)}\,\ell_2\rangle$$

  We now must show $\langle H_1' \mid K_1[\mathcal{CM}^{\rho_2(\mathsf{ref}\,\tau)}\,\ell_1]\rangle \Downarrow, \langle H_2' \mid K_2[\mathcal{CM}^{\rho_2(\mathsf{ref}\,\tau)}\,\ell_2]\rangle \Downarrow$ or that they keep running after $W''.k$ steps. This is wrapped up in a similar fashion to the case above.

□

**Lemma 9.32 (C Dereference)**
If $\Psi; \Delta; \Gamma \vdash \mathbf{v}_1 \approx^{log}_{M+C} \mathbf{v}_2 : \mathbf{ref}\,\boldsymbol{\tau}$, then $\Psi; \Delta; \Gamma \vdash \mathbf{!v}_1 \approx^{log}_{M+C} \mathbf{!v}_2 : \boldsymbol{\tau}$.

**Proof**

First note that $\Psi; \Delta; \Gamma \vdash \mathbf{!v}_1 : \boldsymbol{\tau}$ and $\Psi; \Delta; \Gamma \vdash \mathbf{!v}_2 : \boldsymbol{\tau}$.

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$(W, \rho_1(\gamma_1(\mathbf{!v}_1)), \rho_2(\gamma_2(\mathbf{!v}_2))) = (W, \mathbf{!}\rho_1(\gamma_1(\mathbf{v}_1)), \mathbf{!}\rho_2(\gamma_2(\mathbf{v}_2))) \in \mathcal{E}[\![\boldsymbol{\tau}]\!]\rho$$

We begin by showing $(W, \rho_1(\gamma_1(\mathbf{v}_1)), \rho_2(\gamma_2(\mathbf{v}_2))) \in \mathcal{E}[\![\mathbf{ref}\,\boldsymbol{\tau}]\!]\rho$, $(W, \mathbf{!}[\cdot], \mathbf{!}[\cdot]) \in \text{ContAtom}[\mathbf{ref}\,\boldsymbol{\tau}]\rho \rightsquigarrow [\boldsymbol{\tau}]\rho$.
Using these facts, we applying Lemma 7.13.
Assuming $W' \sqsupseteq_{\text{pub}} W$ and $(W', \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2) \in \mathcal{V}[\![\mathbf{ref}\,\boldsymbol{\tau}]\!]\rho$ it suffices to show that

$$(W', \mathbf{!}\hat{\mathbf{v}}_1, \mathbf{!}\hat{\mathbf{v}}_2) \in \mathcal{E}[\![\boldsymbol{\tau}]\!]\rho$$

We consider the 4 cases of $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$:
For each case start by unfolding down past the $\mathcal{O}$ relation to assume $(W', K_1, K_2) \in \mathcal{K}[\![\boldsymbol{\tau}]\!]\rho$ and $(H_1, H_2) : W'$. We must show $\langle H_1 \mid K_1[\mathbf{v}_1]\rangle \Downarrow \wedge \langle H_2 \mid K_2[\mathbf{v}_2]\rangle \Downarrow$ or running$(W'k, \langle H_1 \mid K_1[\mathbf{v}_1]\rangle) \wedge$ running$(W'.k, \langle H_2 \mid K_2[\mathbf{v}_2]\rangle)$ and continue by applying Lemma **??** and instantiating it with $W' \sqsupseteq W'$, $H_1' = H_1$, and $H_2' = H_2$.

- $\hat{\mathbf{v}}_1 = \boldsymbol{\ell}_1$ and $\hat{\mathbf{v}}_2 = \boldsymbol{\ell}_2$:
  The operational semantics give us

$$\langle H_1 \mid \mathbf{!}\boldsymbol{\ell}_1\rangle \longmapsto^1 \langle H_1 \mid \mathbf{v}_1'\rangle \;\wedge\; \langle H_2 \mid \mathbf{!}\boldsymbol{\ell}_2\rangle \longmapsto^1 \langle H_2 \mid \mathbf{v}_2'\rangle$$

  where $H_1(\boldsymbol{\ell}_1) = \mathbf{v}_1'$ and $H_2(\boldsymbol{\ell}_2) = \mathbf{v}_2'$
  We need to show $\langle H_1 \mid K_1[\mathbf{v}_1']\rangle \Downarrow, \langle H_2 \mid K_2[\mathbf{v}_2']\rangle \Downarrow$ or that they keep running after $W'.k$ steps. This follows by instantiating $(W', \boldsymbol{\ell}_1, \boldsymbol{\ell}_2) \in \mathcal{V}[\![\mathbf{ref}\,\boldsymbol{\tau}]\!]\rho$ with $W'$ to get $(W', \mathbf{v}_1', \mathbf{v}_2') \in \mathcal{V}[\![\boldsymbol{\tau}]\!]\rho$. By Lemma 7.5 this becomes $(W', \mathbf{v}_1', \mathbf{v}_2') \in \mathcal{E}[\![\boldsymbol{\tau}]\!]\rho$, which we can instantiate with $K_1, K_2, H_1, H_2$.

- $\hat{\mathbf{v}}_1 = \boldsymbol{\ell}_1$ and $\hat{\mathbf{v}}_2 = \mathcal{CM}^{\rho_2(\mathsf{ref}\ \tau)}\ \ell_2$:
  The operational semantics give us

  $$\langle H_1 \mid !\boldsymbol{\ell}_1 \rangle \longmapsto^1 \langle H_1 \mid \mathbf{v}'_1 \rangle \ \wedge\ \langle H_2 \mid !\mathcal{CM}^{\rho_2(\mathsf{ref}\ \tau)}\ \ell_2 \rangle \longmapsto^1 \langle H_2 \mid \mathcal{CM}^{\rho_2(\tau)}\ !\ell_2 \rangle \longmapsto^2 \langle H_2 \mid \mathbf{v}''_2 \rangle$$

  where $H_1(\boldsymbol{\ell}_1) = \mathbf{v}'_1$, $H_2(\ell_2) = \mathsf{v}'_2$ and $\mathbf{v}''_2 = \mathbf{CM}^{\tau}(\mathsf{v}'_2)$
  We need to show $\langle H_1 \mid K_1[\mathbf{v}'_1] \rangle \Downarrow, \langle H_2 \mid K_2[\mathbf{v}''_2] \rangle \Downarrow$ or that they keep running after $W'.k$ steps.
  Note that $\Psi; \Delta; \Gamma \vdash !\mathcal{CM}^{\rho_2(\mathsf{ref}\ \tau)}\ \ell_2 : \boldsymbol{\tau}$ tells us that $\exists \tau\ .\ \tau^{\langle \mathcal{C} \rangle} = \boldsymbol{\tau}$, so let us use $\tau^{\langle \mathcal{C} \rangle}$ in place of $\boldsymbol{\tau}$
  We contintue by instantiating $(W', \boldsymbol{\ell}_1, \mathcal{CM}^{\rho_2(\mathsf{ref}\ \tau)}\ \ell_2) \in \mathcal{V}[\![\mathsf{ref}\ \tau^{\langle \mathcal{C} \rangle}]\!]\rho$ with $W'$ to give us $(W', \mathbf{v}'_1, \mathsf{v}'_2) \in \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}, \tau]\!]\rho$. By $\mathbf{v}''_2 = \mathbf{CM}^{\rho_2(\tau)}(\mathsf{v}'_2)$ and the definition of $\mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}, \tau]\!]\rho$, we have $(W', \mathbf{v}'_1, \mathbf{v}''_2) \in \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}]\!]\rho$, Lemma 7.5 then gives us $(W', \mathbf{v}'_1, \mathbf{v}''_2) \in \mathcal{E}[\![\tau^{\langle \mathcal{C} \rangle}]\!]\rho$ which we can instantiate with the proper continuations and heaps for the desired result.
- $\hat{\mathbf{v}}_1 = \mathcal{CM}^{\rho_1(\mathsf{ref}\ \tau)}\ \ell_1$ and $\hat{\mathbf{v}}_2 = \boldsymbol{\ell}_2$:
  The operational semantics give us

  $$\langle H_1 \mid !\mathcal{CM}^{\rho_1(\mathsf{ref}\ \tau)}\ \ell_1 \rangle \longmapsto^1 \langle H_1 \mid \mathcal{CM}^{\rho_1(\tau)}\ !\ell_1 \rangle \longmapsto^2 \langle H_1 \mid \mathbf{v}''_1 \rangle \ \wedge\ \langle H_2 \mid !\boldsymbol{\ell}_2 \rangle \longmapsto^1 \langle H_2 \mid \mathbf{v}'_2 \rangle$$

  where $H_1(\ell_1) = \mathsf{v}'_1$, $H_2(\boldsymbol{\ell}_2) = \mathbf{v}'_2$ and $\mathbf{v}''_1 = \mathbf{CM}^{\rho_1(\tau)}(\mathsf{v}'_1)$
  We need to show $\langle H_1 \mid K_1[\mathbf{v}''_1] \rangle \Downarrow, \langle H_2 \mid K_2[\mathbf{v}'_2] \rangle \Downarrow$ or that they keep running after $W'.k$ steps.
  Since $\Psi; \Delta; \Gamma \vdash !\mathcal{CM}^{\rho_1(\mathsf{ref}\ \tau)}\ \ell_1 : \boldsymbol{\tau}$ tells us that $\exists \tau\ .\ \tau^{\langle \mathcal{C} \rangle} = \boldsymbol{\tau}$, let us use $\tau^{\langle \mathcal{C} \rangle}$ in place of $\boldsymbol{\tau}$
  This follows by instantiating $(W', \mathcal{CM}^{\rho_1(\mathsf{ref}\ \tau)}\ \ell_1, \boldsymbol{\ell}_2) \in \mathcal{V}[\![\mathsf{ref}\ \tau^{\langle \mathcal{C} \rangle}]\!]\rho$ with $W'$, giving us $(W', \mathsf{v}'_1, \mathbf{v}'_2) \in \mathcal{V}[\![\tau, \tau^{\langle \mathcal{C} \rangle}]\!]\rho$.
  From $\mathbf{v}''_1 = \mathbf{CM}^{\rho_1(\tau)}(\mathsf{v}'_1)$ and the definition of $\mathcal{V}[\![\tau, \tau^{\langle \mathcal{C} \rangle}]\!]\rho$ we have $(W', \mathbf{v}''_1, \mathbf{v}'_2) \in \mathcal{V}[\![\boldsymbol{\tau}]\!]\rho$. Lemma 7.5 turns this into $(W', \mathbf{v}''_1, \mathbf{v}'_2) \in \mathcal{E}[\![\boldsymbol{\tau}]\!]\rho$. We wrap up in a similar fashion to the case above.
- $\hat{\mathbf{v}}_1 = \mathcal{CM}^{\rho_1(\mathsf{ref}\ \tau)}\ \ell_1$ and $\hat{\mathbf{v}}_2 = \mathcal{CM}^{\rho_2(\mathsf{ref}\ \tau)}\ \ell_2$:
  The operational semantics give us

  $$\langle H_1 \mid !\mathcal{CM}^{\rho_1(\mathsf{ref}\ \tau)}\ \ell_1 \rangle \longmapsto^1 \langle H_1 \mid \mathcal{CM}^{\rho_1(\tau)}\ !\ell_1 \rangle \longmapsto^2 \langle H_1 \mid \mathbf{v}''_1 \rangle$$

  $$\langle H_2 \mid !\mathcal{CM}^{\rho_2(\mathsf{ref}\ \tau)}\ \ell_2 \rangle \longmapsto^1 \langle H_2 \mid \mathcal{CM}^{\rho_2(\tau)}\ !\ell_2 \rangle \longmapsto^2 \langle H_2 \mid \mathbf{v}''_2 \rangle$$

  where $H_1(\ell_1) = \mathsf{v}'_1$, $H_2(\ell_2) = \mathsf{v}'_2$, $\mathbf{v}''_1 = \mathbf{CM}^{\rho_1(\tau)}(\mathsf{v}'_1)$, and $\mathbf{v}''_2 = \mathbf{CM}^{\rho_2(\tau)}(\mathsf{v}'_2)$
  We need to show $\langle H_1 \mid K_1[\mathbf{v}''_1] \rangle \Downarrow, \langle H_2 \mid K_2[\mathbf{v}''_2] \rangle \Downarrow$ or that they keep running after $W'.k$ steps.
  Since $\Psi; \Delta; \Gamma \vdash !\mathcal{CM}^{\rho_2(\mathsf{ref}\ \tau)}\ \ell_2 : \boldsymbol{\tau}$ tells us that $\exists \tau\ .\ \tau^{\langle \mathcal{C} \rangle} = \boldsymbol{\tau}$, let us use $\tau^{\langle \mathcal{C} \rangle}$ in place of $\boldsymbol{\tau}$
  By instantiating $(W', \mathcal{CM}^{\rho_1(\mathsf{ref}\ \tau)}\ \ell_1, \mathcal{CM}^{\rho_2(\mathsf{ref}\ \tau)}\ \ell_2) \in \mathcal{V}[\![\mathsf{ref}\ \tau^{\langle \mathcal{C} \rangle}]\!]\rho$ with $W'$ gives us $(W', \mathsf{v}'_1, \mathsf{v}'_2) \in \mathcal{V}[\![\tau]\!]\rho$.
  Applying the Bridge Lemma (9.1) provides us with $(W', \mathbf{v}''_1, \mathbf{v}''_2) \in \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}]\!]\rho$. We wrap up with Lemma 7.5 and plugging in the correct continuations and heaps.

□

We omit proofs of the remaining compatibility lemmas for C, as they are identical to the proofs of the corresponding M compatibility lemmas.

**Lemma 9.33 (C Variable)**
If $\mathbf{x} : \boldsymbol{\tau} \in \Gamma$, then $\Psi; \Delta; \Gamma \vdash \mathbf{x} \approx^{log}_{M+C} \mathbf{x} : \boldsymbol{\tau}$.

**Lemma 9.34 (C Unit)**
$\Psi; \Delta; \Gamma \vdash () \approx^{log}_{M+C} () : \mathbf{unit}$

**Lemma 9.35 (C Int)**
$\Psi; \Delta; \Gamma \vdash \mathbf{n} \approx^{log}_{M+C} \mathbf{n} : \mathbf{int}$

**Lemma 9.36 (C Primitive)**
If $\Psi; \Delta; \Gamma \vdash \mathbf{v}_1 \approx^{log}_{M+C} \mathbf{v}_2 : \mathbf{int}$ and $\Psi; \Delta; \Gamma \vdash \mathbf{v}'_1 \approx^{log}_{M+C} \mathbf{v}'_2 : \mathbf{int}$, then $\Psi; \Delta; \Gamma \vdash \mathbf{v}_1\ \mathbf{p}\ \mathbf{v}'_1 \approx^{log}_{M+C} \mathbf{v}_2\ \mathbf{p}\ \mathbf{v}'_2 : \mathbf{int}$.

**Lemma 9.37 (C If0)**
If $\Psi;\Delta;\Gamma \vdash \mathbf{v_1} \approx^{log}_{M+C} \mathbf{v_2} : \mathbf{int}$, $\Psi;\Delta;\Gamma \vdash \mathbf{e_1} \approx^{log}_{M+C} \mathbf{e_2} : \boldsymbol{\tau}$, and $\Psi;\Delta;\Gamma \vdash \mathbf{e_1'} \approx^{log}_{M+C} \mathbf{e_2'} : \boldsymbol{\tau}$, then

$$\Psi;\Delta;\Gamma \vdash \mathbf{if0}\ \mathbf{v_1}\ \mathbf{e_1}\ \mathbf{e_1'} \approx^{log}_{M+C} \mathbf{if0}\ \mathbf{v_2}\ \mathbf{e_2}\ \mathbf{e_2'} : \boldsymbol{\tau}.$$

**Lemma 9.38 (C Fold)**
If $\Psi;\Delta;\Gamma \vdash \mathbf{v_1} \approx^{log}_{M+C} \mathbf{v_2} : \boldsymbol{\tau}[\boldsymbol{\mu\alpha.\tau}/\boldsymbol{\alpha}]$, then $\Psi;\Delta;\Gamma \vdash \mathbf{fold}_{\boldsymbol{\mu\alpha.\tau}}\ \mathbf{v_1} \approx^{log}_{M+C} \mathbf{fold}_{\boldsymbol{\mu\alpha.\tau}}\ \mathbf{v_2} : \boldsymbol{\mu\alpha.\tau}$

**Lemma 9.39 (C Unfold)**
If $\Psi;\Delta;\Gamma \vdash \mathbf{v_1} \approx^{log}_{M+C} \mathbf{v_2} : \boldsymbol{\mu\alpha.\tau}$, then $\Psi;\Delta;\Gamma \vdash \mathbf{unfold}\ \mathbf{v_1} \approx^{log}_{M+C} \mathbf{unfold}\ \mathbf{v_2} : \boldsymbol{\tau}[\boldsymbol{\mu\alpha.\tau}/\boldsymbol{\alpha}]$.

**Lemma 9.40 (C Tuple)**
If $\Psi;\Delta;\Gamma \vdash \mathbf{v_1} \approx^{log}_{M+C} \mathbf{v_2} : \boldsymbol{\tau}$, then $\Psi;\Delta;\Gamma \vdash \langle\overline{\mathbf{v_1}}\rangle \approx^{log}_{M+C} \langle\overline{\mathbf{v_2}}\rangle : \langle\overline{\boldsymbol{\tau}}\rangle$

**Lemma 9.41 (C Reference)**
If $\Psi(\boldsymbol{\ell_1}) = \boldsymbol{\tau}$ and $\Psi(\boldsymbol{\ell_2}) = \boldsymbol{\tau}$, then $\Psi;\Delta;\Gamma \vdash \boldsymbol{\ell_1} \approx^{log}_{M+C} \boldsymbol{\ell_2} : \mathbf{ref}\ \boldsymbol{\tau}$.

**Lemma 9.42 (C New)**
If $\Psi;\Delta;\Gamma \vdash \mathbf{v_1} \approx^{log}_{M+C} \mathbf{v_2} : \boldsymbol{\tau}$, then $\Psi;\Delta;\Gamma \vdash \mathbf{new}\ \mathbf{v_1} \approx^{log}_{M+C} \mathbf{new}\ \mathbf{v_2} : \mathbf{ref}\ \boldsymbol{\tau}$.

**Lemma 9.43 (C Projection)**
If $\Psi;\Delta;\Gamma \vdash \mathbf{v_1} \approx^{log}_{M+C} \mathbf{v_2} : \langle\overline{\boldsymbol{\tau}}\rangle$, then $\Psi;\Delta;\Gamma \vdash \boldsymbol{\pi}_\mathbf{i}(\mathbf{v_1}) \approx^{log}_{M+C} \boldsymbol{\pi}_\mathbf{i}(\mathbf{v_2}) : \boldsymbol{\tau}_\mathbf{i}$.

**Lemma 9.44 (C Let)**
If $\Psi;\Delta;\Gamma \vdash \mathbf{e_1} \approx^{log}_{M+C} \mathbf{e_1'} : \boldsymbol{\tau_1}$ and $\Psi;\Delta;\Gamma, \mathbf{x} : \boldsymbol{\tau_1} \vdash \mathbf{e_2} \approx^{log}_{M+C} \mathbf{e_2'} : \boldsymbol{\tau_2}$,
then $\Psi;\Delta;\Gamma \vdash \mathbf{let}\ \mathbf{x} = \mathbf{e_1}\ \mathbf{in}\ \mathbf{e_2} \approx^{log}_{M+C} \mathbf{let}\ \mathbf{x} = \mathbf{e_1'}\ \mathbf{in}\ \mathbf{e_2'} : \boldsymbol{\tau_2}$.

## 9.3 Fundamental Property and Soundness w.r.t Contextual Equivalence

**Lemma 9.45 (Fundamental Property)**
If $\Psi;\Delta;\Gamma \vdash e : \tau$, then $\Psi;\Delta;\Gamma \vdash e \approx^{log}_{M+C} e : \tau$

**Proof**

By induction on the typing derivation and using the compatibility lemmas. □

**Lemma 9.46 (Weakening)**
If $\Psi;\Delta;\Gamma \vdash e_1 \approx^{log}_{M+C} e_2 : \tau$ and $\Psi \subseteq \Psi'$, $\Delta \subseteq \Delta'$, $\Gamma \subseteq \Gamma'$, then $\Psi';\Delta';\Gamma' \vdash e_1 \approx^{log}_{M+C} e_2 : \tau$.

**Proof**

Let $W \in \mathcal{H}[\![\Psi']\!]$, $\rho' \in \mathcal{D}[\![\Delta']\!]$, and $(W, \gamma') \in \mathcal{G}[\![\Gamma']\!]\rho$.

Let $\rho = \rho'|_\Delta$ and $\gamma = \gamma'|_\Gamma$. Note that $W \in \mathcal{H}[\![\Psi]\!]$ and $\rho \in \mathcal{D}[\![\Delta]\!]$ immediately. By our hypothesis, it suffices to show that $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. Clearly, $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho'$. Since the free type variables of $\Gamma$ are all in $\Delta$, $\mathcal{G}[\![\Gamma]\!]\rho' = \mathcal{G}[\![\Gamma]\!]\rho$, so we are done. □

**Lemma 9.47 (Congruence)**
If $\vdash C : (\Psi;\Delta;\Gamma \vdash \tau) \leadsto (\Psi';\Delta';\Gamma' \vdash \tau')$, $\Psi;\Delta;\Gamma \vdash e_1 \approx^{log}_{M+C} e_2 : \tau$, and we can plug the (expression or value) hole in $C$ with $e_1$ and $e_2$ to get valid expressions—i.e., $\Psi';\Delta';\Gamma' \vdash C[e_1] : \tau'$ and $\Psi';\Delta';\Gamma' \vdash C[e_2] : \tau'$— then $\Psi';\Delta';\Gamma' \vdash C[e_1] \approx^{log}_{M+C} C[e_2] : \tau'$.

**Proof**

The proof is by induction on the type derivation for $C$, using Lemma 9.46 for the cases where $C$ is empty, and the compatibility lemmas for all other cases. □

**Lemma 9.48 (Canonical World)**
If $\vdash H : \Psi$, then for any $k$, $\exists W.\ W.k = k\ \wedge\ W \in \mathcal{H}[\![\Psi]\!]\ \wedge\ (H, H) : W$.

**Proof**

Say that $\Psi = \ell_1 : \tau_1, \ldots, \ell_n : \tau_n$
For $1 \leq i \leq n$ let

$$\theta_i = (\bullet, \{\bullet\}, \{\}, \{\}, \lambda s.\{(W', H_1, H_2) \in \text{HeapAtom}_k \mid (W', H_1(\ell_i), H_2(\ell_i)) \in \mathcal{V}[\![\tau_i]\!]\emptyset\}, \lambda s.\{(\ell_i, \ell_i)\})$$

We construct

$$W = (k, \Psi, \Psi, (\theta_1, \ldots, \theta_n)).$$

We need to show the following:

- For each $i$, $(W, \ell_i, \ell_i) \in \mathcal{V}[\![ref\, \tau_i]\!]\emptyset$,
- $(H, H) : W$.

The first condition follows directly from the definitions. The second condition amounts to showing that $(\triangleright W, H(\ell_i), H(\ell_i)) \in \mathcal{V}[\![\tau_i]\!]\emptyset$. This follows from the Fundamental Property for heap values. □

**Lemma 9.49 (Adequacy)**
If $\Psi; \cdot; \cdot \vdash e_1 \approx^{log}_{M+C} e_2 : \tau$, $\vdash H : \Psi$, then $\langle H \mid e_1 \rangle \Downarrow$ if and only if $\langle H \mid e_2 \rangle \Downarrow$.

**Proof**

We show that $\langle H \mid e_1 \rangle \Downarrow$ implies $\langle H \mid e_2 \rangle \Downarrow$, and the converse holds by an identical argument.

Suppose $\langle H \mid e_1 \rangle \Downarrow^k$. By Lemma 9.48, there is some $W \in \mathcal{H}[\![\Psi]\!]$ such that $(H, H) : W$ and $W.k \geq k$. So by our assumption, $(W, e_1, e_2) \in \mathcal{E}[\![\tau]\!]\emptyset$. We claim that $(W, E, E) \in \mathcal{K}[\![\tau]\!]\emptyset$, where

$$E = \begin{cases} [\cdot] & \tau = \tau \\ [\cdot] & \tau = \boldsymbol{\tau} \end{cases}$$

If the claim holds, then $(W, E[e_1], E[e_2]) = (W, e_1, e_2) \in \mathcal{O}$. Since running$(W.k, \langle H \mid e_1 \rangle)$ contradicts our assumption, we must have $\langle H \mid e_2 \rangle \Downarrow$, as desired.

To prove the claim, let $W' \sqsupseteq_{\text{pub}} W$ and $(W', v_1, v_2) \in \mathcal{V}[\![\tau]\!]\emptyset$. But then

$$(W', E[v_1], E[v_2]) = (W', v_1, v_2) \in \mathcal{O}$$

trivially, so we are done. □

**Lemma 9.50 (Logical Equivalence Implies Contextual Equivalence)**
If $\Psi; \Delta; \Gamma \vdash e_1 \approx^{log}_{M+C} e_2 : \tau$, then $\Psi; \Delta; \Gamma \vdash e_1 \approx^{ctx}_{M+C} e_2 : \tau$.

