You may turn in handwritten solutions to this assignment—but make sure you write clearly and leave lots of whitespace. If you choose to typeset your solutions, the LaTeX sources are available for your use at the course website. (Look in hw2.tex for macros you can use.) A `pdf` file containing the solutions can be submitted via email.

This homework is worth 100 points.

## Parametricity

Recall from class the following *logical relation* for System F (where we assume a call-by-value operational semantics for System F).

**Notation** Below, the metavariable $R$ ranges over relations on closed values (i.e., values with no free type or term variables). The metavariable $\rho$ ranges over finite maps from type variables $\alpha$ to triples $(\tau_1, \tau_2, R)$, where $\tau_1$ and $\tau_2$ are closed types and $R$ is a value relation. Finally, the metavariable $\gamma$ ranges over finite maps from term variables $x$ to pairs of closed values.

If $\rho = \{\alpha_1 \mapsto (\tau_{11}, \tau_{12}, R_1), \ldots, \alpha_n \mapsto (\tau_{n1}, \tau_{n2}, R_n)\}$ and $\gamma = \{x_1 \mapsto (v_{11}, v_{12}), \ldots, x_m \mapsto (v_{m1}, v_{m2})\}$, then

- $\rho_1(\tau)$ denotes $\tau[\tau_{11}/\alpha_1, \ldots, \tau_{n1}/\alpha_n]$ and $\rho_2(\tau)$ denotes $\tau[\tau_{12}/\alpha_1, \ldots, \tau_{n2}/\alpha_n]$
- $\rho_1(e)$ denotes $e[\tau_{11}/\alpha_1, \ldots, \tau_{n1}/\alpha_n]$ and $\rho_2(e)$ denotes $e[\tau_{12}/\alpha_1, \ldots, \tau_{n2}/\alpha_n]$
- $\gamma_1(e)$ denotes $e[v_{11}/x_1, \ldots, v_{m1}/x_m]$ and $\gamma_2(e)$ denotes $e[v_{12}/x_1, \ldots, v_{m2}/x_m]$

$$
\begin{array}{lcl}
\mathrm{Rel}[\tau_1, \tau_2] & = & \{\, R \subseteq \mathit{Val} \times \mathit{Val} \mid \ \forall (v_1, v_2) \in R.\ \vdash v_1 : \tau_1 \ \wedge\ \vdash v_2 : \tau_2 \,\} \\
\mathrm{Atom}[\tau]\rho & = & \{\, (e_1, e_2) \mid \ \vdash e_1 : \rho_1(\tau) \ \wedge\ \vdash e_2 : \rho_2(\tau) \,\} \\
\\
\mathcal{V}\,[\![\alpha]\!]\,\rho & = & \{\, (v_1, v_2) \mid \ \rho(\alpha) = (\tau_1, \tau_2, R) \ \wedge\ (v_1, v_2) \in R \,\} \\
\mathcal{V}\,[\![\tau \to \tau']\!]\,\rho & = & \{\, (\lambda x{:}\rho_1(\tau).\, e_1, \lambda x{:}\rho_2(\tau).\, e_2) \in \mathrm{Atom}[\tau \to \tau']\rho \mid \\
 & & \quad \forall v_1, v_2.\ (v_1, v_2) \in \mathcal{V}\,[\![\tau]\!]\,\rho \implies (e_1[v_1/x], e_2[v_2/x]) \in \mathcal{E}\,[\![\tau']\!]\,\rho \,\} \\
\mathcal{V}\,[\![\forall \alpha.\, \tau]\!]\,\rho & = & \{\, (\Lambda\alpha.\, e_1, \Lambda\alpha.\, e_2) \in \mathrm{Atom}[\forall\alpha.\,\tau]\rho \mid \\
 & & \quad \forall \tau_1, \tau_2, R.\ R \in \mathrm{Rel}[\tau_1, \tau_2] \implies (e_1[\tau_1/\alpha], e_2[\tau_2/\alpha]) \in \mathcal{E}\,[\![\tau]\!]\,\rho[\alpha \mapsto (\tau_1, \tau_2, R)] \,\} \\
\\
\mathcal{E}\,[\![\tau]\!]\,\rho & = & \{\, (e_1, e_2) \in \mathrm{Atom}[\tau]\rho \mid \ \exists v_1, v_2.\ e_1 \longrightarrow^* v_1 \ \wedge\ e_2 \longrightarrow^* v_2 \ \wedge\ (v_1, v_2) \in \mathcal{V}\,[\![\tau]\!]\,\rho \,\} \\
\\
\mathcal{D}\,[\![\cdot]\!] & = & \{\, \emptyset \,\} \\
\mathcal{D}\,[\![\Delta, \alpha]\!] & = & \{\, \rho[\alpha \mapsto (\tau_1, \tau_2, R)] \mid \ \rho \in \mathcal{D}\,[\![\Delta]\!] \ \wedge\ R \in \mathrm{Rel}[\tau_1, \tau_2] \,\} \\
\mathcal{G}\,[\![\cdot]\!]\,\rho & = & \{\, \emptyset \,\} \\
\mathcal{G}\,[\![\Gamma, x : \tau]\!]\,\rho & = & \{\, \gamma[x \mapsto (v_1, v_2)] \mid \ \gamma \in \mathcal{G}\,[\![\Gamma]\!]\,\rho \ \wedge\ (v_1, v_2) \in \mathcal{V}\,[\![\tau]\!]\,\rho \,\} \\
\\
\Delta; \Gamma \vdash e_1 \approx e_2 : \tau & \stackrel{\text{def}}{=} & \forall \rho, \gamma.\ \rho \in \mathcal{D}\,[\![\Delta]\!] \ \wedge\ \gamma \in \mathcal{G}\,[\![\Gamma]\!]\,\rho \implies (\rho_1(\gamma_1(e_1)), \rho_2(\gamma_2(e_2))) \in \mathcal{E}\,[\![\tau]\!]\,\rho
\end{array}
$$

**Theorem** (Parametricity)**.** If $\Delta; \Gamma \vdash e : \tau$ then $\Delta; \Gamma \vdash e \approx e : \tau$.

The proof is by induction on the typing derivation $\Delta; \Gamma \vdash e : \tau$.

- **Case:** $\dfrac{\Delta; \Gamma, x : \tau \vdash e : \tau'}{\Delta; \Gamma \vdash \lambda x{:}\tau.\, e : \tau \to \tau'}$ (Abs)

  **Proof**

  We are required to show that $\Delta; \Gamma \vdash \lambda x{:}\tau.\, e \approx \lambda x{:}\tau.\, e : \tau \to \tau'$.
  Consider arbitrary $\rho$ and $\gamma$ such that

  – $\rho \in \mathcal{D}\,[\![\Delta]\!]$, and
  – $\gamma \in \mathcal{G}\,[\![\Gamma]\!]\,\rho$.

  We are required to show that $(\rho_1(\gamma_1(\lambda x{:}\tau.\, e)), \rho_2(\gamma_2(\lambda x{:}\tau.\, e))) \in \mathcal{E}\,[\![\tau \to \tau']\!]\,\rho$
  $\equiv (\lambda x{:}\rho_1(\tau).\, \rho_1(\gamma_1(e)), \lambda x{:}\rho_2(\tau).\, \rho_2(\gamma_2(e))) \in \mathcal{E}\,[\![\tau \to \tau']\!]\,\rho$.
  Take $v_1 = \lambda x{:}\rho_1(\tau).\, \rho_1(\gamma_1(e))$.
  Take $v_2 = \lambda x{:}\rho_2(\tau).\, \rho_2(\gamma_2(e))$.
  We are required to show that

  – $\lambda x{:}\rho_1(\tau).\, \rho_1(\gamma_1(e)) \longrightarrow^* \lambda x{:}\rho_1(\tau).\, \rho_1(\gamma_1(e))$,
    which is immediate since $\lambda x{:}\rho_1(\tau).\, \rho_1(\gamma_1(e))$ is a value,
  – $\lambda x{:}\rho_2(\tau).\, \rho_2(\gamma_2(e)) \longrightarrow^* \lambda x{:}\rho_2(\tau).\, \rho_2(\gamma_2(e))$,
    which is immediate since $\lambda x{:}\rho_2(\tau).\, \rho_2(\gamma_2(e))$ is a value, and
  – $(\lambda x{:}\rho_1(\tau).\, \rho_1(\gamma_1(e)), \lambda x{:}\rho_2(\tau).\, \rho_2(\gamma_2(e))) \in \mathcal{V}\,[\![\tau \to \tau']\!]\,\rho$,
    which we conclude as follows:

    Consider arbitrary $v_1$ and $v_2$ such that
    * $(v_1, v_2) \in \mathcal{V}\,[\![\tau]\!]\,\rho$.

    We are required to show $(\rho_1(\gamma_1(e))[v_1/x], \rho_2(\gamma_2(e))[v_2/x]) \in \mathcal{E}\,[\![\tau']\!]\,\rho$.
    Applying the induction hypothesis to $\Delta; \Gamma, x : \tau \vdash e : \tau'$, we have that $\Delta; \Gamma, x : \tau \vdash e \approx e : \tau'$.
    Instantiate the latter with $\rho$ and $\gamma[x \mapsto (v_1, v_2)]$. Note that
    * $\rho \in \mathcal{D}\,[\![\Delta]\!]$,
      which follows from above, and
    * $\gamma[x \mapsto (v_1, v_2)] \in \mathcal{G}\,[\![\Gamma, x : \tau]\!]\,\rho$,
      which follows from
      · $\gamma \in \mathcal{G}\,[\![\Gamma]\!]\,\rho$ (follows from above), and
      · $(v_1, v_2) \in \mathcal{V}\,[\![\tau]\!]\,\rho$ (follows from above).

    Hence, $(\rho_1((\gamma[x \mapsto (v_1, v_2)])_1(e)), \rho_2((\gamma[x \mapsto (v_1, v_2)])_2(e))) \in \mathcal{E}\,[\![\tau']\!]\,\rho$.
    Thus, $(\rho_1(\gamma_1(e))[v_1/x], \rho_2(\gamma_2(e))[v_2/x]) \in \mathcal{E}\,[\![\tau']\!]\,\rho$, as we were required to show.

  $\square$

- **Case:** $\dfrac{\Delta; \Gamma \vdash e_1 : \tau_1 \to \tau_2 \qquad \Delta; \Gamma \vdash e_2 : \tau_1}{\Delta; \Gamma \vdash e_1\ e_2 : \tau_2}$ (App)

  **Proof**

  We are required to show that $\Delta; \Gamma \vdash e_1\ e_2 \approx e_1\ e_2 : \tau_2$.
  Consider arbitrary $\rho$ and $\gamma$ such that

  – $\rho \in \mathcal{D}\,[\![\Delta]\!]$, and
  – $\gamma \in \mathcal{G}\,[\![\Gamma]\!]\,\rho$.

We are required to show that $(\rho_1(\gamma_1(e_1\ e_2)), \rho_2(\gamma_1(e_1\ e_2))) \in \mathcal{E}\,[\![\tau_2]\!]\,\rho$
$\equiv (\rho_1(\gamma_1(e_1))\ \rho_1(\gamma_1(e_2)), \rho_2(\gamma_2(e_1))\ \rho_2(\gamma_2(e_2))) \in \mathcal{E}\,[\![\tau_2]\!]\,\rho$.

Applying the induction hypothesis to $\Delta; \Gamma \vdash e_1 : \tau_1 \to \tau_2$, we have that $\Delta; \Gamma \vdash e_1 \approx e_1 : \tau_1 \to \tau_2$.
Instantiate the latter with $\rho$ and $\gamma$. Note that

- $\rho \in \mathcal{D}\,[\![\Delta]\!]$, and
- $\gamma \in \mathcal{G}\,[\![\Gamma]\!]\,\rho$.