**Proof**

Let $\vdash C : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \cdot; \cdot \vdash \tau')$ and $\vdash H : \Psi'$. By congruence, $\Psi'; \cdot; \cdot \vdash C[e_1] \approx^{log}_{M+C} C[e_2] : \tau'$. By adequacy, $\langle H \mid C[e_1] \rangle \Downarrow$ if and only if $\langle H \mid C[e_2] \rangle \Downarrow$, as desired. □

## 9.4 Completeness w.r.t Contextual Equivalence

**Lemma 9.51 (Contextual Equivalence Implies CIU Equivalence)**
If $\Psi; \Delta; \Gamma \vdash e_1 \approx^{ctx}_{M+C} e_2 : \tau$, then $\Psi; \Delta; \Gamma \vdash e_1 \approx^{ciu}_{M+C} e_2 : \tau$.

**Proof**

We have that $\Psi; \Delta; \Gamma \vdash e_1 : \tau$, $\Psi; \Delta; \Gamma \vdash e_2 : \tau$, and

$$\forall C, H, \Psi', \tau'. \vdash C : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \cdot; \cdot \vdash \tau') \ \wedge \ \vdash H : \Psi' \\ \implies (\langle H \mid C[e_1] \rangle \Downarrow \iff \langle H \mid C[e_2] \rangle \Downarrow).$$

We need to show that

$$\forall \delta, \gamma, E, H, \Psi_E, \tau_E.\ \cdot \vdash \delta : \Delta \ \wedge \ \Psi_E; \cdot; \cdot \vdash \gamma : \delta(\Gamma) \ \wedge \ \vdash E : (\Psi; \cdot; \cdot \vdash \tau) \rightsquigarrow (\Psi_E; \cdot; \cdot \vdash \tau_E) \ \wedge \ \vdash H : \Psi_E \\ \implies (\langle H \mid E[\delta(\gamma(e_1))] \rangle \Downarrow \iff \langle H \mid E[\delta(\gamma(e_2))] \rangle \Downarrow).$$

Assume all of the premises in that implication. It suffices to find some $C$ such that co-termination of $\langle H \mid C[e_1] \rangle$ and $\langle H \mid C[e_2] \rangle$ is equivalent to co-termination of $\langle H \mid E[\delta(\gamma(e_1))] \rangle$ and $\langle H \mid E[\delta(\gamma(e_2))] \rangle$. We will need a $C$ such that

$$\vdash C : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi_E; \cdot; \cdot \vdash \tau_E).$$

Let

$$\tau_{\mathsf{E}} = \begin{cases} \tau & \tau_E = \tau \\ \mathsf{L}\langle \boldsymbol{\tau} \rangle & \tau_E = \boldsymbol{\tau} \end{cases} \qquad\qquad \boldsymbol{\tau_{\mathsf{E}}} = \begin{cases} \tau^{\langle \mathbf{C} \rangle} & \tau_E = \tau \\ \boldsymbol{\tau} & \tau_E = \boldsymbol{\tau} \end{cases}$$

$\Delta = \Delta, \boldsymbol{\Delta},\ \Gamma = \Gamma, \boldsymbol{\Gamma},\ \delta^M = \delta|_{\Delta}$.

Now choose $C$ as follows:

$$C = (\boldsymbol{\lambda}[\boldsymbol{\Delta}](\delta(\boldsymbol{\Gamma})).\mathbf{C})\ [\delta(\boldsymbol{\Delta})]\ \delta(\gamma(\mathrm{dom}(\boldsymbol{\Gamma})))$$
$$\mathbf{C} = \mathcal{CM}^{\delta^M(\tau_{\mathsf{E}})}\ ((\lambda[\Delta](\delta^M(\Gamma)).\mathsf{C})\ [\delta(\Delta)]\ \delta^M(\gamma(\mathrm{dom}(\Gamma))))$$
$$\mathsf{C} = \begin{cases} \mathsf{E} & E = \mathsf{E} \\ {}^{\tau_{\mathsf{E}}}\mathcal{MC}\ \mathbf{E} & E = \mathbf{E} \end{cases}$$

By inspection of the operational semantics,

$$\langle H \mid C[e_i] \rangle \longmapsto^* \langle H \mid \mathcal{CM}^{\delta(\tau_{\mathsf{E}})}\ (\mathsf{C}[\delta(\gamma(e_i))]) \rangle.$$

Since this is just a fixed sequence of boundary terms around $E[\delta(\gamma(e_i))]$, we can see that this configuration co-terminates with $\langle H \mid E[\delta(\gamma(e_i))] \rangle$, as desired. □

**Lemma 9.52 (CIU Equivalence Implies Logical Equivalence)**
If $\Psi; \Delta; \Gamma \vdash e_1 \approx^{ciu}_{M+C} e_2 : \tau$, then $\Psi; \Delta; \Gamma \vdash e_1 \approx^{log}_{M+C} e_2 : \tau$.

**Proof**

We have that $\Psi; \Delta; \Gamma \vdash e_1 : \tau$, $\Psi; \Delta; \Gamma \vdash e_2 : \tau$, and

$$\forall \delta, \gamma, E, H, \Psi_E, \tau_E.\ \cdot \vdash \delta : \Delta \ \wedge \ \Psi_E; \cdot; \cdot \vdash \gamma : \delta(\Gamma) \ \wedge \ \vdash E : (\Psi; \Delta; \cdot \vdash \tau) \rightsquigarrow (\Psi_E; \cdot; \cdot \vdash \tau_E) \ \wedge \ \vdash H : \Psi_E \\ \implies (\langle H \mid E[\delta(\gamma(e_1))] \rangle \Downarrow \iff \langle H \mid E[\delta(\gamma(e_2))] \rangle \Downarrow).$$

We need to show that

$$\forall W, \rho, \gamma.\ W \in \mathcal{H}[\![\Psi]\!] \ \wedge \ \rho \in \mathcal{D}[\![\Delta]\!] \ \wedge \ (W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho \implies (W, \rho_1(\gamma_1(e_1)), \rho_2(\gamma_2(e_2))) \in \mathcal{E}[\![\tau]\!]\rho.$$

Assume all the premises of this implication.

Let $(W, E_1, E_2) \in \mathcal{K}[\![\tau]\!]\rho$. We need to show that $(W, E_1[\rho_1(\gamma_1(e_1))], E_2[\rho_2(\gamma_2(e_2))]) \in \mathcal{O}$.

Let $(H_1, H_2) : W$. It suffices to show that

$$\langle H_1 \mid E_1[\rho_1(\gamma_1(e_1))] \rangle \Downarrow \iff \langle H_2 \mid E_2[\rho_2(\gamma_2(e_2))] \rangle \Downarrow.$$

By the Fundamental Property, $\Psi; \Delta; \Gamma \vdash e_1 \approx^{log}_{M+C} e_1 : \tau$. Therefore

$$(W, \rho_1(\gamma_1(e_1)), \rho_2(\gamma_2(e_1))) \in \mathcal{E}[\![\tau]\!]\rho$$

and thus

$$\langle H_1 \mid E_1[\rho_1(\gamma_1(e_1))] \rangle \Downarrow \iff \langle H_2 \mid E_2[\rho_2(\gamma_2(e_1))] \rangle \Downarrow.$$

It remains to show that

$$\langle H_2 \mid E_2[\rho_2(\gamma_2(e_1))] \rangle \Downarrow \iff \langle H_2 \mid E_2[\rho_2(\gamma_2(e_2))] \rangle \Downarrow.$$

But this follows from our hypothesis that $\Psi; \Delta; \Gamma \vdash e_1 \approx^{ciu}_{M+C} e_2 : \tau$. □

# 10 Proofs: Correctness of Closure Conversion

**Lemma 10.1** **(Boundary Cancellation in Logical Equivalence)**
-

- If $\Psi; \Delta; \Gamma \vdash \mathsf{e} : \tau$, then $\Psi; \Delta; \Gamma \vdash \mathsf{e} \approx^{log}_{M+C} {}^{\tau}\mathcal{MC}\,\mathcal{CM}^{\tau}\,\mathsf{e} : \tau$.
- If $\Psi; \Delta; \Gamma \vdash \mathbf{e} : \tau^{\langle \mathcal{C} \rangle}$, then $\Psi; \Delta; \Gamma \vdash \mathbf{e} \approx^{log}_{M+C} \mathcal{CM}^{\tau}\,{}^{\tau}\mathcal{MC}\,\mathbf{e} : \tau^{\langle \mathcal{C} \rangle}$.
- If $\Psi; \Delta; \Gamma \vdash \mathsf{v} : \tau$, then $\Psi; \Delta; \Gamma \vdash \mathsf{v} \approx^{log}_{M+C} {}^{\tau}\mathbf{MC}(\mathbf{CM}^{\tau}(\mathsf{v})) : \tau$.
- If $\Psi; \Delta; \Gamma \vdash \mathbf{v} : \tau^{\langle \mathcal{C} \rangle}$, then $\Psi; \Delta; \Gamma \vdash \mathbf{v} \approx^{log}_{M+C} \mathbf{CM}^{\tau}({}^{\tau}\mathbf{MC}(\mathbf{v})) : \tau^{\langle \mathcal{C} \rangle}$.

**Proof**

We prove the first claim; the others can be proven analogously.

First, note that $\Psi; \Delta; \Gamma \vdash {}^{\tau}\mathcal{MC}\,\mathcal{CM}^{\tau}\,\mathsf{e} : \tau$.

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$.

By the fundamental property, $\Psi; \Delta; \Gamma \vdash \mathsf{e} \approx^{log}_{M+C} \mathsf{e} : \tau$, so $(W, \rho_1(\gamma_1(\mathsf{e})), \rho_2(\gamma_2(\mathsf{e}))) \in \mathcal{E}[\![\tau]\!]\rho$.

By boundary cancellation,

$$(W, \rho_1(\gamma_1(\mathsf{e})), {}^{\rho_2(\tau)}\mathcal{MC}\,\mathcal{CM}^{\rho_2(\tau)}\,\rho_2(\gamma_2(\mathsf{e}))) \in \mathcal{E}[\![\tau]\!]\rho,$$

as desired. □

**Lemma 10.2** **(Context Boundary Cancellation in Logical Equivalence)**
-

1. If the hole in $C$ is $[\cdot]$ then
$\Psi; \Delta; \Gamma \vdash e_1 \approx^{log}_{M+C} C[\mathsf{e_2}] : \tau$ iff $\Psi; \Delta; \Gamma \vdash e_1 \approx^{log}_{M+C} C[{}^{\tau'}\mathcal{MC}\,\mathcal{CM}^{\tau'}\,\mathsf{e_2}] : \tau$.
2. If the hole in $C$ is $[\cdot]$ and $\Psi; \Delta; \Gamma \vdash \mathbf{e_2} : \tau'^{\langle \mathcal{C} \rangle}$ then
$\Psi; \Delta; \Gamma \vdash e_1 \approx^{log}_{M+C} C[\mathbf{e_2}] : \tau$ iff $\Psi; \Delta; \Gamma \vdash e_1 \approx^{log}_{M+C} C[\mathcal{CM}^{\tau'}\,{}^{\tau'}\mathcal{MC}\,\mathbf{e_2}] : \tau$.
3. If the hole in $C$ is $[\cdot]^{\mathsf{v}}$ then
$\Psi; \Delta; \Gamma \vdash e \approx^{log}_{M+C} C[\mathsf{v}] : \tau$ iff $\Psi; \Delta; \Gamma \vdash e \approx^{log}_{M+C} C[{}^{\tau'}\mathbf{MC}(\mathbf{CM}^{\tau'}(\mathsf{v}))] : \tau$.
4. If the hole in $C$ is $[\cdot]^{\mathbf{v}}$ then
$\Psi; \Delta; \Gamma \vdash e \approx^{log}_{M+C} C[\mathbf{v}] : \tau$ iff $\Psi; \Delta; \Gamma \vdash e \approx^{log}_{M+C} C[\mathbf{CM}^{\tau'}({}^{\tau'}\mathbf{MC}(\mathbf{v}))] : \tau$.

**Proof**

For claim 1, by Lemma 10.1 and congruence (Lemma 9.47), we have $\Psi; \Delta; \Gamma \vdash C[\mathsf{e_2}] \approx^{log}_{M+C} C[{}^{\tau'}\mathcal{MC}\,\mathcal{CM}^{\tau'}\,\mathsf{e_2}] : \tau$. The result follows by transitivity.

For claim 2, by Lemma 10.1 and congruence (Lemma 9.47), we have $\Psi; \Delta; \Gamma \vdash C[\mathbf{e_2}] \approx^{log}_{M+C} C[\mathcal{CM}^{\tau'}\,{}^{\tau'}\mathcal{MC}\,\mathbf{e_2}] : \tau$. The result follows by transitivity.

For claim 3, by Lemma 10.1 and congruence (Lemma 9.47), we have $\Psi; \Delta; \Gamma \vdash C[\mathsf{v}] \approx^{log}_{M+C} C[{}^{\tau'}\mathbf{MC}(\mathbf{CM}^{\tau'}(\mathsf{v}))] : \tau$. The result follows by transitivity.

For claim 4, by Lemma 10.1 and congruence (Lemma 9.47), we have $\Psi; \Delta; \Gamma \vdash C[\mathbf{v}] \approx^{log}_{M+C} C[\mathbf{CM}^{\tau'}({}^{\tau'}\mathbf{MC}(\mathbf{v}))] : \tau$. The result follows by transitivity.

□

**Notation for lemmas in this section:** The lemmas in the rest of this section assume the following notation: $\Psi = \overline{\ell : \tau'}$, $\Delta = \overline{\alpha}$, $\Gamma = \overline{x : \tau''}$

**Handling free variables in correctness theorem:** The closure translation of $e$ into $\mathbf{e'}$ will turn free M variables and locations into free C variables and locations. This will cause $\mathbf{e'}$ to not type check under the $\Psi, \Delta, \Gamma$ environments. To remedy this, the correctness statements perform the following substitutions: $\mathbf{e'}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(x)/\mathbf{x}]}$. Notice that $\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell$ and $\mathbf{CM}^{\tau''}(x)$ are both value forms. Because our source and target languages are in monadic form, we cannot substitute $\mathcal{CM}^{\tau''}\,x$—which is *not a value*—for a term variable $\mathbf{x}$. Instead we substitute the value form $\mathbf{CM}^{\tau''}(x)$ for each $\mathbf{x}$.

Overall, the correctness statement should say that if an expression $e$ translates to $\mathbf{e'}$, then $\Psi; \Delta; \Gamma \vdash e \approx^{log}_{M+C} {}^{\tau}\mathcal{MC}\,(\mathbf{e'}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(x)/\mathbf{x}]}) : \tau$. However, since our language is in monadic form, we require a slightly stronger statement (induction hypothesis) for values: if a value $v$ translates to $\mathbf{v'}$, then $\Psi; \Delta; \Gamma \vdash v \approx^{log}_{M+C} {}^{\tau}\mathbf{MC}(\mathbf{v'}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(x)/\mathbf{x}]}) : \tau$. Note that from the latter, by Lemma 7.10, it follows that $\Psi; \Delta; \Gamma \vdash v \approx^{log}_{M+C} {}^{\tau}\mathcal{MC}\,(\mathbf{v'}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(x)/\mathbf{x}]}) : \tau$.

**Lemma 10.3 (Variable)**
If $x : \tau \in \Gamma$, then $\Psi; \Delta; \Gamma \vdash x \approx^{log}_{M+C} {}^{\tau}\mathbf{MC}((\mathbf{CM}^{\tau}(x))) : \tau$.

**Proof**

Follows immediately from Lemma 10.1. □

**Lemma 10.4 (Unit)**
$\Psi; \Delta; \Gamma \vdash () \approx^{log}_{M+C} {}^{\mathsf{unit}}\mathbf{MC}(()) : \mathsf{unit}$.

**Proof**

Follows from Lemmas 7.10 and 7.5. □

**Lemma 10.5 (Int)**
$\Psi; \Delta; \Gamma \vdash n \approx^{log}_{M+C} {}^{\mathsf{int}}\mathbf{MC}(\mathbf{n}) : \mathsf{int}$.

**Proof**

Follows from Lemmas 7.10 and 7.5. □

**Lemma 10.6 (Primitive)**
If $\Psi; \Delta; \Gamma \vdash v \approx^{log}_{M+C} {}^{\mathsf{int}}\mathbf{MC}(\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(x)/\mathbf{x}]}) : \mathsf{int}$
and $\Psi; \Delta; \Gamma \vdash v' \approx^{log}_{M+C} {}^{\mathsf{int}}\mathbf{MC}(\mathbf{v'}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(x)/\mathbf{x}]}) : \mathsf{int}$,
then $\Psi; \Delta; \Gamma \vdash v\ p\ v' \approx^{log}_{M+C} {}^{\mathsf{int}}\mathcal{MC}\,((\mathbf{v}\ \mathbf{p}\ \mathbf{v'})\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(x)/\mathbf{x}]}) : \mathsf{int}$.

**Proof**

Let $\hat{\mathbf{v}} = \mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(x)/\mathbf{x}]}$ and $\hat{\mathbf{v}}' = \mathbf{v'}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(x)/\mathbf{x}]}$
By Lemma 10.2, it suffices to show that

$$\Psi; \Delta; \Gamma \vdash v\ p\ v' \approx^{log}_{M+C} {}^{\mathsf{int}}\mathcal{MC}\,(\mathbf{CM}^{\mathsf{int}}(\mathbf{MC}((\hat{\mathbf{v}}))))\ \mathbf{p}\ (\mathbf{CM}^{\mathsf{int}}(\mathbf{MC}((\hat{\mathbf{v}}'))))) : \mathsf{int}.$$

Note that

$$\Psi; \Delta; \Gamma \vdash v\ p\ v' : \mathsf{int} \text{ and } \Psi; \Delta; \Gamma \vdash {}^{\mathsf{int}}\mathcal{MC}\,(\mathbf{CM}^{\mathsf{int}}(\mathbf{MC}((\hat{\mathbf{v}}))))\ \mathbf{p}\ (\mathbf{CM}^{\mathsf{int}}(\mathbf{MC}((\hat{\mathbf{v}}'))))) : \mathsf{int}.$$

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$\begin{aligned}
&(W, \rho_1(\gamma_1(v\ p\ v')),\\
&\quad \rho_2(\gamma_2({}^{\mathsf{int}}\mathcal{MC}\,(\mathbf{CM}^{\mathsf{int}}(\mathbf{MC}((\hat{\mathbf{v}}))))\ \mathbf{p}\ (\mathbf{CM}^{\mathsf{int}}(\mathbf{MC}((\hat{\mathbf{v}}'))))))\\
&\qquad\qquad = (W, \rho_1(\gamma_1(v))\ p\ \rho_1(\gamma_1(v')),\\
&\qquad\qquad\qquad {}^{\mathsf{int}}\mathcal{MC}\ \rho_2(\gamma_2(\mathbf{CM}^{\mathsf{int}}(\mathbf{MC}((\hat{\mathbf{v}})))))\ \mathbf{p}\ \rho_2(\gamma_2(\mathbf{CM}^{\mathsf{int}}(\mathbf{MC}((\hat{\mathbf{v}}')))))) \in \mathcal{E}[\![\mathsf{int}]\!]\rho.
\end{aligned}$$

By our hypotheses,

$$(W, \rho_1(\gamma_1(\mathsf{v})), \rho_2(\gamma_2({}^{\mathsf{int}}\mathbf{MC}((\hat{\mathbf{v}}))))) \in \mathcal{E}[\![\mathsf{int}]\!]\rho$$

and

$$(W, \rho_1(\gamma_1(\mathsf{v}')), \rho_2(\gamma_2({}^{\mathsf{int}}\mathbf{MC}((\hat{\mathbf{v}}'))))) \in \mathcal{E}[\![\mathsf{int}]\!]\rho.$$

Let $W' \sqsupseteq_{\text{pub}} W$, $(W', \mathsf{m}, \mathsf{m}) \in \mathcal{V}[\![\mathsf{int}]\!]\rho$, and $(W', \mathsf{n}, \mathsf{n}) \in \mathcal{V}[\![\mathsf{int}]\!]\rho$. By Lemma 7.13, it suffices to show that

$$(W', \mathsf{m}\ \mathsf{p}\ \mathsf{n}, {}^{\mathsf{int}}\mathcal{MC}\ (\mathbf{CM}^{\mathsf{int}}(\mathsf{m})\ \mathbf{p}\ \mathbf{CM}^{\mathsf{int}}(\mathsf{n}))) \in \mathcal{E}[\![\mathsf{int}]\!]\rho.$$

Since boundary translations at type int produce the same integers they are given, and since the semantics of primitive operations are the same in M and C, from this point it is clear that we can complete the proof using Lemma 7.10 and Lemma 7.5. □

**Lemma 10.7 (If0)**
If $\Psi; \Delta; \Gamma \vdash \mathsf{v} \approx^{log}_{M+C} {}^{\mathsf{int}}\mathbf{MC}(\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}) : \mathsf{int}$,
$\Psi; \Delta; \Gamma \vdash \mathsf{e}_1 \approx^{log}_{M+C} {}^{\tau}\mathcal{MC}\ (\mathbf{e_1}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}) : \tau$, and
$\Psi; \Delta; \Gamma \vdash \mathsf{e}_2 \approx^{log}_{M+C} {}^{\tau}\mathcal{MC}\ (\mathbf{e_2}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}) : \tau$,
then $\Psi; \Delta; \Gamma \vdash \mathsf{if0}\ \mathsf{v}\ \mathsf{e}_1\ \mathsf{e}_2 \approx^{log}_{M+C} {}^{\tau}\mathcal{MC}\ (\mathbf{if0}\ \mathbf{v}\ \mathbf{e_1}\ \mathbf{e_2})\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]} : \tau$.

**Proof**

By Lemma 10.2, it suffices to show that

$$\begin{aligned}\Psi\Delta\Gamma \vdash \mathsf{if0}\ \mathsf{v}\ \mathsf{e}_1\ \mathsf{e}_2 \approx^{log}_{M+C} {}^{\tau}\mathcal{MC}\ (\mathbf{if0}\ &\mathbf{CM}^{\mathsf{int}}(\mathbf{MC}((\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}))) \\ &(\mathbf{e_1}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}) \\ &(\mathbf{e_2}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]})) \qquad : \tau.\end{aligned}$$

For brevity, let $\hat{\mathbf{v}} = \mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}$, $\hat{\mathbf{e}}_1 = \mathbf{e_1}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}$, and $\hat{\mathbf{e}}_2 = \mathbf{e_2}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}$.
Note that $\Psi; \Delta; \Gamma \vdash \mathsf{if0}\ \mathsf{v}\ \mathsf{e}_1\ \mathsf{e}_2 : \mathsf{int}$ and $\Psi; \Delta; \Gamma \vdash {}^{\tau}\mathcal{MC}\ (\mathbf{if0}\ \mathbf{CM}^{\mathsf{int}}(\mathbf{MC}(\hat{\mathbf{v}}))\ \hat{\mathbf{e}}_1\ \hat{\mathbf{e}}_2) : \tau$.
Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$\begin{aligned}&(W, \rho_1(\gamma_1(\mathsf{if0}\ \mathsf{v}\ \mathsf{e}_1\ \mathsf{e}_2)), \rho_2(\gamma_2({}^{\tau}\mathcal{MC}\ (\mathbf{if0}\ (\mathbf{CM}^{\mathsf{int}}(\mathbf{MC}(\hat{\mathbf{v}})))\ \hat{\mathbf{e}}_1\ \hat{\mathbf{e}}_2)))) \\ &= (W, \mathsf{if0}\ \rho_1(\gamma_1(\mathsf{v}))\ \rho_1(\gamma_1(\mathsf{e}_1))\ \rho_1(\gamma_1(\mathsf{e}_2)), {}^{\tau}\mathcal{MC}\ \mathbf{if0}\ \rho_2(\gamma_2(\mathbf{CM}^{\mathsf{int}}(\mathbf{MC}(\hat{\mathbf{v}}))))\ \rho_2(\gamma_2(\hat{\mathbf{e}}_1))\ \rho_2(\gamma_2(\hat{\mathbf{e}}_2))) \in \mathcal{E}[\![\tau]\!]\rho.\end{aligned}$$

By our first hypothesis, $(W, \rho_1(\gamma_1(\mathsf{v})), \rho_2(\gamma_2({}^{\mathsf{int}}\mathbf{MC}(\hat{\mathbf{v}})))) \in \mathcal{E}[\![\mathsf{int}]\!]\rho$. Let $W' \sqsupseteq_{\text{pub}} W$ and $(W', \mathsf{n}, \mathsf{n}) \in \mathcal{V}[\![\mathsf{int}]\!]\rho$. By Lemma 7.13, it suffices to show that

$$(W', \mathsf{if0}\ \mathsf{n}\ \rho_1(\gamma_1(\mathsf{e}_1))\ \rho_1(\gamma_1(\mathsf{e}_2)), {}^{\tau}\mathcal{MC}\ (\mathbf{if0}\ (\mathbf{CM}^{\mathsf{int}}(\mathsf{n}))\ \rho_2(\gamma_2(\hat{\mathbf{e}}_1))\ \rho_2(\gamma_2(\hat{\mathbf{e}}_2)))) \in \mathcal{E}[\![\tau]\!]\rho.$$

We can complete the proof using a case split on whether $\mathsf{n} = 0$, and then applying Lemma 7.10 and the appropriate one of our hypotheses. □

**Lemma 10.8 (Fold)**
If $\Psi; \Delta; \Gamma \vdash \mathsf{v} \approx^{log}_{M+C} {}^{\tau[\mu\alpha.\tau/\alpha]}\mathbf{MC}((\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]})) : \tau[\mu\alpha.\tau/\alpha]$, then

$$\Psi; \Delta; \Gamma \vdash \mathsf{fold}_{\mu\alpha.\tau}\ \mathsf{v} \approx^{log}_{M+C} {}^{\mu\alpha.\tau}\mathbf{MC}(((\mathbf{fold}_{\mu\alpha.\tau^{\mathcal{C}}}\ \mathbf{v})\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]})) : \mu\alpha.\tau.$$

**Proof**

By Lemma 10.2, it suffices to show that

$$\Psi;\Delta;\Gamma \vdash \mathsf{fold}_{\mu\beta.\tau}\,\mathsf{v} \approx^{log}_{M+C} {}^{\mu\beta.\tau}\mathbf{MC}((\mathbf{fold}_{\mu\beta.\tau^{\langle\mathcal{C}\rangle}}\ \mathbf{CM}^{\tau[\mu\beta.\tau/\beta]}(\mathbf{MC}((\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}))))) : \mu\beta.\tau.$$

Note first that we changed $\mathbf{fold}_{\mu\beta.\tau^{\mathcal{C}}}\ \ldots$ to $\mathbf{fold}_{\mu\beta.\tau^{\langle\mathcal{C}\rangle}}\ \ldots$. We can do this because we applied the $\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}$ substitution and by the type translation definitions $\tau^{\mathcal{C}}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]} = \tau^{\langle\mathcal{C}\rangle}$.
Also note that $\Psi;\Delta;\Gamma \vdash \mathsf{fold}_{\mu\beta.\tau}\,\mathsf{v} : \mu\beta.\tau$ and

$$\Psi;\Delta;\Gamma \vdash {}^{\mu\beta.\tau}\mathbf{MC}((\mathbf{fold}_{\mu\beta.\tau^{\langle\mathcal{C}\rangle}}\ \mathbf{CM}^{\tau[\mu\beta.\tau/\beta]}(\mathbf{MC}((\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}))))) : \mu\beta.\tau.$$

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W,\gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$\begin{aligned}
&(W, \rho_1(\gamma_1(\mathsf{fold}_{\mu\beta.\tau}\,\mathsf{v})), \rho_2(\gamma_2({}^{\mu\beta.\tau}\mathbf{MC}((\mathbf{fold}_{\mu\beta.\tau^{\langle\mathcal{C}\rangle}}\ \mathbf{CM}^{\tau[\mu\beta.\tau/\beta]}(\mathbf{MC}((\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}))))))) \\
&= (W, \mathsf{fold}_{\rho_1(\mu\beta.\tau)}\,\rho_1(\gamma_1(\mathsf{v})), \\
&\quad {}^{\rho_2(\mu\beta.\tau)}\mathbf{MC}(\mathbf{fold}_{\rho_2(\mu\beta.\tau^{\langle\mathcal{C}\rangle})}\ \mathbf{CM}^{\rho_2(\tau[\mu\beta.\tau/\beta])}(\mathbf{MC}((\rho_2(\gamma_2(\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}))))))) \\
&\quad \in \mathcal{E}[\![\mu\beta.\tau]\!]\rho.
\end{aligned}$$

By our hypothesis, $(W, \rho_1(\gamma_1(\mathsf{v})), \rho_2(\gamma_2({}^{\tau[\mu\beta.\tau/\beta]}\mathbf{MC}((\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}))))) \in \mathcal{E}[\![\tau[\mu\beta.\tau/\beta]]\!]\rho$.
Let $W' \sqsupseteq_{\text{pub}} W$ and $(W', \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\tau[\mu\beta.\tau/\beta]]\!]\rho$. By Lemma 7.13, it suffices to show that

$$(W', \mathsf{fold}_{\rho_1(\mu\beta.\tau)}\,\mathsf{v}_1, {}^{\rho_2(\mu\beta.\tau)}\mathbf{MC}(\mathbf{fold}_{\rho_2(\mu\beta.\tau^{\langle\mathcal{C}\rangle})}\ \mathbf{CM}^{\rho_2(\tau[\mu\beta.\tau/\beta])}(\mathsf{v}_2))) \in \mathcal{E}[\![\mu\beta.\tau]\!]\rho.$$

By Lemma 7.1 there are some $\mathbf{v_2}$ and $\mathsf{v}_2'$ such that

$$\mathbf{CM}^{\rho_2(\tau[\mu\alpha.\tau/\beta])}(\mathsf{v}_2) = \mathbf{v_2} \quad \text{and} \quad {}^{\rho_2(\tau[\mu\alpha.\tau/\beta])}\mathbf{MC}(\mathbf{v_2}) = \mathsf{v}_2'.$$

Hence, we have that:

$${}^{\rho_2(\mu\beta.\tau)}\mathbf{MC}(\mathbf{fold}_{\rho_2(\mu\beta.\tau)}\ \mathbf{CM}^{\rho_2(\tau[\mu\alpha.\tau/\beta])}(\mathsf{v}_2)) \ = \ \mathsf{fold}_{\rho_2(\mu\beta.\tau)}\,\mathsf{v}_2'.$$

Thus, by Lemma 7.5, it suffices to show that

$$(W', \mathsf{fold}_{\rho_1(\mu\beta.\tau)}\,\mathsf{v}_1, \mathsf{fold}_{\rho_2(\mu\beta.\tau)}\,\mathsf{v}_2') \in \mathcal{V}[\![\mu\beta.\tau]\!]\rho.$$

This follows from our hypothesis that $(W', \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\tau[\mu\beta.\tau/\beta]]\!]\rho$, by monotonicity and boundary cancellation. □

**Lemma 10.9 (Unfold)**
If $\Psi;\Delta;\Gamma \vdash \mathsf{v} \approx^{log}_{M+C} {}^{\mu\alpha.\tau}\mathbf{MC}(\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}) : \mu\alpha.\tau$, then

$$\Psi;\Delta;\Gamma \vdash \mathsf{unfold}\,\mathsf{v} \approx^{log}_{M+C} {}^{\tau[\mu\alpha.\tau/\alpha]}\mathcal{MC}\ (\mathbf{unfold}\,\mathbf{v})\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]} : \tau[\mu\alpha.\tau/\alpha].$$

**Proof**

By Lemma 10.2, it suffices to show that

$$\Psi;\Delta;\Gamma \vdash \mathsf{unfold}\,\mathsf{v} \approx^{log}_{M+C} {}^{\tau[\mu\beta.\tau/\beta]}\mathcal{MC}\ (\mathbf{unfold}\ \mathbf{CM}^{\mu\beta.\tau}(\mathbf{MC}((\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]})))) : \tau[\mu\beta.\tau/\beta].$$

Note that $\Psi;\Delta;\Gamma \vdash \mathsf{unfold}\,\mathsf{v} : \tau[\mu\beta.\tau/\beta]$ and

$$\Psi;\Delta;\Gamma \vdash {}^{\tau[\mu\beta.\tau/\beta]}\mathcal{MC}\ (\mathbf{unfold}\ \mathbf{CM}^{\mu\beta.\tau}(\mathbf{MC}((\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]})))) : \tau[\mu\beta.\tau/\beta].$$

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W,\gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$(W, \rho_1(\gamma_1(\mathsf{unfold}\,\mathsf{v})), \rho_2(\gamma_2({}^{\tau[\mu\beta.\tau/\beta]}\mathcal{MC}\ (\mathbf{unfold}\ \mathbf{CM}^{\mu\beta.\tau}(\mathbf{MC}((\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]})))))))$$
$$= (W, \mathsf{unfold}\,\rho_1(\gamma_1(\mathsf{v})),$$
$${}^{\rho_2(\tau[\mu\beta.\tau/\beta])}\mathcal{MC}\ \mathbf{unfold}\ \mathbf{CM}^{\rho_2(\mu\beta.\tau)}(\mathbf{MC}((\rho_2(\gamma_2(\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}))))))$$
$$\in \mathcal{E}[\![\tau[\mu\beta.\tau/\beta]]\!]\rho.$$

By our hypothesis, $(W, \rho_1(\gamma_1(\mathsf{v})), \rho_2(\gamma_2({}^{\mu\beta.\tau}\mathbf{MC}((\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}))))) \in \mathcal{E}[\![\mu\beta.\tau]\!]\rho$. Let $W' \sqsupseteq_{\text{pub}} W$ and $(W', \mathsf{fold}_{\rho_1(\mu\beta.\tau)}\,\mathsf{v}_1, \mathsf{fold}_{\rho_2(\mu\beta.\tau)}\,\mathsf{v}_2) \in \mathcal{V}[\![\mu\beta.\tau]\!]\rho$. By Lemma 7.13, it suffices to show that

$$(W', \mathsf{unfold}\,(\mathsf{fold}_{\rho_1(\mu\beta.\tau)}\,\mathsf{v}_1), {}^{\rho_2(\tau[\mu\beta.\tau/\beta])}\mathcal{MC}\ \mathbf{unfold}\ \mathbf{CM}^{\rho_2(\mu\beta.\tau)}(\mathsf{fold}_{\rho_2(\mu\beta.\tau)}\,\mathsf{v}_2)) \in \mathcal{E}[\![\tau[\mu\beta.\tau/\beta]]\!]\rho.$$

By Lemma 7.1 there are some $\mathbf{v_2}$ and $\mathsf{v}_2'$ such that

$$\mathbf{CM}^{\rho_2(\tau[\mu\beta.\tau/\beta])}(\mathsf{v}_2) = \mathbf{v_2} \quad \text{and} \quad {}^{\rho_2(\tau[\mu\beta.\tau/\beta])}\mathbf{MC}(\mathbf{v_2}) = \mathsf{v}_2'.$$

For any $(H_1, H_2){:}\,W'$, the operational semantics give us:

$$\langle H_2 \mid {}^{\rho_2(\tau[\mu\beta.\tau/\beta])}\mathcal{MC}\ \mathbf{unfold}\ \mathbf{CM}^{\rho_2(\mu\beta.\tau)}(\mathsf{fold}_{\rho_2(\mu\beta.\tau)}\,\mathsf{v}_2)\rangle \longmapsto^2 \langle H_2 \mid \mathsf{v}_2'\rangle.$$

The result follows by Lemma 7.10, Lemma 7.5, and boundary cancellation. ☐

**Lemma 10.10 (Location)**
Let $\Psi = \overline{\ell{:}\,\tau}$.
If $\Psi(\ell) = \tau$, then $\Psi;\Delta;\Gamma \vdash \ell \approx^{log}_{M+C} {}^{\mathsf{ref}\,\tau}\mathbf{MC}(\boldsymbol{\ell}[\mathcal{CM}^{\mathsf{ref}\,\tau}\ \ell/\boldsymbol{\ell}]) : \mathsf{ref}\,\tau$.
That is $\Psi;\Delta;\Gamma \vdash \ell \approx^{log}_{M+C} {}^{\mathsf{ref}\,\tau}\mathbf{MC}(\mathcal{CM}^{\mathsf{ref}\,\tau}\ \ell) : \mathsf{ref}\,\tau$.

**Proof**

Note that $\Psi;\Delta;\Gamma \vdash \ell : \mathsf{ref}\,\tau$ and $\Psi;\Delta;\Gamma \vdash {}^{\mathsf{ref}\,\tau}\mathbf{MC}(\mathcal{CM}^{\mathsf{ref}\,\tau}\ \ell) : \mathsf{ref}\,\tau$.

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W,\gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$.

We need to show that $(W, \rho_1(\gamma_1(\ell)), \rho_2(\gamma_2({}^{\mathsf{ref}\,\tau}\mathbf{MC}(\mathcal{CM}^{\mathsf{ref}\,\tau}\ \ell))))$
$= (W, \ell, {}^{\rho_2(\mathsf{ref}\,\tau)}\mathbf{MC}(\mathcal{CM}^{\rho_2(\mathsf{ref}\,\tau)}\ \ell)$
$= (W, \ell, \ell) \in \mathcal{E}[\![\mathsf{ref}\,\tau]\!]\rho$.

The latter is immediate from the compatibility lemma for locations, Lemma 9.17. ☐

**Lemma 10.11 (New)**
If $\Psi;\Delta;\Gamma \vdash \mathsf{v} \approx^{log}_{M+C} {}^{\tau}\mathbf{MC}(\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}) : \tau$, then

$$\Psi;\Delta;\Gamma \vdash \mathsf{new}\,\mathsf{v} \approx^{log}_{M+C} {}^{\mathsf{ref}\,\tau}\mathcal{MC}\ (\mathbf{new}\ \mathbf{v})\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]} : \mathsf{ref}\,\tau.$$

**Proof**

By Lemma 10.2, it suffices to show that

$$\Psi;\Delta;\Gamma \vdash \mathsf{new}\,\mathsf{v} \approx^{log}_{M+C} {}^{\mathsf{ref}\,\tau}\mathcal{MC}\ (\mathbf{new}\ \mathbf{CM}^{\tau}(\mathbf{MC}((\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]})))) : \mathsf{ref}\,\tau.$$

Note that $\Psi;\Delta;\Gamma \vdash \mathsf{new}\,\mathsf{v} : \mathsf{ref}\,\tau$ and

$$\Psi;\Delta;\Gamma \vdash {}^{\mathsf{ref}\,\tau}\mathcal{MC}\ (\mathbf{new}\ \mathbf{CM}^{\tau}(\mathbf{MC}((\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]})))) : \mathsf{ref}\,\tau.$$

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$\begin{aligned}
&(W, \rho_1(\gamma_1(\mathsf{new}\,\mathsf{v})), \rho_2(\gamma_2(^{\mathsf{ref}\,\tau}\mathcal{MC}\ (\mathbf{new}\ \mathbf{CM}^{\tau}(\mathbf{MC}((\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]})))))) \\
&= (W, \mathsf{new}\,\rho_1(\gamma_1(\mathsf{v})), \\
&\quad ^{\rho_2(\mathsf{ref}\,\tau)}\mathcal{MC}\ \mathbf{new}\ \mathbf{CM}^{\rho_2(\tau)}(\mathbf{MC}((\rho_2(\gamma_2(\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}))))) \in \mathcal{E}[\![\mathsf{ref}\,\tau]\!]\rho.
\end{aligned}$$

By our hypothesis, $(W, \rho_1(\gamma_1(\mathsf{v})), \rho_2(\gamma_2(^{\tau}\mathbf{MC}((\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}))))) \in \mathcal{E}[\![\tau]\!]\rho$.
Let $W' \sqsupseteq_{\text{pub}} W$ and $(W', \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\tau]\!]\rho$. By Lemma 7.13, it suffices to show that

$$(W', \mathsf{new}\,\mathsf{v}_1, {}^{\rho_2(\mathsf{ref}\,\tau)}\mathcal{MC}\ \mathbf{new}\ \mathbf{CM}^{\rho_2(\tau)}(\mathsf{v}_2)) \in \mathcal{E}[\![\mathsf{ref}\,\tau]\!]\rho.$$

Applying Lemma 7.9 we let $W'' \sqsupseteq W'$ where $\ell_1 \notin W'.\Psi_1$, $\boldsymbol{\ell_2} \notin W'.\Psi_2$, and

$$\begin{aligned}
W'' &= (W'.k, (W'.\Psi_1, \ell_1 : \tau), (W'.\Psi_2, \boldsymbol{\ell_2} : \tau^{\langle \mathcal{C} \rangle}), (W'.\Theta, \theta)) \\
\theta &= (\bullet, \{\bullet\}, \emptyset, \emptyset, \lambda s.\{\, (\widetilde{W}, \{\ell_1 \mapsto \mathsf{v}_1'\}, \{\boldsymbol{\ell_2} \mapsto \mathbf{v_2'}\}) \in \text{HeapAtom} \mid (\widetilde{W}, \mathsf{v}_1', \mathbf{v_2'}) \in \mathcal{V}[\![\tau, \tau^{\langle \mathcal{C} \rangle}]\!]\rho \,\}, \lambda s.\{(\ell_1, \boldsymbol{\ell_2})\})
\end{aligned}$$

We assume $(H_1, H_2) : W'$ and let $\hat{\mathbf{v}}_2 = \mathbf{CM}^{\rho_2(\tau)}(\mathsf{v}_2)$, by the operational semantics:

$$\langle H_1 \mid \mathsf{new}\,\mathsf{v}_1 \rangle \longmapsto^1 \langle H_1[\ell_1 \mapsto \mathsf{v}_1] \mid \ell_1 \rangle.$$

$$\langle H_2 \mid {}^{\rho_2(\mathsf{ref}\,\tau)}\mathcal{MC}\ \mathbf{new}\ \mathbf{CM}^{\rho_2(\tau)}(\mathsf{v}_2) \rangle \longmapsto^1 \langle H_2[\boldsymbol{\ell_2} \mapsto \hat{\mathbf{v}}_2] \mid {}^{\mathsf{ref}\,\tau}\mathcal{MC}\ \boldsymbol{\ell_2} \rangle.$$

We must show that $(H_1[\ell_1 \mapsto \mathsf{v}_1], H_2[\boldsymbol{\ell_2} \mapsto \hat{\mathbf{v}}_2]) : W''$. This comes down to showing $(W'', \mathsf{v}_1, \hat{\mathbf{v}}_2) \in \mathcal{V}[\![\tau, \tau^{\langle \mathcal{C} \rangle}]\!]\rho$, which follows from the bridge and boundary cancellation lemmas.
Finally, by Lemma 7.5 it then suffices to show $(W'', \ell_1, {}^{\mathsf{ref}\,\tau}\mathcal{MC}\ \boldsymbol{\ell_2}) \in \mathcal{V}[\![\mathsf{ref}\,\tau]\!]\rho$, which follows from our definition of $W''$. □

**Lemma 10.12 (Assignment)**
If $\Psi; \Delta; \Gamma \vdash \mathsf{v} \approx^{log}_{M+C} {}^{\mathsf{ref}\,\tau}\mathbf{MC}(\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}) : \mathsf{ref}\,\tau$ and
$\Psi; \Delta; \Gamma \vdash \mathsf{v}' \approx^{log}_{M+C} {}^{\tau}\mathbf{MC}(\mathbf{v'}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}) : \tau$, then

$$\Psi; \Delta; \Gamma \vdash \mathsf{v} := \mathsf{v}' \approx^{log}_{M+C} {}^{\mathsf{unit}}\mathcal{MC}\ (\mathbf{v} := \mathbf{v'})\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]} : \mathsf{unit}.$$

**Proof**

By Lemma 10.2, it suffices to show that

$$\Psi; \Delta; \Gamma \vdash \mathsf{v} := \mathsf{v}' \approx^{log}_{M+C} {}^{\mathsf{ref}\,\tau}\mathcal{MC}\ (\mathbf{CM}^{\mathsf{ref}\,\tau}(\mathbf{MC}(\mathbf{v})) := \mathbf{CM}^{\tau}(\mathbf{MC}(\mathbf{v'})))\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]} : \mathsf{unit}.$$

For brevity, let $\hat{\mathbf{v}} = \mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}$ and $\hat{\mathbf{v}}' = \mathbf{v'}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}$.
Note that $\Psi; \Delta; \Gamma \vdash \mathsf{v} := \mathsf{v}' : \mathsf{unit}$ and

$$\Psi; \Delta; \Gamma \vdash {}^{\mathsf{unit}}\mathcal{MC}\ (\mathbf{CM}^{\mathsf{ref}\,\tau}(\mathbf{MC}(\hat{\mathbf{v}})) := \mathbf{CM}^{\tau}(\mathbf{MC}((\hat{\mathbf{v}}')))) : \mathsf{unit}.$$

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$\begin{aligned}
&(W, \rho_1(\gamma_1(\mathsf{v} := \mathsf{v}')), \rho_2(\gamma_2(^{\mathsf{unit}}\mathcal{MC}\ (\mathbf{CM}^{\mathsf{ref}\,\tau}(\mathbf{MC}(\hat{\mathbf{v}})) := \mathbf{CM}^{\tau}(\mathbf{MC}((\hat{\mathbf{v}}'))))))) \\
&\quad = (W, \rho_1(\gamma_1(\mathsf{v})) := \rho_1(\gamma_1(\mathsf{v}')), \\
&\qquad ^{\rho_2(\mathsf{unit})}\mathcal{MC}\ (\mathbf{CM}^{\rho_2(\mathsf{ref}\,\tau)}(\mathbf{MC}(\rho_2(\gamma_2(\hat{\mathbf{v}})))) := \mathbf{CM}^{\rho_2(\tau)}(\mathbf{MC}(\rho_2(\gamma_2(\hat{\mathbf{v}}')))))) \in \mathcal{E}[\![\mathsf{unit}]\!]\rho.
\end{aligned}$$

By our hypothesis, $(W, \rho_1(\gamma_1(\mathsf{v})), \rho_2(\gamma_2(^{\mathsf{ref}\,\tau}\mathbf{MC}(\hat{\mathbf{v}})))) \in \mathcal{E}[\![\mathsf{ref}\,\tau]\!]\rho$ and $(W, \rho_1(\gamma_1(\mathsf{v}')), \rho_2(\gamma_2(^{\tau}\mathbf{MC}((\mathbf{v'}))))) \in \mathcal{E}[\![\tau]\!]\rho$.

Let $W' \sqsupseteq_{\text{pub}} W$, $(W', \hat{v}_1, \hat{v}_2) \in \mathcal{V}[\![\mathsf{ref}\ \tau]\!]\rho$, and $(W', \hat{v}'_1, \hat{v}'_2) \in \mathcal{V}[\![\tau]\!]\rho$. By Lemma 7.13, it suffices to show that

$$(W', \hat{v}_1 := \hat{v}'_1, {}^{\rho_2(\mathsf{unit})}\mathcal{MC}\ (\mathbf{CM}^{\rho_2(\mathsf{ref}\ \tau)}(\hat{v}_2) \mathbin{:=} \mathbf{CM}^{\rho_2(\tau)}(\hat{v}'_2))) \in \mathcal{E}[\![\mathsf{unit}]\!]\rho.$$

Note that by Lemma 7.1 there are some $\hat{\mathbf{v}}''_2$ and $\hat{v}''_2$ such that

$$\mathbf{CM}^{\rho_2(\tau)}(\hat{v}'_2) = \hat{\mathbf{v}}''_2 \quad \text{and} \quad {}^{\rho_2(\tau)}\mathbf{MC}(\hat{\mathbf{v}}''_2) = \hat{v}''_2.$$

Also by Lemma 7.1 there are some $\hat{\mathbf{v}}''_1$ and $\hat{v}''_1$ such that

$$\mathbf{CM}^{\rho_1(\tau)}(\hat{v}'_1) = \hat{\mathbf{v}}''_1 \quad \text{and} \quad {}^{\rho_1(\tau)}\mathbf{MC}(\hat{\mathbf{v}}''_2) = \hat{v}''_1.$$

There are 4 possible cases for $\hat{v}_1$ and $\hat{v}_2$.

- $\hat{v}_1 = \ell_1$ and $\hat{v}_2 = \ell_2$:
  Applying Lemma 7.9 we let $W' \sqsupseteq W'$ and assume $(H_1, H_2) : W'$, by the operational semantics:

$$\langle H_1 \mid \ell_1 := \hat{v}'_1 \rangle \longmapsto^1 \langle H_1[\ell_1 \mapsto \hat{v}'_1] \mid () \rangle.$$

$$\begin{aligned}\langle H_2 \mid {}^{\rho_2(\mathsf{unit})}\mathcal{MC}\ (\mathbf{CM}^{\rho_2(\mathsf{ref}\ \tau)}(\ell_2) \mathbin{:=} \mathbf{CM}^{\rho_2(\tau)}(\hat{v}'_2)) \rangle &\longmapsto^2 \langle H_2 \mid {}^{\rho_2(\mathsf{unit})}\mathcal{MCCM}\ (\ell_2 := \hat{v}''_2) \rangle \\ &\longmapsto^3 \langle H_2[\ell_2 \mapsto \hat{v}''_2] \mid () \rangle\end{aligned}$$

  We must show that $(H_1[\ell_1 \mapsto \hat{v}'_1], H_2[\ell_2 \mapsto \hat{v}''_2]) : W'$, which comes down to showing $(W', \hat{v}'_1, \hat{v}''_2) \in \mathcal{V}[\![\tau]\!]\rho$. This follows from the boundary cancellation lemma.