Hence, $(\rho_1(\gamma_1(e_1)), \rho_2(\gamma_2(e_1))) \in \mathcal{E}\,[\![\tau_1 \to \tau_2]\!]\,\rho$.
From the latter, there exist $v_1$ and $v_2$ such that

- $\rho_1(\gamma_1(e_1)) \longrightarrow^* v_{11}$,
- $\rho_2(\gamma_2(e_1)) \longrightarrow^* v_{12}$, and
- $(v_{11}, v_{12}) \in \mathcal{V}\,[\![\tau_1 \to \tau_2]\!]\,\rho$.

From the latter it follows that $v_1 1 = \lambda x{:}\rho_1(\tau_1).\, e_{11}$ and $v_{12} = \lambda x{:}\rho_2(\tau_1).\, e_{12}$.

Applying the induction hypothesis to $\Delta; \Gamma \vdash e_2 : \tau_1$, we have that $\Delta; \Gamma \vdash e_2 \approx e_2 : \tau_1$.
Instantiate the latter with $\rho$ and $\gamma$. Note that

- $\rho \in \mathcal{D}\,[\![\Delta]\!]$, and
- $\gamma \in \mathcal{G}\,[\![\Gamma]\!]\,\rho$.

Hence, $(\rho_1(\gamma_1(e_2)), \rho_2(\gamma_2(e_2))) \in \mathcal{E}\,[\![\tau_1]\!]\,\rho$.
From the latter, there exist $v_{21}$ and $v_{22}$ such that

- $\rho_1(\gamma_1(e_2)) \longrightarrow^* v_{21}$,
- $\rho_2(\gamma_2(e_2)) \longrightarrow^* v_{22}$, and
- $(v_{21}, v_{22}) \in \mathcal{V}\,[\![\tau_1]\!]\,\rho$.

Instantiate $(\lambda x{:}\rho_1(\tau_1).\, e_{11}, \lambda x{:}\rho_2(\tau_1).\, e_{12}) \in \mathcal{V}\,[\![\tau_1 \to \tau_2]\!]\,\rho$ (from above) with $v_{21}$ and $v_{22}$. Note that

- $(v_{21}, v_{22}) \in \mathcal{V}\,[\![\tau_1]\!]\,\rho$.

Hence, $(e_{11}[v_{21}/x], e_{12}[v_{22}/x]) \in \mathcal{E}\,[\![\tau_2]\!]\,\rho$.
From the latter, there exist $v_1$ and $v_2$ such that

- $e_{11}[v_{21}/x] \longrightarrow^* v_1$,
- $e_{12}[v_{22}/x] \longrightarrow^* v_2$, and
- $(v_1, v_2) \in \mathcal{V}\,[\![\tau_2]\!]\,\rho$.

Thus, there exist $v_1$ and $v_2$ such that

- $\rho_1(\gamma_1(e_1))\ \rho_1(\gamma_1(e_2)) \longrightarrow^* v_1$ (from above by operational semantics),
- $\rho_2(\gamma_2(e_1))\ \rho_2(\gamma_2(e_2)) \longrightarrow^* v_2$ (from above by operational semantics), and
- $(v_1, v_2) \in \mathcal{V}\,[\![\tau_2]\!]\,\rho$, which follows from above.

$\square$

**Exercise 1.** (20 pts) Do the TABS and TAPP cases of the proof:

(a) **Case:** $\dfrac{\Delta, \alpha; \Gamma \vdash e : \tau}{\Delta; \Gamma \vdash \Lambda\alpha.\, e : \forall\alpha.\, \tau}$ (TABS)

(b) **Case:** $\dfrac{\Delta; \Gamma \vdash e : \forall\alpha.\, \tau \qquad \Delta \vdash \sigma}{\Delta; \Gamma \vdash e\,[\sigma] : \tau[\sigma/\alpha]}$ (TAPP)

You may make use of the following lemmas:

**Lemma** (Compositionality, or "syntactic substitution equals semantic substitution")**.**
If $\rho \in \mathcal{D}[\![\Delta]\!]$ and $\Delta \vdash \tau'$ and $R = \{\ (v_1', v_2') \mid\ (v_1', v_2') \in \mathcal{V}[\![\tau']\!]\,\rho\ \}$,
then $(v_1, v_2) \in \mathcal{V}[\![\tau[\tau'/\alpha]]\!]\,\rho$ iff $(v_1, v_2) \in \mathcal{V}[\![\tau]\!]\,\rho[\alpha \mapsto (\rho_1(\tau'), \rho_2(\tau'), R)]$.