- $\hat{v}_1 = \ell_1$ and $\hat{v}_2 = {}^{\rho_2(\mathsf{ref}\ \tau)}\mathcal{MC}\ \boldsymbol{\ell}_2$:
  Applying Lemma 7.9 we let $W' \sqsupseteq W'$ and assume $(H_1, H_2) : W'$, by the operational semantics:

$$\langle H_1 \mid \ell_1 := \hat{v}'_1 \rangle \longmapsto^1 \langle H_1[\ell_1 \mapsto \hat{v}'_1] \mid () \rangle.$$

$$\begin{aligned}\langle H_2 \mid {}^{\rho_2(\mathsf{unit})}\mathcal{MC}\ (\mathbf{CM}^{\rho_2(\mathsf{ref}\ \tau)}({}^{\rho_2(\mathsf{ref}\ \tau)}\mathcal{MC}\ \boldsymbol{\ell}_2) \mathbin{:=} \mathbf{CM}^{\rho_2(\tau)}(\hat{v}'_2)) \rangle &\longmapsto^1 \langle H_2[\boldsymbol{\ell}_2 \mapsto \hat{\mathbf{v}}''_2] \mid {}^{\rho_2(\mathsf{unit})}\mathcal{MC}\ () \rangle \\ &\longmapsto^1 \langle H_2[\boldsymbol{\ell}_2 \mapsto \hat{\mathbf{v}}''_2] \mid () \rangle\end{aligned}$$

  We must show that $(H_1[\ell_1 \mapsto \hat{v}'_1], H_2[\boldsymbol{\ell}_2 \mapsto \hat{\mathbf{v}}''_2]) : W'$, which comes down to showing $(W', \hat{v}'_1, \hat{\mathbf{v}}''_2) \in \mathcal{V}[\![\tau, \tau^{\langle \mathcal{C} \rangle}]\!]\rho$. This follows from the boundary cancellation and bridge lemmas.

- $\hat{v}_1 = {}^{\rho_1(\mathsf{ref}\ \tau)}\mathcal{MC}\ \boldsymbol{\ell}_1$ and $\hat{v}_2 = \ell_2$:
  Applying Lemma 7.9 we let $W' \sqsupseteq W'$ and assume $(H_1, H_2) : W'$, by the operational semantics:

$$\begin{aligned}\langle H_1 \mid {}^{\rho_1(\mathsf{ref}\ \tau)}\mathcal{MC}\ \boldsymbol{\ell}_1 := \hat{v}'_1 \rangle &\longmapsto^2 \langle H_1 \mid {}^{\rho_1(\mathsf{unit})}\mathcal{MC}\ (\boldsymbol{\ell}_1 \mathbin{:=} \mathbf{CM}^{\rho_1(\tau)}(\hat{v}'_1)) \rangle \\ &\longmapsto^1 \langle H_1[\boldsymbol{\ell}_1 \mapsto \hat{\mathbf{v}}''_1] \mid {}^{\rho_1(\mathsf{unit})}\mathcal{MC}\ () \rangle \\ &\longmapsto^1 \langle H_1[\boldsymbol{\ell}_1 \mapsto \hat{\mathbf{v}}''_1] \mid () \rangle\end{aligned}$$

$$\begin{aligned}\langle H_2 \mid {}^{\rho_2(\mathsf{unit})}\mathcal{MC}\ (\mathbf{CM}^{\rho_2(\mathsf{ref}\ \tau)}(\ell_2) \mathbin{:=} \mathbf{CM}^{\rho_2(\tau)}(\hat{v}'_2)) \rangle &\longmapsto^2 \langle H_2 \mid {}^{\rho_2(\mathsf{unit})}\mathcal{MCCM}\ (\ell_2 := \hat{v}''_2) \rangle \\ &\longmapsto^3 \langle H_2[\ell_2 \mapsto \hat{v}''_2] \mid () \rangle\end{aligned}$$

  We must show that $(H_1[\boldsymbol{\ell}_1 \mapsto \hat{\mathbf{v}}''_1], H_2[\ell_2 \mapsto \hat{v}''_2]) : W'$, which comes down to showing $(W', \hat{\mathbf{v}}''_1, \hat{v}''_2) \in \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}, \tau]\!]\rho$. This follows from the boundary cancellation and bridge lemmas.

- $\hat{v}_1 = {}^{\rho_1(\mathsf{ref}\ \tau)}\mathcal{MC}\ \boldsymbol{\ell}_1$ and $\hat{v}_2 = {}^{\rho_2(\mathsf{ref}\ \tau)}\mathcal{MC}\ \boldsymbol{\ell}_2$:
  Applying Lemma 7.9 we let $W' \sqsupseteq W'$ and assume $(H_1, H_2) : W'$, by the operational semantics:

$$\begin{aligned}\langle H_1 \mid {}^{\rho_1(\mathsf{ref}\ \tau)}\mathcal{MC}\ \boldsymbol{\ell}_1 := \hat{v}'_1 \rangle &\longmapsto^2 \langle H_1 \mid {}^{\rho_1(\mathsf{unit})}\mathcal{MC}\ (\boldsymbol{\ell}_1 \mathbin{:=} \mathbf{CM}^{\rho_1(\tau)}(\hat{v}'_1)) \rangle \\ &\longmapsto^1 \langle H_1[\boldsymbol{\ell}_1 \mapsto \hat{\mathbf{v}}''_1] \mid {}^{\rho_1(\mathsf{unit})}\mathcal{MC}\ () \rangle \\ &\longmapsto^1 \langle H_1[\boldsymbol{\ell}_1 \mapsto \hat{\mathbf{v}}''_1] \mid () \rangle\end{aligned}$$

$$\langle H_2 \mid {}^{\rho_2(\mathsf{unit})}\mathcal{MC}\ (\mathbf{CM}^{\rho_2(\mathsf{ref}\ \tau)}({}^{\rho_2(\mathsf{ref}\ \tau)}\mathcal{MC}\ \boldsymbol{\ell_2})\ \mathbf{:=}\ \mathbf{CM}^{\rho_2(\tau)}(\hat{\mathsf{v}}_2'))\rangle \longmapsto^1 \langle H_2[\boldsymbol{\ell_2} \mapsto \hat{\mathbf{v}}_2''] \mid {}^{\rho_2(\mathsf{unit})}\mathcal{MC}\ \mathbf{()}\rangle$$
$$\longmapsto^1 \langle H_2[\boldsymbol{\ell_2} \mapsto \hat{\mathbf{v}}_2''] \mid ()\rangle$$

We must show that $(H_1[\boldsymbol{\ell_1} \mapsto \hat{\mathbf{v}}_1''], H_2[\boldsymbol{\ell_2} \mapsto \hat{\mathbf{v}}_2'']) : W'$, which comes down to showing $(W', \hat{\mathbf{v}}_1'', \hat{\mathbf{v}}_2'') \in \mathcal{V}[\![\tau^{\langle \mathcal{C}\rangle}]\!]\rho$. This follows from the bridge lemma.

In all four of the above cases, we must also show that $(W', (), ()) \in \mathcal{E}[\![\mathsf{unit}]\!]\rho$, which follows by Lemma 9.34. ☐

**Lemma 10.13 (Dereference)**
If $\Psi; \Delta; \Gamma \vdash \mathsf{v} \approx^{log}_{M+C} {}^{\mathsf{ref}\ \tau}\mathbf{MC}(\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\ \tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}) : \mathsf{ref}\ \tau$, then

$$\Psi; \Delta; \Gamma \vdash !\mathsf{v} \approx^{log}_{M+C} {}^{\tau}\mathcal{MC}\ (\mathbf{!v})\overline{[\mathcal{CM}^{\mathsf{ref}\ \tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]} : \tau.$$

**Proof**

By Lemma 10.2, we must show that

$$\Psi; \Delta; \Gamma \vdash !\mathsf{v} \approx^{log}_{M+C} {}^{\tau}\mathcal{MC}\ (\mathbf{!CM}({}^{\mathsf{ref}\ \tau}\mathbf{MC}(\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\ \tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}))) : \tau.$$

Note that $\Psi; \Delta; \Gamma \vdash !\mathsf{v} : \tau$ and $\Psi; \Delta; \Gamma \vdash {}^{\tau}\mathcal{MC}\ (\mathbf{!CM}({}^{\mathsf{ref}\ \tau}\mathbf{MC}(\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\ \tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}))) : \tau$. Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$(W, \rho_1(\gamma_1(!\mathsf{v})), \rho_2(\gamma_2({}^{\tau}\mathcal{MC}\ (\mathbf{!CM}(\mathbf{MC}(\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\ \tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}))))) =$$
$$(W, !\rho_1(\gamma_1(\mathsf{v})), {}^{\rho_2(\tau)}\mathcal{MC}\ (\mathbf{!CM}(\mathbf{MC}(\rho_2(\gamma_2(\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\ \tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]})))))) \in \mathcal{E}[\![\tau]\!]\rho.$$

By our hypothesis, $(W, \rho_1(\gamma_1(\mathsf{v})), \rho_2(\gamma_2({}^{\rho_2(\mathsf{ref}\ \tau)}\mathbf{MC}(\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\ \tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]})))) \in \mathcal{E}[\![\mathsf{ref}\ \tau]\!]\rho$. Let $W' \sqsupseteq_{\text{pub}} W$, $(W', \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\mathsf{ref}\ \tau]\!]\rho$. By Lemma 7.13, it suffices to show that

$$(W', !\mathsf{v}_1, {}^{\rho_2(\tau)}\mathcal{MC}\ (\mathbf{!CM}^{\mathsf{ref}\ \tau}(\mathsf{v}_2))) \in \mathcal{E}[\![\tau]\!]\rho.$$

There are 4 possible cases for $\mathsf{v}_1$ and $\mathsf{v}_2$.

- $\mathsf{v}_1 = \ell_1$ and $\mathsf{v}_2 = \ell_2$:
  Applying Lemma 7.10 we assume $(H_1, H_2) : W'$, and using this and $(W', \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\mathsf{ref}\ \tau]\!]\rho$ we have that $H_1(\ell_1) = \mathsf{v}_1'$, $H_2(\ell_2) = \mathsf{v}_2'$, and $(W', \mathsf{v}_1', \mathsf{v}_2') \in \mathcal{V}[\![\tau]\!]\rho$.
  Note that by Lemma 7.1 there are some $\mathbf{v}_2'$ and $\mathsf{v}_2''$ such that

  $$\mathbf{CM}^{\rho_2(\tau)}(\mathsf{v}_2') = \mathbf{v}_2' \quad \text{and} \quad {}^{\rho_2(\tau)}\mathbf{MC}(\mathbf{v}_2') = \mathsf{v}_2''.$$

  By the operational semantics:

  $$\langle H_1 \mid !\ell_1\rangle \longmapsto^1 \langle H_1 \mid \mathsf{v}_1'\rangle.$$
  $$\langle H_2 \mid {}^{\rho_2(\tau)}\mathcal{MC}\ (\mathbf{!}\mathcal{CM}^{\rho_2(\mathsf{ref}\ \tau)}\ \ell_2)\rangle \longmapsto^1 \langle H_2 \mid {}^{\rho_2(\tau)}\mathcal{MCCM}\ !\ell_2\rangle \longmapsto^3 \langle H_2 \mid \mathsf{v}_2''\rangle.$$

  It suffices to show that

  $$(W', \mathsf{v}_1', \mathsf{v}_2'') \in \mathcal{E}[\![\tau]\!]\rho.$$

  This follows by Lemma 7.5 and boundary cancellation.
- $\mathsf{v}_1 = \ell_1$ and $\mathsf{v}_2 = {}^{\rho_2(\mathsf{ref}\ \tau)}\mathcal{MC}\ \boldsymbol{\ell_2}$:
  Applying Lemma 7.10 we assume $(H_1, H_2) : W'$, and using this and $(W', \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\mathsf{ref}\ \tau]\!]\rho$ we have that $H_1(\ell_1) = \mathsf{v}_1'$, $H_2(\boldsymbol{\ell_2}) = \mathbf{v}_2'$, and $(W', \mathsf{v}_1', \mathbf{v}_2') \in \mathcal{V}[\![\tau, \tau^{\langle \mathcal{C}\rangle}]\!]\rho$.
  Note that by Lemma 7.1 there is some $\mathsf{v}_2''$ such that

  $${}^{\rho_2(\tau)}\mathbf{MC}(\mathbf{v}_2') = \mathsf{v}_2''.$$

By the operational semantics:

$$\langle H_1 \mid !\ell_1\rangle \longmapsto^1 \langle H_1 \mid \mathsf{v}_1'\rangle.$$

$$\langle H_2 \mid {}^{\rho_2(\tau)}\mathcal{MC}\,!\mathbf{CM}^{\rho_2(\mathsf{ref}\,\tau)}(\mathcal{MC}\,\ell_2)\rangle \equiv \langle H_2 \mid {}^{\rho_2(\tau)}\mathcal{MC}\,!\ell_2\rangle \longmapsto^1 \langle H_2 \mid {}^{\rho_2(\tau)}\mathcal{MC}\,\mathbf{v}_2'\rangle \longmapsto^1 \langle H_2 \mid \mathsf{v}_2''\rangle.$$

It suffices to show that

$$(W', \mathsf{v}_1', \mathsf{v}_2'') \in \mathcal{E}[\![\tau]\!]\rho.$$

This follows from $(W', \mathsf{v}_1', \mathbf{v}_2') \in \mathcal{V}[\![\tau, \tau^{\langle\mathcal{C}\rangle}]\!]\rho$ by Lemma 7.5.

- $\mathsf{v}_1 = {}^{\rho_1(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell}_1$ and $\mathsf{v}_2 = \ell_2$:
  Applying Lemma 7.10 we assume $(H_1, H_2) : W'$, and using this and $(W', \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\mathsf{ref}\,\tau]\!]\rho$ we have that $H_1(\boldsymbol{\ell}_1) = \mathbf{v}_1'$, $H_2(\ell_2) = \mathsf{v}_2'$, and $(W', \mathbf{v}_1', \mathsf{v}_2') \in \mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}, \tau]\!]\rho$.
  Note that by Lemma 7.1 there is some $\mathsf{v}_1''$ such that

  $${}^{\rho_1(\tau)}\mathbf{MC}(\mathbf{v}_1') = \mathsf{v}_1''.$$

  Also by Lemma 7.1 there are some $\mathbf{v}_2'$ and $\mathsf{v}_2''$ such that

  $$\mathbf{CM}^{\rho_2(\tau)}(\mathsf{v}_2') = \mathbf{v}_2' \quad \text{and} \quad {}^{\rho_2(\tau)}\mathbf{MC}(\mathbf{v}_2') = \mathsf{v}_2''.$$

  By the operational semantics:

  $$\langle H_1 \mid !({}^{\rho_1(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell}_1)\rangle \longmapsto^1 \langle H_1 \mid {}^{\rho_1(\tau)}\mathcal{MC}\,\mathbf{v}_1'\rangle \longmapsto^1 \langle H_1 \mid \mathsf{v}_1''\rangle.$$

  $$\langle H_2 \mid {}^{\rho_2(\tau)}\mathcal{MC}\,(!\mathcal{CM}^{\rho_2(\mathsf{ref}\,\tau)}\,\ell_2)\rangle \longmapsto^1 \langle H_2 \mid {}^{\rho_2(\tau)}\mathcal{MCCM}\,!\ell_2\rangle \longmapsto^3 \langle H_2 \mid \mathsf{v}_2''\rangle.$$

  It suffices to show that

  $$(W', \mathsf{v}_1'', \mathsf{v}_2'') \in \mathcal{E}[\![\tau]\!]\rho.$$

  This follows from $(W', \mathbf{v}_1', \mathsf{v}_2') \in \mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}, \tau]\!]\rho$ by boundary cancellation and Lemma 7.5.
- $\mathsf{v}_1 = {}^{\rho_1(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell}_1$ and $\mathsf{v}_2 = {}^{\rho_2(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell}_2$:
  Applying Lemma 7.10 we assume $(H_1, H_2) : W'$, and using this and $(W', \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\mathsf{ref}\,\tau]\!]\rho$ we have that $H_1(\boldsymbol{\ell}_1) = \mathbf{v}_1'$, $H_2(\boldsymbol{\ell}_2) = \mathbf{v}_2'$, and $(W', \mathbf{v}_1', \mathbf{v}_2') \in \mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}]\!]\rho$.
  Note that by Lemma 7.1 there are some $\mathsf{v}_1''$ and $\mathsf{v}_2''$ such that

  $${}^{\rho_1(\tau)}\mathbf{MC}(\mathbf{v}_1') = \mathsf{v}_1'' \quad \text{and} \quad {}^{\rho_2(\tau)}\mathbf{MC}(\mathbf{v}_2') = \mathsf{v}_2''.$$

  By the operational semantics:

  $$\langle H_1 \mid !({}^{\rho_1(\mathsf{ref}\,\tau)}\mathcal{MC}\,\boldsymbol{\ell}_1)\rangle \longmapsto^1 \langle H_1 \mid {}^{\rho_1(\tau)}\mathcal{MC}\,\mathbf{v}_1'\rangle \longmapsto^1 \langle H_1 \mid \mathsf{v}_1''\rangle.$$

  $$\langle H_2 \mid {}^{\rho_2(\tau)}\mathcal{MC}\,!\mathbf{CM}^{\rho_2(\mathsf{ref}\,\tau)}(\mathcal{MC}\,\boldsymbol{\ell}_2)\rangle \equiv \langle H_2 \mid {}^{\rho_2(\tau)}\mathcal{MC}\,!\boldsymbol{\ell}_2\rangle \longmapsto^1 \langle H_2 \mid {}^{\rho_2(\tau)}\mathcal{MC}\,\mathbf{v}_2'\rangle \longmapsto^1 \langle H_2 \mid \mathsf{v}_2''\rangle.$$

  It suffices to show that

  $$(W', \mathsf{v}_1'', \mathsf{v}_2'') \in \mathcal{E}[\![\tau]\!]\rho.$$

  This follows from $(W', \mathbf{v}_1', \mathbf{v}_2') \in \mathcal{V}[\![\tau^{\langle\mathcal{C}\rangle}]\!]\rho$ by the bridge lemma and Lemma 7.5.

□

**Lemma 10.14 (Tuple)**
If $\Psi; \Delta; \Gamma \vdash \mathsf{v} \approx^{log}_{M+C} {}^{\tau}\mathbf{MC}(\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}) : \tau$, then

$$\Psi; \Delta; \Gamma \vdash \langle\overline{\mathsf{v}}\rangle \approx^{log}_{M+C} {}^{\langle\overline{\tau}\rangle}\mathbf{MC}(\langle\overline{\mathbf{v}}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}\rangle) : \langle\overline{\tau}\rangle$$

**Proof**

By Lemma 10.2, it suffices to show that

$$\Psi;\Delta;\Gamma \vdash \langle\overline{v}\rangle \approx^{log}_{M+C} {}^{\langle\overline{\tau}\rangle}\mathbf{MC}((\overline{\langle\mathbf{CM}^{\tau}(\mathbf{MC}((\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(x)/\mathbf{x}]})))\rangle})) : \langle\overline{\tau}\rangle.$$

Note that $\Psi;\Delta;\Gamma \vdash \langle\overline{v}\rangle : \langle\overline{\tau}\rangle$ and

$$\Psi;\Delta;\Gamma \vdash {}^{\langle\overline{\tau}\rangle}\mathbf{MC}((\overline{\langle\mathbf{CM}^{\langle\overline{\tau}\rangle}(\mathbf{MC}((\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(x)/\mathbf{x}]})))\rangle})) : \langle\overline{\tau}\rangle.$$

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W,\gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$\begin{aligned}&(W, \rho_1(\gamma_1(\langle\overline{v}\rangle)), \rho_2(\gamma_2({}^{\langle\overline{\tau}\rangle}\mathbf{MC}((\overline{\langle\mathbf{CM}^{\tau}(\mathbf{MC}((\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(x)/\mathbf{x}]})))\rangle}))))) \\ &= (W, \langle\overline{\rho_1(\gamma_1(v))}\rangle, {}^{\rho_2(\langle\overline{\tau}\rangle)}\mathbf{MC}((\overline{\langle\mathbf{CM}^{\rho_2(\tau)}(\mathbf{MC}((\rho_2(\gamma_2(\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(x)/\mathbf{x}]})))))\rangle}))) \in \mathcal{E}[\![\langle\overline{\tau}\rangle]\!]\rho.\end{aligned}$$

By our hypothesis,

$$\overline{(W, \rho_1(\gamma_1(v)), \rho_2(\gamma_2({}^{\tau}\mathbf{MC}((\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(x)/\mathbf{x}]}))))) \in \mathcal{E}[\![\tau]\!]\rho}.$$

Let $W' \sqsupseteq_{\text{pub}} W$ and $\overline{(W', v_1, v_2) \in \mathcal{V}[\![\tau]\!]\rho}$. By Lemma 7.13, it suffices to show that

$$(W', \langle\overline{v_1}\rangle, {}^{\rho_2(\langle\overline{\tau}\rangle)}\mathbf{MC}(\langle\overline{\mathbf{CM}^{\rho_2(\tau)}(v_2)}\rangle))) \in \mathcal{E}[\![\langle\overline{\tau}\rangle]\!]\rho.$$

We have this by Lemma 7.10, Lemma 7.5, and boundary cancellation. ☐

**Lemma 10.15 (Projection)**
If $\Psi;\Delta;\Gamma \vdash v \approx^{log}_{M+C} {}^{\langle\overline{\tau}\rangle}\mathbf{MC}(\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(x)/\mathbf{x}]}) : \langle\overline{\tau}\rangle$, then

$$\Psi;\Delta;\Gamma \vdash \pi_i(v) \approx^{log}_{M+C} {}^{\tau_i}\mathcal{MC}\ \boldsymbol{\pi_i}(\mathbf{v})\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(x)/\mathbf{x}]} : \tau_i$$

.

**Proof**

By Lemma 10.2, it suffices to show that

$$\Psi;\Delta;\Gamma \vdash \pi_i(v) \approx^{log}_{M+C} {}^{\tau_i}\mathcal{MC}\ (\boldsymbol{\pi_i}(\mathbf{CM}^{\langle\overline{\tau}\rangle}(\mathbf{MC}((\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(x)/\mathbf{x}]}))))) : \tau_i.$$

Note that $\Psi;\Delta;\Gamma \vdash \pi_i(v) : \tau_i$ and $\Psi;\Delta;\Gamma \vdash {}^{\tau_i}\mathcal{MC}\ (\boldsymbol{\pi_i}(\mathbf{CM}^{\langle\overline{\tau}\rangle}(\mathbf{MC}((\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(x)/\mathbf{x}]}))))) : \tau_i$.
Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W,\gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$\begin{aligned}&(W, \rho_1(\gamma_1(\pi_i(v))), \rho_2(\gamma_2({}^{\tau_i}\mathcal{MC}\ (\boldsymbol{\pi_i}(\mathbf{CM}^{\langle\overline{\tau}\rangle}(\mathbf{MC}((\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(x)/\mathbf{x}]}))))))))) \\ &= (W, \pi_i(\rho_1(\gamma_1(v))), {}^{\rho_2(\tau_i)}\mathcal{MC}\ \boldsymbol{\pi_i}(\mathbf{CM}^{\rho_2(\langle\overline{\tau}\rangle)}(\mathbf{MC}((\rho_2(\gamma_2(\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(x)/\mathbf{x}]}))))))) \in \mathcal{E}[\![\tau_i]\!]\rho.\end{aligned}$$

By our hypothesis, $(W, \rho_1(\gamma_1(v)), \rho_2(\gamma_2({}^{\langle\overline{\tau}\rangle}\mathbf{MC}((\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(x)/\mathbf{x}]}))))) \in \mathcal{E}[\![\langle\overline{\tau}\rangle]\!]\rho$.
Let $W' \sqsupseteq_{\text{pub}} W$ and $(W', \langle\overline{v_1}\rangle, \langle\overline{v_2}\rangle) \in \mathcal{V}[\![\langle\overline{\tau}\rangle]\!]\rho$. By Lemma 7.13, it suffices to show that

$$(W', \pi_i((\langle\overline{v_1}\rangle)), {}^{\rho_2(\tau_i)}\mathcal{MC}\ \boldsymbol{\pi_i}(\mathbf{CM}^{\rho_2(\langle\overline{\tau}\rangle)}(\langle\overline{v_2}\rangle))) \in \mathcal{E}[\![\tau_i]\!]\rho.$$

We have this by Lemma 7.10, Lemma 7.5, and boundary cancellation. ☐

**Lemma 10.16 (Let)**
If $\Psi;\Delta;\Gamma \vdash e_1 \approx^{log}_{M+C} {}^{\tau_1}\mathcal{MC}\ \mathbf{e_1}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(x)/\mathbf{x}]} : \tau_1$ and
$\Psi;\Delta;\Gamma, y : \tau_1 \vdash e_2 \approx^{log}_{M+C} {}^{\tau_1}\mathcal{MC}\ \mathbf{e_2}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(x)/\mathbf{x}]}[\mathcal{CM}^{\tau_1}\ y/\mathbf{y}] : \tau_2$, then

$$\Psi;\Delta;\Gamma \vdash \mathsf{let}\,y = e_1\ \mathsf{in}\ e_2 \approx^{log}_{M+C} {}^{\tau_2}\mathcal{MC}\ (\mathbf{let\,y = e_1\ in\ e_2})\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(x)/\mathbf{x}]} : \tau_2.$$

**Proof**

For brevity, let

$$\hat{\mathbf{e}}_1 = \mathcal{CM}^{\tau_1}\mathcal{MC}\ (\mathbf{e_1}\overline{[\mathcal{CM}^{\mathsf{ref}\ \tau'}\ \ell/\ell]}\overline{[\lceil\alpha\rceil/\alpha]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]})$$

$$\hat{\mathbf{e}}_2 = \mathbf{e_2}\overline{[\mathcal{CM}^{\mathsf{ref}\ \tau'}\ \ell/\ell]}\overline{[\lceil\alpha\rceil/\alpha]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}$$

By Lemma 10.2, it suffices to show that

$$\Psi;\Delta;\Gamma \vdash \mathsf{let}\,\mathsf{y} = \mathsf{e}_1\ \mathsf{in}\ \mathsf{e}_2 \approx^{log}_{M+C}\ {}^{\mathsf{ref}\ \tau}\mathcal{MC}\ (\mathbf{let}\ \mathbf{y} = \hat{\mathbf{e}}_1\ \mathbf{in}\ \hat{\mathbf{e}}_2[\mathbf{CM}^{\tau_1}(\mathbf{MC}(\mathbf{y}))/\mathbf{y}]) : \tau_2$$

Note that $\Psi;\Delta;\Gamma \vdash \mathsf{let}\,\mathsf{y} = \mathsf{e}_1\ \mathsf{in}\ \mathsf{e}_2 : \tau_2$ and $\Psi;\Delta;\Gamma \vdash {}^{\tau_2}\mathcal{MC}\ (\mathbf{let}\ \mathbf{y} = \hat{\mathbf{e}}_1\ \mathbf{in}\ \hat{\mathbf{e}}_2[\mathbf{CM}^{\tau_1}(\mathbf{MC}(\mathbf{y}))/\mathbf{y}]) : \tau_2$
Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W,\gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$\begin{aligned}&(W, \rho_1(\gamma_1(\mathsf{let}\,\mathsf{x} = \mathsf{e}_1\ \mathsf{in}\ \mathsf{e}_2)), \rho_2(\gamma_2({}^{\tau_2}\mathcal{MC}\ (\mathbf{let}\ \mathbf{y} = \hat{\mathbf{e}}_1\ \mathbf{in}\ \hat{\mathbf{e}}_2[\mathbf{CM}^{\rho_2(\tau_1)}(\mathbf{MC}(\mathbf{y}))/\mathbf{y}])))) \\ &\quad = (W, \mathsf{let}\,\mathsf{y} = \rho_1(\gamma_1(\mathsf{e}_1))\ \mathsf{in}\ \rho_1(\gamma_1(\mathsf{e}_2)), \\ &\qquad {}^{\rho_2(\tau_2)}\mathcal{MC}\ \mathbf{let}\ \mathbf{y} = \rho_2(\gamma_2(\hat{\mathbf{e}}_1))\ \mathbf{in}\ \rho_2(\gamma_2(\hat{\mathbf{e}}_2[\mathbf{CM}^{\rho_2(\tau_1)}(\mathbf{MC}(\mathbf{y}))/\mathbf{y}]))) \in \mathcal{E}[\![\tau_2]\!]\rho.\end{aligned}$$

By our hypothesis, $(W, \rho_1(\gamma_1(\mathsf{e}_1)), \rho_2(\gamma_2({}^{\tau_1}\mathcal{MC}\ \mathbf{e_1}\overline{[\mathcal{CM}^{\mathsf{ref}\ \tau'}\ \ell/\ell]}\overline{[\lceil\alpha\rceil/\alpha]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}))) \in \mathcal{E}[\![\tau_1]\!]\rho$
Let $W' \sqsupseteq_{\text{pub}} W$, $(W', \mathsf{v}_1, \mathsf{v}_1') \in \mathcal{V}[\![\tau_1]\!]\rho$. By Lemma 7.13, it suffices to show that

$$(W', \mathsf{let}\,\mathsf{y} = \mathsf{v}_1\ \mathsf{in}\ \rho_1(\gamma_1(\mathsf{e}_2)), {}^{\rho_2(\tau_2)}\mathcal{MC}\ (\mathbf{let}\ \mathbf{y} = \mathcal{CM}^{\rho_2(\tau_1)}\ \mathsf{v}_1'\ \mathbf{in}\ \rho_2(\gamma_2(\hat{\mathbf{e}}_2[\mathbf{CM}^{\rho_2(\tau_1)}(\mathbf{MC}(\mathbf{y}))/\mathbf{y}])))) \in \mathcal{E}[\![\tau_2]\!]\rho.$$

Applying Lemma 7.10 we assume $(H_1, H_2) : W'$ and let

$$\hat{\mathbf{v}}_1' = \mathbf{CM}^{\rho_2(\tau_1)}(\mathsf{v}_1') \quad \text{and} \quad \hat{\mathsf{v}}_1'' = {}^{\rho_2(\tau_1)}\mathbf{MC}(\hat{\mathbf{v}}_1')$$

By the operational semantics:

$$\langle H_1 \mid \mathsf{let}\,\mathsf{y} = \mathsf{v}_1\ \mathsf{in}\ \rho_1(\gamma_1(\mathsf{e}_2))\rangle \longmapsto^1 \langle H_1 \mid \rho_1(\gamma_1(\mathsf{e}_2))[\mathsf{v}_1/\mathsf{y}]\rangle.$$

$$\begin{aligned}&\langle H_2 \mid {}^{\rho_2(\tau_2)}\mathcal{MC}\ (\mathbf{let}\ \mathbf{y} = \hat{\mathbf{v}}_1'\ \mathbf{in}\ \rho_2(\gamma_2(\hat{\mathbf{e}}_2[\mathbf{CM}^{\rho_2(\tau_1)}(\mathbf{MC}(\mathbf{y}))/\mathbf{y}])))\rangle \\ &\quad \longmapsto^1 \langle H_2 \mid {}^{\rho_2(\tau_2)}\mathcal{MC}\ \rho_2(\gamma_2(\hat{\mathbf{e}}_2[\mathbf{CM}^{\rho_2(\tau_1)}(\mathbf{MC}(\hat{\mathbf{v}}_1'))/\mathbf{y}])))\rangle\end{aligned}$$

By Lemma 7.10, it suffices to show that

$$(W', \rho_1(\gamma_1(\mathsf{e}_2[\mathsf{v}_1/\mathsf{y}])), {}^{\rho_2(\tau_2)}\mathcal{MC}\ \rho_2(\gamma_2(\hat{\mathbf{e}}_2))[\mathbf{CM}^{\rho_2(\tau_1)}(\mathbf{MC}(\hat{\mathbf{v}}_1'))/\mathbf{y}]) \in \mathcal{E}[\![\tau_2]\!]\rho$$

By the fact that $\mathbf{CM}^{\rho_2(\tau_1)}(\mathbf{MC}(\hat{\mathbf{v}}_1')) = \mathbf{CM}^{\rho_2(\tau_1)}(\hat{\mathsf{v}}_1'')$ and Lemma 7.11, it suffices to show that

$$(W', \rho_1(\gamma_1(\mathsf{e}_2[\mathsf{v}_1/\mathsf{y}])), {}^{\rho_2(\tau_2)}\mathcal{MC}\ \rho_2(\gamma_2(\hat{\mathbf{e}}_2))[\mathbf{CM}^{\rho_2(\tau_1)}(\hat{\mathsf{v}}_1'')/\mathbf{y}]) \in \mathcal{E}[\![\tau_2]\!]\rho$$

This follows by instantiating the second premise with $W' \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W', \gamma[\mathsf{y} \mapsto (\mathsf{v}_1, \hat{\mathsf{v}}_1'')) \in \mathcal{G}[\![\Gamma, \mathsf{y} : \tau_1]\!]\rho$. It remains to show that $(W', \mathsf{v}_1, \hat{\mathsf{v}}_1'') \in \mathcal{V}[\![\tau_1]\!]\rho$ which follows from boundary cancellation and $(W', \mathsf{v}_1, \mathsf{v}_1') \in \mathcal{V}[\![\tau_1]\!]\rho$ □

**Lemma 10.17 (Pack)**
If $\Psi;\Delta;\Gamma \vdash \mathsf{v} \approx^{log}_{M+C}\ {}^{\tau[\tau'/\alpha]}\mathbf{MC}(\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\ \tau'}\ \ell/\ell]}\overline{[\lceil\alpha\rceil/\alpha]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}) : \tau[\tau'/\alpha]$ and $\tau'^{\mathcal{C}} = \boldsymbol{\tau'}$
then

$$\Psi;\Delta;\Gamma \vdash \mathsf{pack}\ \langle\tau', \mathsf{v}\rangle\ \mathsf{as}\ \exists\alpha.\tau \approx^{log}_{M+C}\ {}^{\exists\alpha.\tau}\mathbf{MC}((\mathbf{pack}\ \langle\boldsymbol{\tau'}, \mathbf{v}\rangle\ \mathbf{as}\ \exists\alpha.\tau^{\mathcal{C}})\overline{[\mathcal{CM}^{\mathsf{ref}\ \tau'}\ \ell/\ell]}\overline{[\lceil\alpha\rceil/\alpha]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}) : \exists\alpha.\tau.$$

**Proof**

First, the attentive reader might be worried that the C pack uses the compiler type translation and the boundary that is wrapping it expects types to be translated by the operational type translation. This is not an issue since once we apply the $\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}$ substitution to $\exists\alpha.\tau^{\mathcal{C}}$ we are left with $\exists\alpha.\tau^{\langle\mathcal{C}\rangle}$

By Lemma 10.2, it suffices to show that

$$\Psi;\Delta;\Gamma\vdash \mathsf{pack}\,\langle\tau',\mathsf{v}\rangle\;\mathsf{as}\;\exists\alpha.\tau \approx^{log}_{M+C}$$
$$^{\exists\alpha.\tau}\mathbf{MC}((\mathbf{pack}\,\langle\boldsymbol{\tau'},\mathbf{CM}^{\tau[\tau'/\alpha]}(\mathbf{MC}(\mathsf{v}))\rangle\;\mathbf{as}\;\exists\alpha.\tau^{\langle\mathcal{C}\rangle})\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}):\tau_2$$

Let $W\in\mathcal{H}[\![\Psi]\!]$, $\rho\in\mathcal{D}[\![\Delta]\!]$, and $(W,\gamma)\in\mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$(W,\rho_1(\gamma_1(\mathsf{pack}\,\langle\tau',\mathsf{v}\rangle\;\mathsf{as}\;\exists\alpha.\tau)),$$
$$\rho_2(\gamma_2(^{\exists\alpha.\tau}\mathbf{MC}((\mathbf{pack}\,\langle\boldsymbol{\tau'},\mathbf{CM}^{\tau[\tau'/\alpha]}(\mathbf{MC}(\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}))\rangle\;\mathbf{as}\;\exists\alpha.\tau^{\langle\mathcal{C}\rangle})))))$$
$$=(W,\mathsf{pack}\,\langle\rho_1(\tau'),\rho_1(\gamma_1(\mathsf{v}))\rangle\;\mathsf{as}\;\rho_1(\exists\alpha.\tau),$$
$$^{\rho_2(\exists\alpha.\tau)}\mathbf{MC}(\mathbf{pack}\,\langle\rho_2(\boldsymbol{\tau'}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}),\rho_2(\gamma_2(\mathbf{CM}^{\tau[\tau'/\alpha]}(\mathbf{MC}(\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}))))\rangle\;\mathbf{as}\;)$$
$$\rho_2(\exists\alpha.\tau^{\langle\mathcal{C}\rangle}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}))\in\mathcal{E}[\![\exists\alpha.\tau]\!]\rho$$

By our hypothesis, $(W,\rho_1(\gamma_1(\mathsf{v})),\rho_2(\gamma_2(^{\tau[\tau'/\alpha]}\mathbf{MC}(\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}))))\in\mathcal{E}[\![\tau[\tau'/\alpha]]\!]\rho$
Let $W'\sqsupseteq_{\text{pub}} W$, $(W',\mathsf{v}_1,\mathsf{v}_1')\in\mathcal{V}[\![\tau[\tau'/\alpha]]\!]\rho$. By Lemma 7.13, it suffices to show that

$$(W',\mathsf{pack}\,\langle\rho_1(\tau'),\mathsf{v}_1\rangle\;\mathsf{as}\;\rho_1(\exists\alpha.\tau),$$
$$^{\rho_2(\exists\alpha.\tau)}\mathbf{MC}(\mathbf{pack}\,\langle\rho_2(\boldsymbol{\tau'}),\mathbf{CM}^{\rho_2(\tau[\tau'/\alpha])}(\mathsf{v}_1')\rangle\;\mathbf{as}\;\rho_2(\exists\alpha.\tau^{\langle\mathcal{C}\rangle}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]})))\in\mathcal{E}[\![\exists\alpha.\tau]\!]\rho$$

Note that $\;^{\rho_2(\exists\alpha.\tau)}\mathbf{MC}(\mathbf{pack}\,\langle\rho_2(\boldsymbol{\tau'}),\mathbf{CM}^{\rho_2(\tau[\tau'/\alpha])}(\mathsf{v}_1')\rangle\;\mathbf{as}\;\rho_2(\exists\alpha.\tau^{\langle\mathcal{C}\rangle}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}))$
$$=\;\mathsf{pack}\,\langle\rho_2(\tau'),\mathbf{MC}(\mathbf{CM}^{\tau[\tau'/\alpha]}(\mathsf{v}_1'))\rangle\;\mathsf{as}\;\rho_2(\exists\alpha.\tau)$$

Hence, it suffices to show

$$(W',\mathsf{pack}\,\langle\rho_1(\tau'),\mathsf{v}_1\rangle\;\mathsf{as}\;\rho_1(\exists\alpha.\tau),\mathsf{pack}\,\langle\rho_2(\tau'),\mathbf{MC}(\mathbf{CM}^{\tau[\tau'/\alpha]}(\mathsf{v}_1'))\rangle\;\mathsf{as}\;\rho_2(\exists\alpha.\tau))\in\mathcal{E}[\![\exists\alpha.\tau]\!]\rho$$

which follows from boundary cancellation (Lemma 8.2) and compatibility lemmas (Lemma 9.13). ☐

**Lemma 10.18 (Unpack)**
If

$$\Psi;\Delta;\Gamma\vdash\mathsf{v}\approx^{log}_{M+C}\;^{\exists\alpha.\tau}\mathbf{MC}(\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}):\exists\alpha.\tau.$$

$$\Psi;\Delta,\beta;\Gamma,\mathsf{y}:\tau\vdash\mathsf{e}\approx^{log}_{M+C}\;^{\tau'}\mathcal{MC}\,\mathbf{e}[\lceil\beta\rceil/\boldsymbol{\beta}][\mathbf{CM}^{\tau}(\mathsf{y})/\mathbf{y}]\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}:\hat{\tau}$$

then

$$\Psi;\Delta;\Gamma\vdash\mathsf{unpack}\,\langle\beta,\mathsf{y}\rangle=\mathsf{v}\;\mathsf{in}\;\mathsf{e}\approx^{log}_{M+C}\;^{\tau'}\mathcal{MC}\;(\mathbf{unpack}\,\langle\boldsymbol{\beta},\mathbf{y}\rangle=\mathbf{v}\;\mathbf{in}\;\mathbf{e})\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}:\hat{\tau}.$$

**Proof**

For brevity, let $\hat{\mathbf{v}}=\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}$ and $\hat{\mathbf{e}}=\mathbf{e}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau'}\,\ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}$.
By Lemma 10.2, it suffices to show that

$$\cdot;\overline{\alpha};\overline{\mathsf{x}:\tau'}\vdash\mathsf{unpack}\,\langle\beta,\mathsf{y}\rangle=\mathsf{v}\;\mathsf{in}\;\mathsf{e}\approx^{log}_{M+C}\;^{\hat{\tau}}\mathcal{MC}\;(\mathbf{unpack}\,\langle\boldsymbol{\beta},\mathbf{y}\rangle=(\mathbf{CM}^{\exists\beta.\tau}(\mathbf{MC}(\hat{\mathbf{v}})))\;\mathbf{in}\;\hat{\mathbf{e}}[\mathbf{CM}^{\tau}(\mathbf{MC}(\mathbf{y}))/\mathbf{y}]):\hat{\tau}.$$

Note that $\cdot;\overline{\alpha};\overline{\mathsf{x}:\tau'}\vdash\mathsf{unpack}\,\langle\beta,\mathsf{y}\rangle=\mathsf{v}\;\mathsf{in}\;\mathsf{e}:\hat{\tau}$ and

$$\cdot;\overline{\alpha};\overline{\mathsf{x}:\tau'}\vdash\;^{\hat{\tau}}\mathcal{MC}\;(\mathbf{unpack}\,\langle\boldsymbol{\beta},\mathbf{y}\rangle=(\mathbf{CM}^{\exists\beta.\tau}(\mathbf{MC}(\hat{\mathbf{v}})))\;\mathbf{in}\;\hat{\mathbf{e}}[\mathbf{CM}^{\tau}(\mathbf{MC}(\mathbf{y}))/\mathbf{y}]):\hat{\tau}.$$

Let $W \in \text{World}$, $\rho \in \mathcal{D}[\![\overline{\alpha}]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\overline{x : \tau}]\!]\rho$. We need to show that

$$(W, \rho_1(\gamma_1(\mathsf{unpack}\ \langle\beta, \mathsf{y}\rangle = \mathsf{v}\ \mathsf{in}\ \mathsf{e})),\ \rho_2(\gamma_2({}^{\hat{\tau}}\mathcal{MC}\ (\mathbf{unpack}\ \langle\boldsymbol{\beta}, \mathbf{y}\rangle = (\mathbf{CM}^{\exists\beta.\tau}(\mathbf{MC}(\hat{\mathbf{v}})))\ \mathbf{in}\ \hat{\mathbf{e}}[\mathbf{CM}^{\tau}(\mathbf{MC}(\mathbf{y}))/\mathbf{y}]))))$$
$$= (W, \mathsf{unpack}\ \langle\beta, \mathsf{y}\rangle = \rho_1(\gamma_1(\mathsf{v}))\ \mathsf{in}\ \rho_1(\gamma_1(\mathsf{e})),\ {}^{\rho_2(\hat{\tau})}\mathcal{MC}\ (\mathbf{unpack}\ \langle\boldsymbol{\beta}, \mathbf{y}\rangle = (\mathbf{CM}^{\rho_2(\exists\beta.\tau)}(\mathbf{MC}(\rho_2(\gamma_2(\hat{\mathbf{v}})))))\ \mathbf{in}\ \rho_2(\gamma_2(\hat{\mathbf{e}}))[\mathbf{CM}^{\rho_2(\tau)}(\mathbf{MC}(\mathbf{y}))/\mathbf{y}]) \in \mathcal{E}[\![\hat{\tau}]\!]\rho.$$

By our first hypothesis, $(W, \rho_1(\gamma_1(\mathsf{v})), {}^{\rho_2(\exists\beta.\tau)}\mathbf{MC}(\rho_2(\gamma_2(\hat{\mathbf{v}})))) \in \mathcal{E}[\![\exists\beta.\tau]\!]\rho$. Let $W' \sqsupseteq W$ and

$$(W', \mathsf{pack}\ \langle\tau_1, \mathsf{v}_1\rangle\ \mathsf{as}\ \rho_1(\exists\beta.\tau), \mathsf{pack}\ \langle\tau_2, \mathsf{v}_2\rangle\ \mathsf{as}\ \rho_2(\exists\beta.\tau)) \in \mathcal{V}[\![\exists\beta.\tau]\!]\rho.$$

By Lemma 7.13, it suffices to show that

$$(W', \mathsf{unpack}\ \langle\beta, \mathsf{y}\rangle = (\mathsf{pack}\ \langle\tau_1, \mathsf{v}_1\rangle\ \mathsf{as}\ \rho_1(\exists\beta.\tau))\ \mathsf{in}\ \rho_1(\gamma_1(\mathsf{e})),\ {}^{\rho_2(\hat{\tau})}\mathcal{MC}\ (\mathbf{unpack}\ \langle\boldsymbol{\beta}, \mathbf{y}\rangle = (\mathbf{CM}^{\rho_2(\exists\beta.\tau)}(\mathsf{pack}\ \langle\tau_2, \mathsf{v}_2\rangle))\ \mathbf{in}\ \rho_2(\gamma_2(\hat{\mathbf{e}}))[\mathbf{CM}^{\rho_2(\tau)}(\mathbf{MC}(\mathbf{y}))/\mathbf{y}]) \in \mathcal{E}[\![\hat{\tau}]\!]\rho.$$

By Lemma 7.1, there are some $\mathbf{v_2}$ and $\mathsf{v}_2'$ such that

$$\mathbf{CM}^{\rho_2(\tau[\tau_2/\beta])}(\mathsf{v}_2) = \mathbf{v_2} \quad \text{and} \quad {}^{\rho_2(\tau[\tau_2/\beta])}\mathbf{MC}(\mathbf{v_2}) = \mathsf{v}_2'.$$

Applying Lemma 7.10, we assume that $(H_1, H_2) : W'$. By the operational semantics we have that

$$\langle H_1 \mid \mathsf{unpack}\ \langle\beta, \mathsf{y}\rangle = (\mathsf{pack}\ \langle\tau_1, \mathsf{v}_1\rangle\ \mathsf{as}\ \rho_1(\exists\beta.\tau))\ \mathsf{in}\ \rho_1(\gamma_1(\mathsf{e}))\rangle \longmapsto \langle H_1 \mid \rho_1(\gamma_1(\mathsf{e}))[\tau_1/\beta][\mathsf{v}_1/\mathsf{y}]\rangle$$

and

$$\langle H_2 \mid {}^{\rho_2(\hat{\tau})}\mathcal{MC}\ (\mathbf{unpack}\ \langle\boldsymbol{\beta}, \mathbf{y}\rangle = (\mathbf{CM}^{\rho_2(\exists\beta.\tau)}(\mathsf{pack}\ \langle\tau_2, \mathsf{v}_2\rangle))\ \mathbf{in}\ \rho_2(\gamma_2(\hat{\mathbf{e}}))[\mathbf{CM}^{\rho_2(\tau)}(\mathbf{MC}(\mathbf{y}))/\mathbf{y}])\rangle$$
$$\longmapsto^2 \langle H_2 \mid {}^{\rho_2(\hat{\tau})}\mathcal{MC}\ (\rho_2(\gamma_2(\hat{\mathbf{e}}')))[\tau_2^{\langle\mathcal{C}\rangle}/\boldsymbol{\beta}][\mathbf{CM}^{\rho_2(\tau)}(\mathbf{MC}(\mathbf{v_2}))/\mathbf{y}])\rangle.$$

By Lemma 7.10, it suffices to show that

$$(W', \rho_1(\gamma_1(\mathsf{e}))[\tau_1/\beta][\mathsf{v}_1/\mathsf{y}], {}^{\rho_2(\hat{\tau})}\mathcal{MC}\ (\rho_2(\gamma_2(\hat{\mathbf{e}}')))[\tau_2^{\langle\mathcal{C}\rangle}/\boldsymbol{\beta}][\mathbf{CM}^{\rho_2(\tau)}(\mathbf{MC}(\mathbf{v_2}))/\mathbf{y}])) \in \mathcal{E}[\![\hat{\tau}]\!]\rho.$$

By Lemma 7.11, it suffices to show that

$$(W', \rho_1(\gamma_1(\mathsf{e}))[\tau_1/\beta][\mathsf{v}_1/\mathsf{y}], {}^{\rho_2(\hat{\tau})}\mathcal{MC}\ (\rho_2(\gamma_2(\hat{\mathbf{e}})))[\tau_2^{\langle\mathcal{C}\rangle}/\boldsymbol{\beta}][\mathbf{CM}^{\rho_2(\tau)}(\mathsf{v}_2')/\mathbf{y}])) \in \mathcal{E}[\![\hat{\tau}]\!]\rho.$$

By definition of $\mathcal{V}[\![\exists\beta.\tau]\!]\rho$, there is some $\text{VR} \in \text{MMValRel}$ such that $\text{VR}.\tau_1 = \tau_1$, $\text{VR}.\tau_2 = \tau_2$, and $(W', \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\tau]\!]\rho[\beta \mapsto \text{VR}]$. By boundary cancellation, $(W', \mathsf{v}_1, \mathsf{v}_2') \in \mathcal{V}[\![\tau]\!]\rho[\beta \mapsto \text{VR}]$. Therefore $\rho[\beta \mapsto \text{VR}] \in \mathcal{D}[\![\overline{\alpha}, \beta]\!]$ and $(W', \gamma[\mathsf{y} \mapsto (\mathsf{v}_1, \mathsf{v}_2')]) \in \mathcal{G}[\![\overline{x : \tau'}, \mathsf{y} : \tau]\!]\rho$. Hence we can apply our second hypothesis to get exactly this result. □