**Lemma** (Well-typed inhabitants)**.** If $(v_1, v_2) \in \mathcal{V}[\![\tau]\!]\,\rho$, then $\vdash v_1 : \rho_1(\tau)$ and $\vdash v_2 : \rho_2(\tau)$.

---

Let us add product types, existential types, and record types to System F.

$$
\begin{array}{lcl}
\tau & ::= & \ldots \mid \tau_1 \times \tau_2 \mid \exists\alpha.\,\tau \mid \{\ell_1 : \tau_1, \ldots, \ell_n : \tau_n\} \\
e & ::= & \ldots \mid \langle e_1, e_2\rangle \mid \mathsf{fst}\, e \mid \mathsf{snd}\, e \mid \mathsf{pack}\,\langle\tau', e\rangle\,\mathsf{as}\,\exists\alpha.\,\tau \mid \mathsf{unpack}\,\langle\alpha, x\rangle = e_1\,\mathsf{in}\,e_2 \mid \\
 & & \{\ell_1 = e_1, \ldots, \ell_n = e_n\} \mid e.\ell \\
v & ::= & \ldots \mid \langle v_1, v_2\rangle \mid \mathsf{pack}\,\langle\tau', v\rangle\,\mathsf{as}\,\exists\alpha.\,\tau \mid \{\ell_1 = v_1, \ldots, \ell_n = v_n\} \\
E & ::= & \ldots \mid \langle E, e_2\rangle \mid \langle v_1, E\rangle \mid \mathsf{fst}\, E \mid \mathsf{snd}\, E \mid \mathsf{pack}\,\langle\tau', E\rangle\,\mathsf{as}\,\exists\alpha.\,\tau \mid \mathsf{unpack}\,\langle\alpha, x\rangle = E\,\mathsf{in}\,e_2 \mid \\
 & & \{\ell_1 = v_1, \ldots, \ell_i = E, \ldots \ell_n = e_n\} \mid E.\ell
\end{array}
$$

$\boxed{e \longrightarrow e'}$

$$
\begin{array}{lcl}
\mathsf{fst}\,\langle v_1, v_2\rangle & \longrightarrow & v_1 \\
\mathsf{snd}\,\langle v_1, v_2\rangle & \longrightarrow & v_2 \\
\mathsf{unpack}\,\langle\alpha, x\rangle = (\mathsf{pack}\,\langle\tau', v\rangle\,\mathsf{as}\,\exists\alpha.\,\tau)\,\mathsf{in}\,e_2 & \longrightarrow & e_2[\tau'/\alpha][v/x] \\
(\{\ell_1 = v_1, \ldots, \ell_i = v_i, \ldots \ell_n = v_n\}).\ell_i & \longrightarrow & v_i
\end{array}
$$

$\boxed{\Delta; \Gamma \vdash e : \tau}$

$$\frac{\Delta; \Gamma \vdash e_1 : \tau_1 \qquad \Delta; \Gamma \vdash e_2 : \tau_2}{\Delta; \Gamma \vdash \langle e_1, e_2\rangle : \tau_1 \times \tau_2}\ (\textsc{Pair}) \qquad \frac{\Delta; \Gamma \vdash e : \tau_1 \times \tau_2}{\Delta; \Gamma \vdash \mathsf{fst}\, e : \tau_1}\ (\textsc{Fst}) \qquad \frac{\Delta; \Gamma \vdash e : \tau_1 \times \tau_2}{\Delta; \Gamma \vdash \mathsf{snd}\, e : \tau_2}\ (\textsc{Snd})$$

$$\frac{\Delta \vdash \sigma \qquad \Delta; \Gamma \vdash e : \tau[\sigma/\alpha]}{\Delta; \Gamma \vdash \mathsf{pack}\,\langle\sigma, e\rangle\,\mathsf{as}\,\exists\alpha.\,\tau : \exists\alpha.\,\tau}\ (\textsc{Pack})$$