**Lemma 10.19 (Function)**
Let $k' \geq k$, $m' \geq m$, $\Psi = \overline{\ell : \tau''}$, $\Delta = \beta_1, \ldots, \beta_{k'}$, and $\Gamma = \mathsf{y}_1 : \tau_1, \ldots, \mathsf{y}_{m'} : \tau_{m'}$.
If $\text{fv}(\lambda[\overline{\alpha}](\overline{x : \tau}).\mathsf{e}) = \mathsf{y}_1, \ldots, \mathsf{y}_m$, $\text{ftv}(\lambda[\overline{\alpha}](\overline{x : \tau}).\mathsf{e}) = \beta_1, \ldots, \beta_k$, $\boldsymbol{\tau}_{\mathbf{env}} = \langle\boldsymbol{\tau_1}^{\mathcal{C}}, \ldots, \boldsymbol{\tau_m}^{\mathcal{C}}\rangle$,

$$\Psi; (\Delta, \overline{\alpha}); (\Gamma, \overline{x : \tau}) \vdash \mathsf{e} \approx^{log}_{M+C} {}^{\tau'}\mathcal{MC}\ \mathbf{e}\overline{[\mathcal{CM}^{\mathsf{ref}\ \tau''}\ \ell/\boldsymbol{\ell}]}[\lceil\beta_1\rceil/\boldsymbol{\beta_1}] \cdots [\lceil\beta_{k'}\rceil/\boldsymbol{\beta_{k'}}]\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}$$
$$[\mathbf{CM}^{\tau_1}(\mathsf{y}_1)/\mathbf{y_1}] \cdots [\mathbf{CM}^{\tau_{m'}}(\mathsf{y}_{m'})/\mathbf{y_{m'}}]\overline{[\mathbf{CM}^{\tau}(\mathsf{x})/\mathbf{x}]} : \tau',$$

$\mathbf{v_f} = \boldsymbol{\lambda}[\boldsymbol{\beta_1}, \ldots, \boldsymbol{\beta_k}, \overline{\boldsymbol{\alpha}}](\mathbf{z} : \boldsymbol{\tau}_{\mathbf{env}}, \overline{\mathbf{x} : \boldsymbol{\tau}^{\mathcal{C}}}).\mathbf{let}\ \mathbf{y_1} = \boldsymbol{\pi}_1(\mathbf{z})\ \mathbf{in}\ \ldots\mathbf{let}\ \mathbf{y_m} = \boldsymbol{\pi}_{\mathbf{m}}(\mathbf{z})\ \mathbf{in}\ \mathbf{e}$, and

$$\mathbf{v} = \mathbf{pack}\ \langle\boldsymbol{\tau}_{\mathbf{env}}, \langle\mathbf{v_f}[\boldsymbol{\beta_1}] \cdots [\boldsymbol{\beta_k}], \langle\mathbf{y_1}, \ldots, \mathbf{y_m}\rangle\rangle\rangle\ \mathbf{as}\ \exists\boldsymbol{\alpha}'.\langle(\forall[\overline{\boldsymbol{\alpha}}].(\boldsymbol{\alpha}', \overline{\boldsymbol{\tau}^{\mathcal{C}}}) \to \boldsymbol{\tau}'^{\mathcal{C}}), \boldsymbol{\alpha}'\rangle,$$

then

$$\Psi; \Delta; \Gamma \vdash \lambda[\overline{\alpha}](\overline{x : \tau}).\mathsf{e} \approx^{log}_{M+C} {}^{\forall[\overline{\alpha}].(\overline{\tau}) \to \tau'}\mathbf{MC}(\mathbf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\ \tau''}\ \ell/\boldsymbol{\ell}]}[\lceil\beta_1\rceil/\boldsymbol{\beta_1}] \cdots [\lceil\beta_{k'}\rceil/\boldsymbol{\beta_{k'}}])$$
$$[\mathbf{CM}^{\tau_1}(\mathsf{y}_1)/\mathbf{y_1}] \cdots [\mathbf{CM}^{\tau_{m'}}(\mathsf{y}_{m'})/\mathbf{y_{m'}}] : \forall[\overline{\alpha}].(\overline{x : \tau}) \to \tau'.$$

**Proof**

Let

$$\hat{\mathbf{e}} = \mathbf{let\ y_1} = \mathcal{CM}^{\tau_1}\,\mathcal{MC}\ \boldsymbol{\pi_1}(\mathbf{z})\ \mathbf{in}\ \ldots \mathbf{let\ y_m} = \mathcal{CM}^{\tau_m}\,\mathcal{MC}\ \boldsymbol{\pi_m}(\mathbf{z})\ \mathbf{in\ e}\overline{[\mathbf{CM}^{\tau}(\mathbf{MC}(\mathbf{x}))/\mathbf{x}]}$$

$$\mathbf{v_f'} = \boldsymbol{\lambda}[\boldsymbol{\beta_1},\ldots,\boldsymbol{\beta_k},\overline{\alpha}](\mathbf{z}\colon\boldsymbol{\tau_{env}},\overline{\mathbf{x}\colon\tau^{\mathcal{C}}}).\hat{\mathbf{e}}$$

and

$$\mathbf{v'} = \mathbf{pack}\ \langle\boldsymbol{\tau_{env}},\langle\mathbf{v_f'}[\boldsymbol{\beta_1}]\cdots[\boldsymbol{\beta_k}],\langle\mathbf{y_1},\ldots,\mathbf{y_m}\rangle\rangle\rangle\ \mathbf{as}\ \exists\boldsymbol{\alpha'}.\langle(\forall[\overline{\alpha}].(\boldsymbol{\alpha'},\overline{\tau^{\mathcal{C}}})\to\tau'^{\mathcal{C}}),\boldsymbol{\alpha'}\rangle.$$

By Lemma 10.2, it suffices to show that

$$\Psi;\Delta;\Gamma\vdash\lambda[\overline{\alpha}](\overline{\mathsf{x}\colon\tau}).\mathsf{e}\approx^{log}_{M+C}{}^{\forall[\overline{\alpha}].(\overline{\tau})\to\tau'}\mathbf{MC}(\mathbf{v'}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau''}\,\ell/\boldsymbol{\ell}]}[\lceil\beta_1\rceil/\boldsymbol{\beta_1}]\cdots[\lceil\beta_{k'}\rceil/\boldsymbol{\beta_{k'}}])$$
$$[\mathbf{CM}^{\tau_1}(\mathsf{y_1})/\mathbf{y_1}]\cdots[\mathbf{CM}^{\tau_{m'}}(\mathsf{y_{m'}})/\mathbf{y_{m'}}]\colon\forall[\overline{\alpha}].(\overline{\mathsf{x}\colon\tau})\to\tau'.$$

Note that $\Psi;\Delta;\Gamma\vdash\lambda[\overline{\alpha}](\overline{\mathsf{x}\colon\tau}).\mathsf{e}\colon\forall[\overline{\alpha}].(\overline{\tau})\to\tau'$ and

$$\Psi;\Delta;\Gamma\vdash{}^{\forall[\overline{\alpha}].(\overline{\tau})\to\tau'}\mathbf{MC}(\mathbf{v'}\overline{[\mathcal{CM}^{\mathsf{ref}\,\tau''}\,\ell/\boldsymbol{\ell}]}[\lceil\beta_1\rceil/\boldsymbol{\beta_1}]\cdots[\lceil\beta_{k'}\rceil/\boldsymbol{\beta_{k'}}])$$
$$[\mathbf{CM}^{\tau_1}(\mathsf{y_1})/\mathbf{y_1}]\cdots[\mathbf{CM}^{\tau_{m'}}(\mathsf{y_{m'}})/\mathbf{y_{m'}}]\colon\forall[\overline{\alpha}].(\overline{\mathsf{x}\colon\tau})\to\tau'.$$

Let $W\in\mathcal{H}[\![\Psi]\!]$, $\rho\in\mathcal{D}[\![\Delta]\!]$, and $(W,\gamma)\in\mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$(W,\rho_1(\gamma_1(\lambda[\overline{\alpha}](\overline{\mathsf{x}\colon\tau}).\mathsf{e})),\rho_2(\gamma_2({}^{\forall[\overline{\alpha}].(\overline{\tau})\to\tau'}\mathbf{MC}(\mathbf{v'}\overline{[\mathcal{CM}^{\rho_2(\mathsf{ref}\,\tau'')}\,\ell/\boldsymbol{\ell}]}[\lceil\beta_1\rceil/\boldsymbol{\beta_1}]\cdots[\lceil\beta_{k'}\rceil/\boldsymbol{\beta_{k'}}])$$
$$[\mathbf{CM}^{\tau_1}(\mathsf{y_1})/\mathbf{y_1}]\cdots[\mathbf{CM}^{\tau_{m'}}(\mathsf{y_{m'}})/\mathbf{y_{m'}}])))$$
$$=(W,\lambda[\overline{\alpha}](\overline{\mathsf{x}\colon\tau}).\rho_1(\gamma_1(\mathsf{e})),{}^{\forall[\overline{\alpha}].(\overline{\tau})\to\tau'}\mathbf{MC}(\rho_2(\gamma_2(\mathbf{v'}))\overline{[\mathcal{CM}^{\rho_2(\mathsf{ref}\,\tau'')}\,\ell/\boldsymbol{\ell}]}[\rho_2(\beta_1)^{\langle\mathcal{C}\rangle}/\boldsymbol{\beta_1}]\cdots[\rho_2(\beta_{k'})^{\langle\mathcal{C}\rangle}/\boldsymbol{\beta_{k'}}])$$
$$[\mathbf{CM}^{\rho_2(\tau_1)}(\gamma_2(\mathsf{y_1}))/\mathbf{y_1}]\cdots[\mathbf{CM}^{\rho_2(\tau_{m'})}(\gamma_2(\mathsf{y_{m'}}))/\mathbf{y_{m'}}])$$
$$\in\mathcal{E}[\![\forall[\overline{\alpha}].(\overline{\tau})\to\tau']\!]\rho.$$

Note that

$$\rho_2(\gamma_2(\mathbf{v'}))\overline{[\mathcal{CM}^{\rho_2(\mathsf{ref}\,\tau'')}\,\ell/\boldsymbol{\ell}]}[\rho_2(\beta_1)^{\langle\mathcal{C}\rangle}/\boldsymbol{\beta_1}]\cdots[\rho_2(\beta_{k'})^{\langle\mathcal{C}\rangle}/\boldsymbol{\beta_{k'}}][\mathbf{CM}^{\rho_2(\tau_1)}(\gamma_2(\mathsf{y_1}))/\mathbf{y_1}]\cdots[\mathbf{CM}^{\rho_2(\tau_{m'})}(\gamma_2(\mathsf{y_{m'}}))/\mathbf{y_{m'}}]$$
$$=\mathbf{pack}\langle\boldsymbol{\tau_{env}}[\rho_2(\beta_1)^{\langle\mathcal{C}\rangle}/\boldsymbol{\beta_1}]\cdots[\rho_2(\beta_{k'})^{\langle\mathcal{C}\rangle}/\boldsymbol{\beta_{k'}}],$$
$$\langle\rho_2(\gamma_2(\mathbf{v_f'}))\overline{[\mathcal{CM}^{\rho_2(\mathsf{ref}\,\tau'')}\,\ell/\boldsymbol{\ell}]}[\rho_2(\beta_1)^{\langle\mathcal{C}\rangle}]\cdots[\rho_2(\beta_{k'})^{\langle\mathcal{C}\rangle}],\langle\mathbf{CM}^{\rho_2(\tau_1)}(\gamma_2(\mathsf{y_1})),\ldots,\mathbf{CM}^{\rho_2(\tau_1)}(\gamma_2(\mathsf{y_m}))\rangle\rangle\rangle.$$

By Lemma 7.1, there are some $\mathbf{v_1},\ldots,\mathbf{v_m}$ and $\mathsf{v_1'},\ldots,\mathsf{v_m'}$ such that for each $1\le i\le m$

$$\mathbf{CM}^{\rho_2(\tau_i)}(\gamma_1(\mathsf{y_i}))=\mathbf{v_i}\quad\text{and}\quad{}^{\rho_2(\tau_i)}\mathbf{MC}(\mathbf{v_i})=\mathsf{v_i'}.$$

Let

$$\hat{\mathbf{v}}_{\mathbf{f}}'=\rho_2(\gamma_2(\mathbf{v_f'}))\overline{[\mathcal{CM}^{\rho_2(\mathsf{ref}\,\tau'')}\,\ell/\boldsymbol{\ell}]}$$

$$\hat{\mathbf{v}}'=\mathbf{pack}\ \langle\boldsymbol{\tau_{env}}[\rho_2(\beta_1)^{\langle\mathcal{C}\rangle}/\boldsymbol{\beta_1}]\cdots[\rho_2(\beta_{k'})^{\langle\mathcal{C}\rangle}/\boldsymbol{\beta_{k'}}],\langle\hat{\mathbf{v}}_{\mathbf{f}}'[\rho_2(\beta_1)^{\langle\mathcal{C}\rangle}]\cdots[\rho_2(\beta_{k'})^{\langle\mathcal{C}\rangle}],\langle\mathbf{v_1},\ldots,\mathbf{v_m}\rangle\rangle\rangle.$$

By Lemma 7.10 and Lemma 7.5, it suffices to show that

$$(W,\lambda[\overline{\alpha}](\overline{\mathsf{x}\colon\tau}).\rho_1(\gamma_1(\mathsf{e})),$$
$$\lambda[\overline{\alpha}](\overline{\mathsf{x}\colon\tau}).({}^{\tau'}\mathcal{MC}\ \mathbf{unpack}\ \langle\boldsymbol{\beta'},\mathbf{z}\rangle=\hat{\mathbf{v}}'\ \mathbf{in\ let}\ (\mathbf{f},\hat{\mathbf{y}})=(\boldsymbol{\pi_1}(\mathbf{z}),\boldsymbol{\pi_2}(\mathbf{z}))\ \mathbf{in\ f}\ [\overline{\lceil\alpha\rceil}]\ (\hat{\mathbf{y}},\overline{\mathbf{CM}^{\tau}(\mathsf{x})})))$$
$$\in\mathcal{V}[\![\forall[\overline{\alpha}].(\overline{\tau})\to\tau']\!]\rho.$$

Let $W' \sqsupseteq W$, $\overline{\mathrm{VR} \in \mathrm{MMValRel}}$, and $\overline{(W', \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2) \in \mathcal{V}[\![\tau]\!]\rho[\alpha \mapsto \mathrm{VR}]}$. For convenience, let $\overline{\hat{\tau}_1 = \mathrm{VR}.\tau_1}$ and $\overline{\hat{\tau}_2 = \mathrm{VR}.\tau_2}$. We need to show that

$$\begin{aligned}&(W', (\lambda[\overline{\alpha}](\overline{\mathsf{x}:\tau}).\rho_1(\gamma_1(\mathsf{e})))\ [\overline{\hat{\tau}_1}]\ \overline{\hat{\mathsf{v}}_1},\\ &\quad (\lambda[\overline{\alpha}](\overline{\mathsf{x}:\tau}).{}^{\tau'}\mathcal{MC}\ \textbf{unpack}\ \langle \boldsymbol{\beta}', \mathbf{z}\rangle = \hat{\mathbf{v}}'\ \textbf{in let}\ (\mathbf{f}, \hat{\mathbf{y}}) = (\pi_1(\mathbf{z}), \pi_2(\mathbf{z}))\ \textbf{in f}\ [\overline{\lceil \alpha \rceil}]\ (\hat{\mathbf{y}}, \overline{\mathbf{CM}^{\tau}(\mathsf{x})}))\ [\overline{\hat{\tau}_2}]\ \overline{\hat{\mathsf{v}}_2})\\ &\quad \in \mathcal{E}[\![\tau']\!]\rho\overline{[\alpha \mapsto \mathrm{VR}]}.\end{aligned}$$

By Lemma 7.10, it suffices to show that

$$\begin{aligned}&(W', \rho_1(\gamma_1(\mathsf{e}))\overline{[\hat{\tau}_1/\alpha]}\overline{[\hat{\mathsf{v}}_1/\mathsf{x}]},\\ &\quad {}^{\tau'\overline{[\hat{\tau}_2/\alpha]}}\mathcal{MC}\ (\hat{\mathbf{v}}'_{\mathbf{f}}[\rho_2(\beta_1)^{\langle\mathcal{C}\rangle}] \cdots [\rho_2(\beta_{\mathsf{k}'})^{\langle\mathcal{C}\rangle}]\ [\overline{\hat{\tau}_2{}^{\langle\mathcal{C}\rangle}}]\ \langle \mathbf{v_1}, \ldots, \mathbf{v_m}\rangle, \overline{\mathbf{CM}^{\tau\overline{[\hat{\tau}_2/\alpha]}}(\hat{\mathsf{v}}_2)})) \in \mathcal{E}[\![\tau']\!]\rho\overline{[\alpha \mapsto \mathrm{VR}]}.\end{aligned}$$

By Lemma 7.1, there are some $\overline{\hat{\mathbf{v}}_2}$ and $\overline{\hat{\mathsf{v}}'_2}$ such that

$$\overline{\mathbf{CM}^{\tau\overline{[\hat{\tau}_2/\alpha]}}(\hat{\mathsf{v}}_2) = \hat{\mathbf{v}}_2} \quad \text{and} \quad \overline{{}^{\tau\overline{[\hat{\tau}_2/\alpha]}}\mathbf{MC}(\hat{\mathbf{v}}_2) = \hat{\mathsf{v}}'_2}.$$

Hence, it suffices to show that

$$\begin{aligned}&(W', \rho_1(\gamma_1(\mathsf{e}))\overline{[\hat{\tau}_1/\alpha]}\overline{[\hat{\mathsf{v}}_1/\mathsf{x}]},\\ &\quad {}^{\tau'\overline{[\hat{\tau}_2/\alpha]}}\mathcal{MC}\ (\hat{\mathbf{v}}'_{\mathbf{f}}[\rho_2(\beta_1)^{\langle\mathcal{C}\rangle}] \cdots [\rho_2(\beta_{\mathsf{k}'})^{\langle\mathcal{C}\rangle}]\ [\overline{\hat{\tau}_2{}^{\langle\mathcal{C}\rangle}}]\ \langle \mathbf{v_1}, \ldots, \mathbf{v_m}\rangle, \overline{\hat{\mathbf{v}}_2})) \in \mathcal{E}[\![\tau']\!]\rho\overline{[\alpha \mapsto \mathrm{VR}]}.\end{aligned}$$

Note that for some $(H_1, H_2) : W'$

$$\begin{aligned}&\langle H_2 \mid {}^{\tau'\overline{[\hat{\tau}_2/\alpha]}}\mathcal{MC}\ (\mathbf{v}'_{\mathbf{f}}[\rho_2(\beta_1)^{\langle\mathcal{C}\rangle}] \cdots [\rho_2(\beta_{\mathsf{k}'})^{\langle\mathcal{C}\rangle}]\ [\overline{\hat{\tau}_2{}^{\langle\mathcal{C}\rangle}}]\ \langle \mathbf{v_1}, \ldots, \mathbf{v_m}\rangle, \overline{\hat{\mathbf{v}}_2})\rangle\\ &\quad \longmapsto^{*} \langle H_2 \mid {}^{\tau'\overline{[\hat{\tau}_2/\alpha]}}\mathcal{MC}\ \mathbf{e}\overline{[\mathcal{CM}^{\rho_2(\text{ref}\ \tau'')}\ \ell/\boldsymbol{\ell}]}[(\mathbf{CM}^{\tau_1}(\mathbf{MC}(\mathbf{v_1})))/\mathbf{y_1}] \cdots [(\mathbf{CM}^{\tau_m}(\mathbf{MC}(\mathbf{v_1})))/\mathbf{y_m}]\\ &\qquad\qquad \overline{[\mathbf{CM}^{\tau}(\mathbf{MC}(\hat{\mathbf{v}}_2))/\mathbf{x}]}[\rho_2(\beta_1)^{\langle\mathcal{C}\rangle}/\boldsymbol{\beta_1}] \cdots [\rho_2(\beta_{\mathsf{k}'})^{\langle\mathcal{C}\rangle}/\boldsymbol{\beta_{k'}}]\overline{[\hat{\tau}_2{}^{\langle\mathcal{C}\rangle}/\boldsymbol{\alpha}]}\rangle\end{aligned}$$

By Lemma 7.10 and multiple uses of Lemma 7.11, it suffices to show

$$\begin{aligned}&(W', \rho_1(\gamma_1(\mathsf{e}))\overline{[\hat{\tau}_1/\alpha]}\overline{[\hat{\mathsf{v}}_1/\mathsf{x}]},\\ &\quad {}^{\tau'\overline{[\hat{\tau}_2/\alpha]}}\mathcal{MC}\ \mathbf{e}\overline{[\mathcal{CM}^{\rho_2(\text{ref}\ \tau'')}\ \ell/\boldsymbol{\ell}]}[\mathbf{CM}^{\tau_1}(\mathsf{v}'_1)/\mathbf{y_1}] \cdots [\mathbf{CM}^{\tau_m}(\mathsf{v}'_m)/\mathbf{y_m}]\overline{[\mathbf{CM}^{\tau}(\hat{\mathsf{v}}'_2)/\mathbf{x}]}\\ &\qquad [\rho_2(\beta_1)^{\langle\mathcal{C}\rangle}/\boldsymbol{\beta_1}] \cdots [\rho_2(\beta_{\mathsf{k}'})^{\langle\mathcal{C}\rangle}/\boldsymbol{\beta_{k'}}]\overline{[\hat{\tau}_2{}^{\langle\mathcal{C}\rangle}/\boldsymbol{\alpha}]}) \in \mathcal{E}[\![\tau']\!]\rho\overline{[\alpha \mapsto \mathrm{VR}]}.\end{aligned}$$

We have that $\rho\overline{[\alpha \mapsto \mathrm{VR}]} \in \mathcal{D}[\![\Delta, \overline{\alpha}]\!]$, and by monotonicity and boundary cancellation, that

$$\begin{aligned}&(W', \emptyset[\mathsf{y_1} \mapsto (\gamma_1(\mathsf{y_1}), \mathsf{v}'_1)] \cdots [\mathsf{y_m} \mapsto (\gamma_1(\mathsf{y_m}), \mathsf{v}'_m)][\mathsf{y_{m+1}} \mapsto \gamma(\mathsf{y_{m+1}})] \cdots [\mathsf{y_{m'}} \mapsto \gamma(\mathsf{y_{m'}})]\overline{[\mathsf{x} \mapsto (\hat{\mathsf{v}}_1, \hat{\mathsf{v}}'_2)]})\\ &\qquad\qquad \in \mathcal{G}[\![\Gamma, \overline{\mathsf{x}:\tau}]\!]\rho\overline{[\alpha \mapsto \mathrm{VR}]}.\end{aligned}$$

Therefore we can apply our hypothesis to get exactly the needed result. □

**Lemma 10.20 (Application)**
If $\Psi; \Delta; \Gamma \vdash \mathsf{v_0} \approx^{log}_{M+C} {}^{\forall[\overline{\alpha}].(\overline{\tau_1}) \to \tau_2}\mathbf{MC}(\mathbf{v_0}\overline{[\mathcal{CM}^{\text{ref}\ \tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}) : \forall[\overline{\alpha}].(\overline{\tau_1}) \to \tau_2$, $\overline{\Delta \vdash \tau}$, and

$$\overline{\Psi; \Delta; \Gamma \vdash \mathsf{v} \approx^{log}_{M+C} {}^{\tau_1\overline{[\tau/\alpha]}}\mathbf{MC}(\mathbf{v}\overline{[\mathcal{CM}^{\text{ref}\ \tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}) : \tau_1\overline{[\tau/\alpha]}},$$

then

$$\begin{aligned}&\Psi; \Delta; \Gamma \vdash \mathsf{v_0}\ [\overline{\tau}]\ \overline{\mathsf{v}} \approx^{log}_{M+C}\\ &{}^{\tau_2\overline{[\tau/\alpha]}}\mathcal{MC}\ (\textbf{unpack}\ \langle \boldsymbol{\beta}', \mathbf{z}\rangle = \mathbf{v_0}\ \textbf{in let}\ (\mathbf{f}, \mathbf{y}) = (\pi_1(\mathbf{z}), \pi_2(\mathbf{z}))\ \textbf{in f}\ [\overline{\tau^{\mathcal{C}}}]\ (\mathbf{y}, \overline{\mathbf{v}}))\overline{[\mathcal{CM}^{\text{ref}\ \tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil\alpha\rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]} : \tau_2\overline{[\tau/\alpha]}\end{aligned}$$