$$\frac{\Delta; \Gamma \vdash e_1 : \exists\alpha.\,\tau \qquad \Delta \vdash \tau_2 \qquad \Delta, \alpha; \Gamma, x : \tau \vdash e_2 : \tau_2}{\Delta; \Gamma \vdash \mathsf{unpack}\,\langle\alpha, x\rangle = e_1\,\mathsf{in}\,e_2 : \tau_2}\ (\textsc{Unpack})$$

$$\frac{\Delta; \Gamma \vdash e_i : \tau_i \quad \text{forall } i \in \{1, \ldots, n\}}{\Delta; \Gamma \vdash \{\ell_1 = e_1, \ldots, \ell_n = e_n\} : \{\ell_1 : \tau_1, \ldots, \ell_n : \tau_n\}}\ (\textsc{Record})$$

$$\frac{\Delta; \Gamma \vdash e : \{\ell_1 : \tau_1, \ldots, \ell_i : \tau_i, \ldots, \ell_n : \tau_n\}}{\Delta; \Gamma \vdash e.\ell_i : \tau_i}\ (\textsc{Proj})$$

The logical relation for the extended language is exactly as above, except that we must also define when values of product, existential, and record types are related.

The value relation for products and existential types is defined as follows.

$$
\begin{array}{lcl}
\mathcal{V}\,[\![\tau_1 \times \tau_2]\!]\,\rho & = & \{\, (\langle v_1, v_2\rangle, \langle v_1', v_2'\rangle) \in \mathrm{Atom}[\tau_1 \times \tau_2]\rho \mid \ (v_1, v_1') \in \mathcal{V}\,[\![\tau_1]\!]\,\rho \ \wedge \ (v_2, v_2') \in \mathcal{V}\,[\![\tau_2]\!]\,\rho \,\} \\
\mathcal{V}\,[\![\exists\alpha.\,\tau]\!]\,\rho & = & \{\, (\mathsf{pack}\,\langle \tau_1, v_1\rangle\,\mathsf{as}\,\exists\alpha.\,\rho_1(\tau), \mathsf{pack}\,\langle \tau_2, v_2\rangle\,\mathsf{as}\,\exists\alpha.\,\rho_2(\tau)) \in \mathrm{Atom}[\exists\alpha.\,\tau]\rho \mid \\
 & & \quad \exists R.\ R \in \mathrm{Rel}[\tau_1, \tau_2] \ \wedge \ (v_1, v_2) \in \mathcal{V}\,[\![\tau]\!]\,\rho[\alpha \mapsto (\tau_1, \tau_2, R)] \,\}
\end{array}
$$

**Recommended:** Do the PACK and UNPACK cases of the proof of parametricity.

**Exercise 2.** (5 pts) Define the value relation for record types.

---

## Free Theorems

**Exercise 3.** (15 pts) Let $\vdash e : \forall\alpha.\,\alpha \to \alpha \to \alpha$, $\vdash \tau$, $\vdash v_1 : \tau$, and $\vdash v_2 : \tau$.
Prove the following: If $e\,[\tau]\ v_1\ v_2 \longrightarrow^* v$ then either $v = v_1$ or $v = v_2$.

**Exercise 4.** (30 pts) Let $\vdash e : \forall\alpha.\,\alpha \to \alpha$, $\vdash f : \tau \to \tau'$, and $\vdash v : \tau$.
Prove the following: $\cdot;\cdot \vdash f\ (e\,[\tau]\ v) \approx e\,[\tau']\ (f\ v) : \tau'$.

---

## Representation Independence

Parametricity is the essence of *representation independence* for abstract data types.

A relation $R \in \mathrm{Rel}[\tau_1, \tau_2]$ captures the invariant relationship between two packages of type $\exists\alpha.\,\tau$: $\mathsf{pack}\,\langle \tau_1, v_1\rangle\,\mathsf{as}\,\exists\alpha.\,\tau$ and $\mathsf{pack}\,\langle \tau_2, v_2\rangle\,\mathsf{as}\,\exists\alpha.\,\tau$ if and only if:

$$(v_1, v_2) \in \mathcal{V}\,[\![\tau]\!]\,\emptyset[\alpha \mapsto (\tau_1, \tau_2, R)]$$

When two packages $e_1$ and $e_2$ are logically related, they are indistinguishable by any client — that is, the execution of a client $e_c$ cannot depend on whether $e_1$ or $e_2$ is the provided package. In other words, if $\alpha; x : \tau \vdash e_c : \tau_c$, then

$$\cdot;\cdot \vdash \mathsf{unpack}\,\langle \alpha, x\rangle = e_1\ \mathsf{in}\ e_c \approx \mathsf{unpack}\,\langle \alpha, x\rangle = e_2\ \mathsf{in}\ e_c : \tau_c$$

For the exercises that follow, we must define when two values of type $\mathsf{int}$ are related. We use the metavariable $n$ for integers and extend the logical relation to integer types as follows:

$$\mathcal{V}\,[\![\mathsf{int}]\!]\,\rho \ = \ \{\, (n, n) \in \mathrm{Atom}[\mathsf{int}]\rho \,\}$$

**Exercise 5.** (15 pts) For each of the following pairs of packages $e_1$ and $e_2$, say whether they are logically related (contextually equivalent). If so, provide a proof. If not, provide a program context that can tell them apart. (You may assume that the logical relation defined earlier in this document is sound with respect to contextual equivalence.)

(a)

$$
\begin{array}{lcl}
\tau & = & \exists\alpha.\,\alpha \times (\alpha \to \mathsf{bool}) \\
e_1 & = & \mathsf{pack}\,\langle \mathsf{int}, \langle 0, \lambda x{:}\mathsf{int}.\,(\mathsf{eq?}\ x\ 1)\rangle\rangle\ \mathsf{as}\ \tau \\
e_2 & = & \mathsf{pack}\,\langle \mathsf{bool}, \langle \mathsf{true}, \lambda x{:}\mathsf{bool}.\,\mathsf{not}\ x\rangle\rangle\ \mathsf{as}\ \tau
\end{array}
$$

(b)

$$\begin{array}{rcl} \tau & = & \exists\alpha.\, \alpha \times (\alpha \to \mathsf{int}) \\ e_1 & = & \mathsf{pack}\ \langle \mathsf{string}, \langle \text{“hello”}, \lambda x{:}\mathsf{string}.\ (\mathsf{string\text{-}length}\ x)\rangle\rangle\ \mathsf{as}\ \tau \\ e_2 & = & \mathsf{pack}\ \langle \mathsf{int}, \langle 4, \lambda x{:}\mathsf{int}.\ x\rangle\rangle\ \mathsf{as}\ \tau \end{array}$$

**Exercise 6.** (15 pts) Show that the following two implementations of counters are logically related.

$$\begin{array}{rcl} \mathsf{Counter} & = & \exists\alpha.\, \{\mathsf{new} : \alpha, \\ & & \quad \mathsf{inc} : \alpha \to \alpha, \\ & & \quad \mathsf{get} : \alpha \to \mathsf{int}\} \\ \\ \mathsf{c1} & = & \{\mathsf{new} = 0, \\ & & \ \mathsf{inc} = \lambda x : \mathsf{int}.\ x + 1, \\ & & \ \mathsf{get} = \lambda x : \mathsf{int}.\ x\} \\ \\ \mathsf{cntr1} & = & \mathsf{pack}\ \langle \mathsf{int}, \mathsf{c1}\rangle\ \mathsf{as\ Counter} \\ \\ \mathsf{c2} & = & \{\mathsf{new} = 0, \\ & & \ \mathsf{inc} = \lambda x : \mathsf{int}.\ x - 1, \\ & & \ \mathsf{get} = \lambda x : \mathsf{int}.\ 0 - x\} \\ \\ \mathsf{cntr2} & = & \mathsf{pack}\ \langle \mathsf{int}, \mathsf{c2}\rangle\ \mathsf{as\ Counter} \end{array}$$

Prove the following: $\cdot; \cdot \vdash \mathsf{cntr1} \approx \mathsf{cntr2} : \mathsf{Counter}$.

You will need to use the value relation for record types from Exercise 2.