**Proof**

Let $\hat{\mathsf{v}}_0 = \mathsf{v}_0\overline{[\mathcal{CM}^{\mathsf{ref}\ \tau'}\ \ell/\ell]}\overline{[\lceil\alpha\rceil/\alpha]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathsf{x}]}$ and $\hat{\mathsf{v}} = \overline{\mathsf{v}\overline{[\mathcal{CM}^{\mathsf{ref}\ \tau'}\ \ell/\ell]}\overline{[\lceil\alpha\rceil/\alpha]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathsf{x}]}}$. By Lemma 10.2, it suffices to show that

$$\Psi;\Delta;\Gamma \vdash \mathsf{v}_0\ [\overline{\tau}]\ \overline{\mathsf{v}} \approx^{log}_{M+C}\ {}^{\tau_2\overline{[\tau/\alpha]}}\mathcal{MC}\ (\mathbf{unpack}\ \langle\beta',\mathsf{z}\rangle = \mathbf{CM}^{\forall[\overline{\alpha}].(\overline{\tau_1})\to\tau_2}(\mathbf{MC}(\hat{\mathsf{v}}_0))$$
$$\mathbf{in\ let}\ (\mathsf{f},\mathsf{y}) = (\pi_1(\mathsf{z}),\pi_2(\mathsf{z}))\ \mathbf{in\ f}\ [\overline{\tau^{\mathcal{C}}}]\ (\mathsf{y}, \overline{\mathbf{CM}^{\tau_1\overline{[\tau/\alpha]}}(\mathbf{MC}(\hat{\mathsf{v}}))}) : \tau_2\overline{[\tau/\alpha]}.$$

Note that $\Psi;\Delta;\Gamma \vdash \mathsf{v}_0\ [\overline{\tau}]\ \overline{\mathsf{v}} : \tau_2$ and

$$\Psi;\Delta;\Gamma \vdash {}^{\tau_2\overline{[\tau/\alpha]}}\mathcal{MC}\ (\mathbf{unpack}\ \langle\beta',\mathsf{z}\rangle = \mathbf{CM}^{\forall[\overline{\alpha}].(\overline{\tau_1})\to\tau_2}(\mathbf{MC}(\hat{\mathsf{v}}_0))$$
$$\mathbf{in\ let}\ (\mathsf{f},\mathsf{y}) = (\pi_1(\mathsf{z}),\pi_2(\mathsf{z}))\ \mathbf{in\ f}\ [\overline{\tau^{\mathcal{C}}}]\ (\mathsf{y}, \overline{\mathbf{CM}^{\tau_1\overline{[\tau/\alpha]}}(\mathbf{MC}(\hat{\mathsf{v}}))})) : \tau_2\overline{[\tau/\alpha]}.$$

Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W,\gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$(W, \rho_1(\gamma_1(\mathsf{v}_0\ [\overline{\tau}]\ \overline{\mathsf{v}})), \rho_2(\gamma_2({}^{\tau_2\overline{[\tau/\alpha]}}\mathcal{MC}\ (\mathbf{unpack}\ \langle\beta',\mathsf{z}\rangle = \mathbf{CM}^{\forall[\overline{\alpha}].(\overline{\tau_1})\to\tau_2}(\mathbf{MC}(\hat{\mathsf{v}}_0))$$
$$\mathbf{in\ let}\ (\mathsf{f},\mathsf{y}) = (\pi_1(\mathsf{z}),\pi_2(\mathsf{z}))\ \mathbf{in\ f}\ [\overline{\tau^{\mathcal{C}}}]\ (\mathsf{y}, \overline{\mathbf{CM}^{\tau_1\overline{[\tau/\alpha]}}(\mathbf{MC}(\hat{\mathsf{v}}))})))))$$
$$= (W, \rho_1(\gamma_1(\mathsf{v}_0))\ [\overline{\tau}]\ \overline{\rho_1(\gamma_1(\mathsf{v}))}, {}^{\rho_2(\tau_2\overline{[\tau/\alpha]})}\mathcal{MC}\ (\mathbf{unpack}\ \langle\beta',\mathsf{z}\rangle = \mathbf{CM}^{\forall[\overline{\alpha}].(\overline{\tau_1})\to\tau_2}(\mathbf{MC}(\rho_2(\gamma_2(\hat{\mathsf{v}}_0))))$$
$$\mathbf{in\ let}\ (\mathsf{f},\mathsf{y}) = (\pi_1(\mathsf{z}),\pi_2(\mathsf{z}))\ \mathbf{in\ f}\ [\overline{\tau^{\mathcal{C}}}]\ (\mathsf{y}, \overline{\mathbf{CM}^{\tau_1\overline{[\tau/\alpha]}}(\mathbf{MC}(\rho_2(\gamma_2(\hat{\mathsf{v}}))))})))$$
$$\in \mathcal{E}[\![\tau_2\overline{[\tau/\alpha]}]\!]\rho.$$

Let $W' \sqsupseteq_{\mathrm{pub}} W$, $(W',\mathsf{v}_1,\mathsf{v}_2) \in \mathcal{V}[\![\forall[\overline{\alpha}].(\overline{\tau_1})\to\tau_2]\!]\rho$, and $\overline{(W',\hat{\mathsf{v}}_1,\hat{\mathsf{v}}_2) \in \mathcal{V}[\![\tau_1\overline{[\tau/\alpha]}]\!]\rho}$. By Lemma 7.13, it suffices to show that

$$(W', \mathsf{v}_1\ [\overline{\tau}]\ \overline{\hat{\mathsf{v}}_1}, {}^{\rho_2(\tau_2\overline{[\tau/\alpha]})}\mathcal{MC}\ (\mathbf{unpack}\ \langle\beta',\mathsf{z}\rangle = \mathbf{CM}^{\rho_2(\forall[\overline{\alpha}].(\overline{\tau_1})\to\tau_2)}(\mathsf{v}_2)$$
$$\mathbf{in\ let}\ (\mathsf{f},\mathsf{y}) = (\pi_1(\mathsf{z}),\pi_2(\mathsf{z}))\ \mathbf{in\ f}\ [\overline{\tau^{\mathcal{C}}}]\ (\mathsf{y}, \overline{\mathbf{CM}^{\rho_2(\tau_1\overline{[\tau/\alpha]})}(\hat{\mathsf{v}}_2)}))) \in \mathcal{E}[\![\tau_2\overline{[\tau/\alpha]}]\!]\rho.$$

By Lemma 7.1 there are some $\overline{\hat{\mathsf{v}}_2}$ and $\overline{\hat{\mathsf{v}}_2'}$ such that

$$\overline{\mathbf{CM}^{\rho_2(\tau_1\overline{[\tau/\alpha]})}(\hat{\mathsf{v}}_2) = \hat{\mathsf{v}}_2} \quad\text{and}\quad \overline{{}^{\rho_2(\tau_1\overline{[\mathsf{L}\langle\tau^{\langle\mathcal{C}\rangle}\rangle/\alpha]})}\mathbf{MC}(\hat{\mathsf{v}}_2) = \hat{\mathsf{v}}_2'}.$$

Let

$$\hat{\mathsf{v}} = \lambda[\overline{\alpha}](\mathsf{z} : \mathsf{unit}, \overline{\mathsf{y} : \rho_2(\tau_1)^{\mathcal{C}}}).\mathcal{CM}^{\rho_2(\tau_2)\overline{[\mathsf{L}\langle\alpha\rangle/\alpha]}}\ (\mathsf{v}_2\ [\overline{\mathsf{L}\langle\alpha\rangle}]\ \overline{{}^{\rho_2(\tau_1)\overline{[\mathsf{L}\langle\alpha\rangle/\alpha]}}\mathcal{MC}\ \mathsf{y}})$$

and note that for any $(H_1,H_2) : W'$

$$\langle H_2 \mid {}^{\rho_2(\tau_2\overline{[\tau/\alpha]})}\mathcal{MC}\ (\mathbf{unpack}\ \langle\beta',\mathsf{z}\rangle = \mathbf{CM}^{\rho_2(\forall[\overline{\alpha}].(\overline{\tau_1})\to\tau_2)}(\mathsf{v}_2)$$
$$\mathbf{in\ let}\ (\mathsf{f},\mathsf{y}) = (\pi_1(\mathsf{z}),\pi_2(\mathsf{z}))\ \mathbf{in\ f}\ [\overline{\tau^{\mathcal{C}}}]\ (\mathsf{y}, \overline{\mathbf{CM}^{\rho_2(\tau_1\overline{[\tau/\alpha]})}(\hat{\mathsf{v}}_2)})))\rangle$$
$$\longmapsto\ \langle H_2 \mid {}^{\rho_2(\tau_2\overline{[\tau/\alpha]})}\mathcal{MC}\ (\mathbf{unpack}\ \langle\beta',\mathsf{z}\rangle = (\mathbf{pack}\ \langle\mathsf{unit}, \langle\hat{\mathsf{v}}, ()\rangle\rangle\ \mathbf{as}\ \rho_2(\forall[\overline{\alpha}].(\overline{\tau_1})\to\tau_2)^{\langle\mathcal{C}\rangle})$$
$$\mathbf{in\ let}\ (\mathsf{f},\mathsf{y}) = (\pi_1(\mathsf{z}),\pi_2(\mathsf{z}))\ \mathbf{in\ f}\ [\overline{\tau^{\mathcal{C}}}]\ (\mathsf{y}, \overline{\mathbf{CM}^{\rho_2(\tau_1\overline{[\tau/\alpha]})}(\hat{\mathsf{v}}_2)})))\rangle$$
$$\longmapsto^3 \langle H_2 \mid {}^{\rho_2(\tau_2\overline{[\tau/\alpha]})}\mathcal{MC}\ \hat{\mathsf{v}}\ [\overline{\tau^{\langle\mathcal{C}\rangle}}]\ (), \overline{\mathbf{CM}^{\rho_2(\tau_1\overline{[\tau/\alpha]})}(\hat{\mathsf{v}}_2)}\rangle$$
$$\longmapsto^* \langle H_2 \mid {}^{\rho_2(\tau_2\overline{[\tau/\alpha]})}\mathcal{MC}\ \hat{\mathsf{v}}\ [\overline{\tau^{\langle\mathcal{C}\rangle}}]\ (), \overline{\hat{\mathsf{v}}_2}\rangle$$
$$\longmapsto\ \langle H_2 \mid {}^{\rho_2(\tau_2\overline{[\tau/\alpha]})}\mathcal{MC}\ \mathcal{CM}^{\rho_2(\tau_2)\overline{[\mathsf{L}\langle\tau^{\langle\mathcal{C}\rangle}\rangle/\alpha]}}\ (\mathsf{v}_2\ [\overline{\mathsf{L}\langle\tau^{\langle\mathcal{C}\rangle}\rangle}]\ \overline{{}^{\rho_2(\tau_1)\overline{[\mathsf{L}\langle\tau^{\langle\mathcal{C}\rangle}\rangle/\alpha]}}\mathcal{MC}\ \hat{\mathsf{v}}_2})\rangle$$
$$\longmapsto^* \langle H_2 \mid {}^{\rho_2(\tau_2\overline{[\tau/\alpha]})}\mathcal{MC}\ \mathcal{CM}^{\rho_2(\tau_2)\overline{[\mathsf{L}\langle\tau^{\langle\mathcal{C}\rangle}\rangle/\alpha]}}\ (\mathsf{v}_2\ [\overline{\mathsf{L}\langle\tau^{\langle\mathcal{C}\rangle}\rangle}]\ \overline{\hat{\mathsf{v}}_2'})\rangle$$

By Lemma 7.9, it suffices to show that

$$(W', \mathsf{v}_1\ [\overline{\tau}]\ \overline{\hat{\mathsf{v}}_1}, {}^{\rho_2(\tau_2\overline{[\tau/\alpha]})}\mathcal{MC}\ \mathcal{CM}^{\rho_2(\tau_2)\overline{[\mathsf{L}\langle\tau^{\langle\mathcal{C}\rangle}\rangle/\alpha]}}\ (\mathsf{v}_2\ [\overline{\mathsf{L}\langle\tau^{\langle\mathcal{C}\rangle}\rangle}]\ \overline{\hat{\mathsf{v}}_2'})) \in \mathcal{E}[\![\tau_2\overline{[\tau/\alpha]}]\!]\rho.$$

From here let $\overline{R = \begin{bmatrix} \mathcal{V}[\![\tau, \tau]\!]\rho & \mathcal{V}[\![\tau, \tau^{\langle \mathcal{C} \rangle}]\!]\rho \\ \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}, \tau]\!]\rho & \mathcal{V}[\![\tau^{\langle \mathcal{C} \rangle}, \tau^{\langle \mathcal{C} \rangle}]\!]\rho \end{bmatrix}}$ and note that by Lemma 9.4

$$\mathcal{E}[\![\tau_2\overline{[\tau/\alpha]}]\!]\rho = \mathcal{E}[\![\tau_2]\!]\rho\overline{[\alpha \mapsto (\rho_1(\tau), \rho_2(\tau), R)]}$$

Let

$$\overline{\mathrm{VR} = \mathrm{opaqueR}(\rho_1(\tau), \rho_2(\tau), R)}.$$

Using $\overline{\mathrm{VR}}$ and $\overline{(W', \hat{\mathsf{v}}_1, \hat{\mathsf{v}}_2') \in \mathcal{V}[\![\tau_1]\!]\rho\overline{[\alpha \mapsto \mathrm{VR}]}}$ (which we have by Lemma 9.4 and boundary cancellation), we can instantiate $(W', \mathsf{v}_1, \mathsf{v}_2) \in \mathcal{V}[\![\forall[\overline{\alpha}].(\overline{\tau_1}) \to \tau_2]\!]\rho$ to get

$$(W', \mathsf{v}_1\ [\overline{\tau}]\ \overline{\hat{\mathsf{v}}_1}, \mathsf{v}_2\ [\overline{\mathsf{L}\langle \tau^{\langle \mathcal{C} \rangle} \rangle}]\ \overline{\hat{\mathsf{v}}_2'}) \in \mathcal{E}[\![\tau_2]\!]\rho\overline{[\alpha \mapsto \mathrm{VR}]}.$$

The result follows by boundary cancellation and Lemma 7.25. □

**Theorem 10.21 (Closure Conversion is Semantics Preserving)**
If $\Psi; \Delta; \Gamma \vdash \mathsf{e} : \tau \rightsquigarrow \mathbf{e}$, then

$$\Psi; \Delta; \Gamma \vdash \mathsf{e} \approx^{log}_{M+C}\ {}^{\tau}\mathcal{MC}\ (\mathbf{e}\overline{[\mathcal{CM}^{\mathsf{ref}\ \tau'}\ \ell/\boldsymbol{\ell}]}\overline{[\lceil \alpha \rceil/\boldsymbol{\alpha}]}\overline{[\mathbf{CM}^{\tau''}(\mathsf{x})/\mathbf{x}]}) : \tau.$$

**Proof**

By induction on the compiler judgment, using the preceding lemmas. □

# 11 Adding First-Class Continuations to Target Language

In this section, we describe the changes we need to make to the technical development in Sections 1 through 10 in order to verify the correctness of closure conversion from M to the target language $C^{\mathsf{cont}}$ which extends C with first-class continuations.

## 11.1 Source Language: M

No changes.

## 11.2 Closure-Conversion Target Language: $C^{\mathsf{cont}}$

Extend the syntax for C as follows:

$\tau ::= \ldots \mid \mathsf{cont}\,\tau$

$\mathsf{v} ::= \ldots \mid \mathsf{cont}_\tau\,\mathsf{E}$

$\mathsf{e} ::= \ldots \mid \mathsf{cont}_\tau\,\mathsf{E} \mid \mathsf{call/cc}_\tau(\mathsf{x.\,e}) \mid \mathsf{throw}_\tau\,\mathsf{v}\,\mathsf{to}\,\mathsf{v}$

The grammar for $\mathsf{E}$, $\mathsf{H}$, $\Psi$, $\Delta$, and $\Gamma$ remains unchanged.

### 11.2.1 Well-Formed Types $\boxed{\Delta \vdash \tau}$

Add the following rule:

$$\frac{\Delta \vdash \tau}{\Delta \vdash \mathsf{cont}\,\tau}$$

### 11.2.2 Well-Typed Terms $\boxed{\Psi;\Delta;\Gamma \vdash \mathsf{e} : \tau}$

Add the following rule to C for continuation typing:

$$\frac{\vdash \mathsf{E} : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi;\Delta;\Gamma \vdash \tau')}{\Psi;\Delta;\Gamma \vdash \mathsf{E} \div \tau}$$

Add the following rules for type-checking terms:

$$\frac{\Psi;\Delta;\Gamma \vdash \mathsf{E} \div \tau}{\Psi;\Delta;\Gamma \vdash \mathsf{cont}_\tau\,\mathsf{E} : \mathsf{cont}\,\tau} \qquad \frac{\Psi;\Delta;\Gamma, \mathsf{x} : \mathsf{cont}\,\tau \vdash \mathsf{e} : \tau}{\Psi;\Delta;\Gamma \vdash \mathsf{call/cc}_\tau(\mathsf{x.\,e}) : \tau} \qquad \frac{\Psi;\Delta;\Gamma \vdash \mathsf{v}' : \tau' \qquad \Psi;\Delta;\Gamma \vdash \mathsf{v} : \mathsf{cont}\,\tau'}{\Psi;\Delta;\Gamma \vdash \mathsf{throw}_\tau\,\mathsf{v}'\,\mathsf{to}\,\mathsf{v} : \tau}$$

### 11.2.3 Reduction Relation $\boxed{\langle \mathsf{H} \mid \mathsf{e} \rangle \longmapsto \langle \mathsf{H}' \mid \mathsf{e}' \rangle}$

Add the following reduction rules:

$$\begin{array}{ll} \langle \mathsf{H} \mid \mathsf{E}[\mathsf{call/cc}_\tau(\mathsf{x.\,e})] \rangle & \longmapsto \langle \mathsf{H} \mid \mathsf{E}[\mathsf{e}[\mathsf{cont}_\tau\,\mathsf{E}/\mathsf{x}]] \rangle \\ \langle \mathsf{H} \mid \mathsf{E}[\mathsf{throw}_\tau\,\mathsf{v}\,\mathsf{to}\,\mathsf{cont}_\tau\,\mathsf{E}'] \rangle & \longmapsto \langle \mathsf{H} \mid \mathsf{E}'[\mathsf{v}] \rangle \end{array}$$

## 11.3 Closure-Conversion: M to $C^{\mathsf{cont}}$

No changes.

## 11.4 Multi-Language Semantics: M + $C^{\mathsf{cont}}$

Below we present the grammar for the multi-language between M and $C^{\mathsf{cont}}$. This grammar should now be read as an extension to the grammar for M (from Section 1) and $C^{\mathsf{cont}}$ (from Section 11.2).

Notice that we have added terms of the form $\mathbf{cont}_{\boldsymbol{\tau}}\ \mathsf{E}$ to the multi-language. This is because the multi-language between M and $C^{\mathsf{cont}}$ must support terms of the form $\mathbf{cont}_{\boldsymbol{\tau}}\ E$ and not just $\mathbf{cont}_{\boldsymbol{\tau}}\ \mathbf{E}$. (Recall that $\mathbf{E}$ is a context with a result of type $\boldsymbol{\tau}$, while $E$ is a context with a result of type $\tau$.) The need for $\mathbf{cont}_{\boldsymbol{\tau}}\ E$ arises because the reduction rule for $\mathbf{call/cc}$ (see below) will capture the entire multi-language context $E$.

$$
\begin{array}{llll}
\tau ::= \cdots \mid \mathsf{L}\langle\boldsymbol{\tau}\rangle & & \tau ::= \tau \mid \boldsymbol{\tau} \\
\mathsf{e} ::= \cdots \mid {}^{\tau}\mathcal{MC}\,\mathbf{e} & & e ::= \mathsf{e} \mid \mathbf{e} \\
\mathsf{v} ::= \cdots \mid {}^{\mathsf{ref}\,\tau}\mathcal{MC}\,\boldsymbol{\ell} \mid {}^{\mathsf{L}\langle\boldsymbol{\tau}\rangle}\mathcal{MC}\,\mathbf{v} & & v ::= \mathsf{v} \mid \mathbf{v} \\
\mathsf{E} ::= \cdots \mid {}^{\tau}\mathcal{MC}\,\mathbf{E} & & E ::= \mathsf{E} \mid \mathbf{E} \\
\boldsymbol{\tau} ::= \cdots \mid \lceil\alpha\rceil & & H ::= (\mathsf{H}, \mathbf{H}) \\
\mathbf{e} ::= \cdots \mid \mathbf{cont}_{\boldsymbol{\tau}}\ \mathsf{E} \mid \mathcal{CM}^{\tau}\,\mathsf{e} & & \Psi ::= (\Psi, \boldsymbol{\Psi}) \\
\mathbf{v} ::= \cdots \mid \mathbf{cont}_{\boldsymbol{\tau}}\ \mathsf{E} \mid \mathcal{CM}^{\mathsf{ref}\,\tau}\,\ell & & \Delta ::= \cdot \mid \Delta, \alpha \mid \Delta, \boldsymbol{\alpha} \\
\mathbf{E} ::= \cdots \mid \mathcal{CM}^{\tau}\,\mathsf{E} & & \Gamma ::= \cdot \mid \Gamma, \mathsf{x}\!:\!\tau \mid \Gamma, \mathbf{x}\!:\!\boldsymbol{\tau}
\end{array}
$$

**Specification Type Translation & Value Translation** The specification type translation $\tau^{\langle \mathcal{C}\rangle}$ and value translation are exactly as before (since both of these are directed by source types and we haven't made any extensions to source-language types).

### 11.4.1 Well-Typed Terms $\boxed{\Psi;\Delta;\Gamma \vdash e : \tau}$

Add the following rule to the multi-language for continuation typing:

$$\frac{\vdash E : (\Psi;\Delta;\Gamma \vdash \tau) \rightsquigarrow (\Psi;\Delta;\Gamma \vdash \tau')}{\Psi;\Delta;\Gamma \vdash E \div \tau}$$

Adapt the corresponding judgments and typing rules for M and $C^{\mathsf{cont}}$ by changing all the environments to the apprroprriate multi-language environments. Change the $C^{\mathsf{cont}}$ typing rule for $\mathbf{cont}_{\boldsymbol{\tau}}\ \mathbf{E}$ to the following:

$$\frac{\Psi;\Delta;\Gamma \vdash E \div \boldsymbol{\tau}}{\Psi;\Delta;\Gamma \vdash \mathbf{cont}_{\boldsymbol{\tau}}\ E : \mathbf{cont}\ \boldsymbol{\tau}}$$

Include the typing rules for boundaries as specified in Section 4.5.

### 11.4.2 Reduction Relation $\boxed{\langle H \mid e\rangle \longmapsto \langle H' \mid e'\rangle}$

As before, lift the M and $C^{\mathsf{cont}}$ reduction rules to the new configuration, replacing evalation contexts $\mathsf{E}$ with $E$. We show the lifted rules for $\mathbf{call/cc}$ and $\mathbf{throw}$ below:

$$
\begin{array}{ll}
\langle H \mid E[\mathbf{call/cc}_{\boldsymbol{\tau}}(\mathbf{x}.\,\mathbf{e})]\rangle & \longmapsto \langle H \mid E[\mathbf{e}[\mathbf{cont}_{\boldsymbol{\tau}}\ E/\mathbf{x}]]\rangle \\
\langle H \mid E[\mathbf{throw}_{\boldsymbol{\tau}}\ \mathbf{v}\ \mathbf{to}\ \mathbf{cont}_{\boldsymbol{\tau}'}\ E']\rangle & \longmapsto \langle H \mid E'[\mathbf{v}]\rangle
\end{array}
$$

Include the boundary rules exactly as specified in Section 4.7.

## 11.5 Contexts and Contextual Equivalence

The grammar for $\mathsf{C}^{\mathsf{v}}$ and $\mathsf{C}$ is exactly as before (see Section 5). Extend the grammar for $\mathbf{C}^{\mathbf{v}}$ and $\mathbf{C}$ as follows:

$$
\begin{array}{l}
\mathbf{C}^{\mathbf{v}} ::= \ldots \mid \mathbf{throw}_{\boldsymbol{\tau}}\ \mathbf{C}^{\mathbf{v}}\ \mathbf{to}\ \mathbf{v} \mid \mathbf{throw}_{\boldsymbol{\tau}}\ \mathbf{v}\ \mathbf{to}\ \mathbf{C}^{\mathbf{v}} \\
\mathbf{C} ::= \ldots \mid \mathbf{call/cc}_{\boldsymbol{\tau}}(\mathbf{x}.\,\mathbf{C}) \\
C ::= \mathsf{C} \mid \mathbf{C}
\end{array}
$$

### 11.5.1 Well-Typed Context $\boxed{\vdash C : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \tau')}$

Add the following rules:

$$\frac{\vdash \mathbf{C^v} : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \boldsymbol{\tau'}) \qquad \Psi'; \Delta'; \Gamma' \vdash \mathbf{v} : \mathbf{cont}\, \boldsymbol{\tau'}}{\vdash \mathbf{throw}_{\boldsymbol{\tau}}\, \mathbf{C^v}\ \mathbf{to}\ \mathbf{v} : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \boldsymbol{\tau})}$$

$$\frac{\Psi'; \Delta'; \Gamma' \vdash \mathbf{v} : \boldsymbol{\tau'} \qquad \vdash \mathbf{C^v} : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \mathbf{cont}\, \boldsymbol{\tau'})}{\vdash \mathbf{throw}_{\boldsymbol{\tau}}\, \mathbf{v}\ \mathbf{to}\ \mathbf{C^v} : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \boldsymbol{\tau})}$$

$$\frac{\vdash \mathbf{C} : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \Delta'; \Gamma', \mathbf{x} : \mathbf{cont}\, \boldsymbol{\tau} \vdash \boldsymbol{\tau})}{\vdash \mathbf{call/cc}_{\boldsymbol{\tau}}(\mathbf{x}.\, \mathbf{C}) : (\Psi; \Delta; \Gamma \vdash \tau) \rightsquigarrow (\Psi'; \Delta'; \Gamma' \vdash \boldsymbol{\tau})}$$

### 11.5.2 Contextual Equivalence

No changes.

### 11.5.3 CIU Equivalence

No changes.

## 11.6 Logical Relation

In the presence of **call/cc**, private transitions must be disallowed (see Dreyer et al. [1]). We can enforce this by requiring that the set of all transitions $\delta$ be the same as the set of public transitiona $\pi$. Specifically, extend the definition of $\text{Island}_n$ with one additional requirement:

$$\text{Island}_n \stackrel{\text{def}}{=} \{\, \theta = (s, S, \delta, \pi, \text{HR}, \text{bij}) \mid \ \ldots\ \wedge\ \pi = \delta \,\}$$

We must also define when two values of the form $\mathbf{cont}_{\boldsymbol{\tau}}\, E$ are related. We extend the logical relation with the following:

$$\begin{aligned} \mathcal{V}[\![\mathbf{cont}\, \boldsymbol{\tau}]\!]\rho \quad = \quad & \{\, (W, \mathbf{cont}_{\rho_1(\boldsymbol{\tau})}\, E_1, \mathbf{cont}_{\rho_2(\boldsymbol{\tau})}\, E_2, \in \text{ValAtom}[\mathbf{cont}\, \boldsymbol{\tau}]\rho \mid \\ & \quad \forall W', \mathbf{v_1}, \mathbf{v_2}.\ W' \sqsupseteq W\ \wedge \\ & \qquad (W', \mathbf{v_1}, \mathbf{v_2}) \in \mathcal{V}[\![\boldsymbol{\tau}]\!]\rho \implies (W', E_1[\mathbf{v_1}], E_2[\mathbf{v_2}]) \in \mathcal{O} \,\} \end{aligned}$$

## 11.7 Proofs: Basic Properties

**Lemma 7.6 (Monotonicity)**: proof of Part 1 requires an additional case.
If $\rho \in \mathcal{D}[\![\Delta]\!]$, $\Delta \vdash \tau$, $\Delta \vdash \boldsymbol{\tau}$ and $W' \sqsupseteq W$

1. $(W, \mathbf{v_1}, \mathbf{v_2}) \in \mathcal{V}[\![\boldsymbol{\tau}]\!]\rho \implies (W', \mathbf{v_1}, \mathbf{v_2}) \in \mathcal{V}[\![\boldsymbol{\tau}]\!]\rho$
2. As before.
3. As before.
4. As before.

Proof 1. Case **cont** $\boldsymbol{\tau}$ By transitivity of world extension.

**Lemma 7.14**: proof of Part 1 requires an additional case.
If $\rho[\alpha \mapsto \text{VR}] \in \mathcal{D}[\![\Delta, \alpha]\!]$ and $\alpha \notin \text{ftv}\, \boldsymbol{\tau}$, then

1. $\mathcal{V}[\![\boldsymbol{\tau}]\!]\rho = \mathcal{V}[\![\boldsymbol{\tau}]\!]\rho[\alpha \mapsto \text{VR}]$
2. As before.

3. As before.

Proof 1. Case **cont** $\tau$  Follows from the induction hypothesis for $\tau$.

The proofs of Part 1 of **Lemmas 7.18, 7.21, and 7.25** also need an additional proof case for **cont** $\tau$. All of these cases follow from the induction hypothesis for $\tau$, exactly as in the proof of the additional case for Lemma 7.14 shown above.

## 11.8 Proofs: Boundary Cancellation

No changes.

## 11.9 Proofs: Soundness and Completeness

The proof of Part 1 of **Lemma**  (Syntactic type substitution is equivalent to semantic type substitution in C) needs an additional proof case for **cont** $\tau$. The proof follows from the induction hypothesis for $\tau$, exactly as in the proof of the additional case for Lemma 7.14 shown above.

### 11.9.1 Compatibility Lemmas

We now prove compatibility lemmas for the new constructs in our language: **cont E**, **call/cc**, and **throw**.

To state compatibility for **cont E**, we will need to specify when two open continuations should be considered logically related.

**Definition 11.1 (Logical Relation: Continuations)**

$$\begin{aligned}\Psi;\Delta;\Gamma \vdash E_1 \approx^{log}_{M+C} E_2 \div \tau \stackrel{\text{def}}{=} \;& \Psi;\Delta;\Gamma \vdash E_1 \div \tau \;\wedge\; \Psi;\Delta;\Gamma \vdash E_2 \div \tau \;\wedge \\ & \forall W,\rho,\gamma.\; W \in \mathcal{H}[\![\Psi]\!] \;\wedge\; \rho \in \mathcal{D}[\![\Delta]\!] \;\wedge\; (W,\gamma) \in \mathcal{G}[\![\Gamma]\!]\rho \\ & \quad \Longrightarrow (W, \rho_1(\gamma_1(E_1)), \rho_2(\gamma_2(E_2))) \in \mathcal{K}[\![\tau]\!]\rho\end{aligned}$$

**Lemma 11.2 (C Cont)**
If $\Psi;\Delta;\Gamma \vdash E_1 \approx^{log}_{M+C} E_2 \div \tau$
then $\Psi;\Delta;\Gamma \vdash \mathbf{cont}_\tau\, E_1 \approx^{log}_{M+C} \mathbf{cont}_\tau\, E_2 : \mathbf{cont}\,\tau$.

**Proof**

Note that $\Psi;\Delta;\Gamma \vdash \mathbf{cont}_\tau\, E_1 : \mathbf{cont}\,\tau$ and $\Psi;\Delta;\Gamma \vdash \mathbf{cont}_\tau\, E_2 : \mathbf{cont}\,\tau$.
Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W,\gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$\begin{aligned}&(W, \rho_1(\gamma_1(\mathbf{cont}_\tau\, E_1)), \rho_2(\gamma_2(\mathbf{cont}_\tau\, E_2))) \\ &\qquad = (W, \mathbf{cont}_{\rho_1(\tau)}\, \rho_1(\gamma_1(E_1)), \mathbf{cont}_{\rho_2(\tau)}\, \rho_2(\gamma_2(E_2))) \in \mathcal{E}[\![\mathbf{cont}\,\tau]\!]\rho.\end{aligned}$$

By Lemma 7.5, it suffices to show

$$(W, \mathbf{cont}_{\rho_1(\tau)}\, \rho_1(\gamma_1(E_1)), \mathbf{cont}_{\rho_2(\tau)}\, \rho_2(\gamma_2(E_2))) \in \mathcal{V}[\![\mathbf{cont}\,\tau]\!]\rho.$$

The above follows from $(W, \rho_1(\gamma_1(E_1)), \rho_2(\gamma_2(E_2))) \in \mathcal{K}[\![\tau]\!]\rho$, which follows from our premise. ☐

**Lemma 11.3 (C Call/cc)**
If $\Psi;\Delta;\Gamma, \mathbf{x} : \mathbf{cont}\,\tau \vdash \mathbf{e_1} \approx^{log}_{M+C} \mathbf{e_2} : \tau$
then $\Psi;\Delta;\Gamma \vdash \mathbf{call/cc}_\tau(\mathbf{x}.\,\mathbf{e_1}) \approx^{log}_{M+C} \mathbf{call/cc}_\tau(\mathbf{x}.\,\mathbf{e_2}) : \tau$

**Proof**

Note that $\Psi; \Delta; \Gamma \vdash \mathsf{call/cc}_{\tau}(\mathsf{x}.\, \mathsf{e}_1) : \tau$ and $\Psi; \Delta; \Gamma \vdash \mathsf{call/cc}_{\tau}(\mathsf{x}.\, \mathsf{e}_2) : \tau$.
Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$(W, \rho_1(\gamma_1(\mathsf{call/cc}_{\tau}(\mathsf{x}.\, \mathsf{e}_1))), \rho_2(\gamma_2(\mathsf{call/cc}_{\tau}(\mathsf{x}.\, \mathsf{e}_2)))) \\ = (W, \mathsf{call/cc}_{\rho_1(\tau)}(\mathsf{x}.\, \rho_1(\gamma_1(\mathsf{e}_1))), \mathsf{call/cc}_{\rho_2(\tau)}(\mathsf{x}.\, \rho_2(\gamma_2(\mathsf{e}_2)))) \in \mathcal{E}[\![\tau]\!]\rho.$$

Let $(W, E_1, E_2) \in \mathcal{K}[\![\tau]\!]\rho$. We have to show

$$(W, E_1[\mathsf{call/cc}_{\rho_1(\tau)}(\mathsf{x}.\, \rho_1(\gamma_1(\mathsf{e}_1)))], E_2[\mathsf{call/cc}_{\rho_2(\tau)}(\mathsf{x}.\, \rho_2(\gamma_2(\mathsf{e}_2)))]) \in \mathcal{O}.$$

By Lemma 7.7 and the operational semantics, it suffices to show

$$(W, E_1[(\rho_1(\gamma_1(\mathsf{e}_1)))[\mathsf{cont}_{\rho_1(\tau)}\, E_1/\mathsf{x}]], E_2[(\rho_2(\gamma_2(\mathsf{e}_2)))[\mathsf{cont}_{\rho_2(\tau)}\, E_2/\mathsf{x}]]) \in \mathcal{O}.$$

From $(W, E_1, E_2) \in \mathcal{K}[\![\tau]\!]\rho$, it follows that $(W, \mathsf{cont}_{\rho_1(\tau)}\, E_1, \mathsf{cont}_{\rho_2(\tau)}\, E_1) \in \mathcal{V}[\![\mathsf{cont}\ \tau]\!]\rho$.
Let $\gamma' = \gamma[\mathsf{x} \mapsto (\mathsf{cont}_{\rho_1(\tau)}\, E_1, \mathsf{cont}_{\rho_2(\tau)}\, E_1)]$. It follows that $(W, \gamma') \in \mathcal{G}[\![\Gamma, \mathsf{x} : \mathsf{cont}\ \tau]\!]\rho$.
Instantiating the premise with $W$, $\rho$, and $\gamma'$ gives us $(W, \rho_1(\gamma_1'(\mathsf{e}_!)), \rho_2(\gamma_2'(\mathsf{e}_2))) \in \mathcal{E}[\![\tau]\!]\rho$.
From the latter, since $(W, E_1, E_2) \in \mathcal{K}[\![\tau]\!]\rho$, we have that

$$(W, E_1[\rho_1(\gamma_1'(\mathsf{e}_!))], E_2[\rho_2(\gamma_2'(\mathsf{e}_2))]) \in \mathcal{O}$$

Since $E_1$ and $E_2$ contain no free type variables, the latter is equivalent to what we wanted to show:

$$(W, E_1[(\rho_1(\gamma_1(\mathsf{e}_1)))[\mathsf{cont}_{\rho_1(\tau)}\, E_1/\mathsf{x}]], E_2[(\rho_2(\gamma_2(\mathsf{e}_2)))[\mathsf{cont}_{\rho_2(\tau)}\, E_2/\mathsf{x}]]) \in \mathcal{O}.$$

□

**Lemma 11.4 (C Throw)**
If $\Psi; \Delta; \Gamma \vdash \mathsf{v}_1' \approx^{log}_{M+C} \mathsf{v}_2' : \tau'$ and $\Psi; \Delta; \Gamma \vdash \mathsf{v}_1 \approx^{log}_{M+C} \mathsf{v}_2 : \mathsf{cont}\ \tau'$
then $\Psi; \Delta; \Gamma \vdash \mathsf{throw}_{\tau}\ \mathsf{v}_1'\ \mathsf{to}\ \mathsf{v}_1 \approx^{log}_{M+C} \mathsf{throw}_{\tau}\ \mathsf{v}_2'\ \mathsf{to}\ \mathsf{v}_2 : \tau$.

**Proof**

Note that $\Psi; \Delta; \Gamma \vdash \mathsf{throw}_{\tau}\ \mathsf{v}_1'\ \mathsf{to}\ \mathsf{v}_1 : \tau$ and $\Psi; \Delta; \Gamma \vdash \mathsf{throw}_{\tau}\ \mathsf{v}_2'\ \mathsf{to}\ \mathsf{v}_2 : \tau$.
Let $W \in \mathcal{H}[\![\Psi]\!]$, $\rho \in \mathcal{D}[\![\Delta]\!]$, and $(W, \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$. We need to show that

$$(W, \rho_1(\gamma_1(\mathsf{throw}_{\tau}\ \mathsf{v}_1'\ \mathsf{to}\ \mathsf{v}_1)), \rho_2(\gamma_2(\mathsf{throw}_{\tau}\ \mathsf{v}_2'\ \mathsf{to}\ \mathsf{v}_2))) \\ = (W, \mathsf{throw}_{\rho_1(\tau)}\ \rho_1(\gamma_1(\mathsf{v}_1'))\ \mathsf{to}\ \rho_1(\gamma_1(\mathsf{v}_1)), \mathsf{throw}_{\rho_2(\tau)}\ \rho_2(\gamma_2(\mathsf{v}_2'))\ \mathsf{to}\ \rho_2(\gamma_2(\mathsf{v}_2))) \in \mathcal{E}[\![\tau]\!]\rho.$$

From our first premise, we have

$$(W, \rho_1(\gamma_1(\mathsf{v}_1')), \rho_2(\gamma_2(\mathsf{v}_2'))) \in \mathcal{E}[\![\tau']\!]\rho.$$

Applying Lemma 7.13, we assume $W' \sqsupseteq_{\text{pub}} W$ and $(W', \hat{\mathsf{v}}_1', \hat{\mathsf{v}}_2') \in \mathcal{V}[\![\tau']\!]\rho$, and need to show

$$(W', \mathsf{throw}\ \hat{\mathsf{v}}_1'\ \mathsf{to}\ \rho_1(\gamma_1(\mathsf{v}_1)), \mathsf{throw}\ \hat{\mathsf{v}}_2'\ \mathsf{to}\ \rho_2(\gamma_2(\mathsf{v}_2))) \in \mathcal{E}[\![\tau]\!]\rho.$$

Instantiating our second premise with $W'$, noting that by monotonicity we have $W' \in \mathcal{H}[\![\Psi]\!]$ and $(W', \gamma) \in \mathcal{G}[\![\Gamma]\!]\rho$, we get

$$(W', \rho_1(\gamma_1(\mathsf{v}_1)), \rho_2(\gamma_2(\mathsf{v}_2))) \in \mathcal{E}[\![\mathsf{cont}\ \tau']\!]\rho.$$

Applying Lemma 7.13 again, we assume $W'' \sqsupseteq_{\text{pub}} W'$ and $(W'', \mathsf{cont}\ E_1, \mathsf{cont}\ E_2) \in \mathcal{V}[\![\mathsf{cont}\ \tau']\!]\rho$, and need to show

$$(W'', \textbf{throw } \hat{\mathbf{v}}'_1 \textbf{ to cont } E_1, \textbf{throw } \hat{\mathbf{v}}'_2 \textbf{ to cont } E_2) \in \mathcal{E}[\![\boldsymbol{\tau}]\!]\rho.$$

Let $(W'', E'_1, E'_2) \in \mathcal{K}[\![\boldsymbol{\tau}]\!]\rho$. We need to show

$$(W'', E'_1[\textbf{throw } \hat{\mathbf{v}}'_1 \textbf{ to cont } E_1], E'_2[\textbf{throw } \hat{\mathbf{v}}'_2 \textbf{ to cont } E_2]) \in \mathcal{O}.$$

By Lemma 7.7 and the operational semantics, it suffices to show

$$(W'', E_1[\hat{\mathbf{v}}'_1], E_2[\hat{\mathbf{v}}'_2]) \in \mathcal{O}.$$

The latter follows by instantiating $(W'', \textbf{cont } E_1, \textbf{cont } E_2) \in \mathcal{V}[\![\textbf{cont } \boldsymbol{\tau}']\!]\rho$ with $(W'', \hat{\mathbf{v}}'_1, \hat{\mathbf{v}}'_2) \in \mathcal{V}[\![\boldsymbol{\tau}']\!]\rho$, which we have by monotonicity. □

## 11.10 Proofs: Correctness of Closure Conversion

No changes.

## 11.11 Example

The following expressions $\mathsf{e}_1$ and $\mathsf{e}_2$ are contextually equivalent in the source language M; that is, there is no M context $\mathsf{C}$ can can distinguish between them.

$$\begin{aligned}
\tau = \quad & ((\mathsf{unit}) \to \mathsf{unit}) \to \mathsf{int} \\
\mathsf{e}_1 = \quad & \mathsf{let}\, \mathsf{x} = \mathsf{new}\, 0 \;\mathsf{in} \\
& \lambda(\mathsf{f} : _).\mathsf{let}\, _ = \mathsf{x} := 0 \;\mathsf{in}\; \mathsf{let}\, _ = \mathsf{f}\,() \;\mathsf{in}\; \mathsf{let}\, _ = \mathsf{x} := 2 \;\mathsf{in}\; \mathsf{let}\, _ = \mathsf{f}\,() \;\mathsf{in}\; !\mathsf{x} \\
\mathsf{e}_2 = \quad & \lambda(\mathsf{f} : _).\mathsf{let}\, _ = \mathsf{f}\,() \;\mathsf{in}\; \mathsf{let}\, _ = \mathsf{f}\,() \;\mathsf{in}\; 2
\end{aligned}$$

The terms $\mathbf{e_1}$ and $\mathbf{e_2}$ below, of translation type $\boldsymbol{\tau} = \tau^{\langle \mathcal{C} \rangle}$, are the compiled versions of $\mathsf{e}_1$ and $\mathsf{e}_2$:

$$\begin{aligned}
\boldsymbol{\tau} = \quad & \exists\beta.\langle ((\beta, \exists\alpha.\langle ((\alpha, \mathbf{unit}) \to \mathbf{unit})\,, \alpha \rangle) \to \mathbf{int})\,, \beta \rangle \\
\mathbf{e_1} = \quad & \mathbf{let\; x = new\; 0\; in} \\
& \mathbf{pack}\ \langle \langle \mathbf{ref\;int} \rangle, \langle \boldsymbol{\lambda}(\mathbf{env} : \langle \mathbf{ref\;int} \rangle, \mathbf{fc} : (\mathsf{unit}) \to \mathsf{unit}^{\langle \mathcal{C} \rangle}).\mathbf{e}_1', \langle \mathbf{x} \rangle \rangle \rangle \;\mathbf{as}\; ((\mathsf{unit}) \to \mathsf{unit}) \to \mathsf{int}^{\langle \mathcal{C} \rangle} \\
& \text{where } \mathbf{e}_1' \;=\; \mathbf{let\; x'} = \boldsymbol{\pi}_1(\mathbf{env}) \;\mathbf{in\; x'} := \mathbf{0};\, \mathbf{unpack}\ \langle \beta, \mathbf{fp} \rangle = \mathbf{fc\; in} \\
& \qquad\qquad \mathbf{let\; (f, y)} = (\boldsymbol{\pi}_1(\mathbf{fp}), \boldsymbol{\pi}_2(\mathbf{fp})) \;\mathbf{in\; f\;(y, ())};\, \mathbf{x'} := \mathbf{2};\, \mathbf{f\;(y, ())};\, !\mathbf{x'} \\
\mathbf{e_2} = \quad & \mathbf{pack}\ \langle \langle \rangle, \langle \boldsymbol{\lambda}(\mathbf{env} : \langle \rangle, \mathbf{fc} : (\mathsf{unit}) \to \mathsf{unit}^{\langle \mathcal{C} \rangle}).\mathbf{e}_2', \langle \rangle \rangle \rangle \;\mathbf{as}\; ((\mathsf{unit}) \to \mathsf{unit}) \to \mathsf{int}^{\langle \mathcal{C} \rangle} \\
& \text{where } \mathbf{e}_2' \;=\; \mathbf{unpack}\ \langle \beta, \mathbf{fp} \rangle = \mathbf{fc\; in} \\
& \qquad\qquad \mathbf{let\; (f, y)} = (\boldsymbol{\pi}_1(\mathbf{fp}), \boldsymbol{\pi}_2(\mathbf{fp})) \;\mathbf{in\; f\;(y, ())};\, \mathbf{f\;(y, ())};\, \mathbf{2}
\end{aligned}$$

While $\mathsf{e}_1$ and $\mathsf{e}_2$ are contextually equivalent in the source language, their compiled versions $\mathbf{e_1}$ and $\mathbf{e_2}$ can be distinguished by the target language context $\mathbf{C}$ shown below. The context $\mathbf{C}$ does this by using $\mathbf{call/cc}$ to capture the continuation $\mathbf{k}$ of the second call to $\mathbf{f}$, after which it sets $\mathbf{x}$ back to $\mathbf{0}$ and then throws control back to $\mathbf{k}$.

First, we show a version of $\mathbf{C}$ written in a non-closure-converted (hence, more readable) form, as written, say, in an M-like language with call/cc:

$$\begin{aligned}
C = \; & \mathsf{let}\, \mathsf{g} = [\cdot] \;\mathsf{in}\; \mathsf{let}\, \mathsf{b} = \mathsf{new}\, 1 \;\mathsf{in} \\
& \mathsf{let}\ \mathsf{f} \;=\; \lambda(_).\ \mathsf{let}\ \mathsf{bv} \;=\; !\mathsf{b}\ \mathsf{in}\ \mathsf{if0}\ \mathsf{bv}\ (\mathsf{call/cc}_{\mathsf{unit}}(\mathsf{k}.\, \mathsf{g}\,(\lambda(_).\mathsf{throw}_{\mathsf{unit}}\,()\,\mathsf{to}\,\mathsf{k}))) \\
& \qquad\qquad\qquad\qquad\qquad\qquad (\mathsf{b} := 0)\ \mathsf{in} \\
& \mathsf{g}\,\mathsf{f}
\end{aligned}$$

The above context $C$ can be expressed in our target language C as follows:

$$\begin{aligned}
\mathbf{C} = \; & \mathbf{let\; gc} \;=\; [\cdot] \;\mathbf{in} \\
& \mathbf{unpack}\ \langle \beta, \mathbf{gp} \rangle = \mathbf{gc\; in} \\
& \mathbf{let\; (g, z)} = (\boldsymbol{\pi}_1(\mathbf{gp}), \boldsymbol{\pi}_2(\mathbf{gp})) \;\mathbf{in} \\
& \mathbf{let\; b} \;=\; \mathbf{new\; 1\; in} \\
& \mathbf{let\; fc} \;=\; \mathbf{pack}\ \langle \langle \mathbf{ref\;int} \rangle,\ \langle (\boldsymbol{\lambda}(\mathbf{y} : \langle \mathbf{ref\;int} \rangle, _ : \mathbf{unit}). \\
& \qquad\qquad\qquad\qquad \mathbf{let\; b'} \;=\; \boldsymbol{\pi}_1(\mathbf{y}) \;\mathbf{in} \\
& \qquad\qquad\qquad\qquad \mathbf{let\; bv} \;=\; !\mathbf{b'}\;\mathbf{in} \\
& \qquad\qquad\qquad\qquad \mathbf{if0\; bv}\ (\mathbf{call/cc}_{\mathbf{unit}}(\mathbf{k.\; g\; (z, p)})) \\
& \qquad\qquad\qquad\qquad\qquad (\mathbf{b'} := \mathbf{0})), \\
& \qquad\qquad\qquad \langle \mathbf{b} \rangle \rangle \rangle \;\mathbf{as}\; (\mathsf{unit}) \to \mathsf{unit}^{\langle \mathcal{C} \rangle} \;\mathbf{in} \\
& \mathbf{g\;(z, fc)}
\end{aligned}$$

$$\begin{aligned}
\text{where } \mathbf{p} = \; & \mathbf{pack}\ \langle \langle \mathbf{cont\;unit} \rangle, \\
& \quad \langle \boldsymbol{\lambda}(\mathbf{k'} : \langle \mathbf{cont\;unit} \rangle,\ _ : \mathbf{unit}). \\
& \qquad\qquad \mathbf{throw}_{\mathbf{unit}}\,()\;\mathbf{to}\;\boldsymbol{\pi}_1(\mathbf{k'}) \\
& \quad \langle \mathbf{k} \rangle \rangle \;\mathbf{as}\; (\mathsf{unit}) \to \mathsf{unit}^{\langle \mathcal{C} \rangle}
\end{aligned}$$

Note that:

$$\mathsf{C}[\mathsf{e_1}] \longmapsto^* \mathbf{0} \qquad\qquad \mathsf{C}[\mathsf{e_2}] \longmapsto^* \mathbf{2}$$

# References


[1] Derek Dreyer, Georg Neis, and Lars Birkedal. The impact of higher-order state and control effects on local relational reasoning. *Journal of Functional Programming*, 22(4&5):477–528, 2012.

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