CS1800

10/20 - Fri :)

Admin

- HW5 out next Fri
- Back to recitations next week

Agenda

1. Basic Probability
2. Expected Value
3. Variance

Counting

Sets

1. Basic Probability

- Set cardinality $|S|$ – # of distinct elements
- Cartesian Product $A \times B$ – $\{(a,b) \mid a \in A \wedge b \in B\}$
- Product rule $m \cdot n$ – "and"
- Sum rule $m + n$ – "or"
- Pigeonhole – makes guarantees

Probability

↳ measures likelihood of events

- experiment – procedure, ~~w~~ repeatable, well-defined set of outcomes
- sample space – set of all possible outcomes of an experiment $S = \{s_1, s_2, \ldots, s_n\}$
- event space – subset of sample space $E \subseteq S$

Die – 6 sides

1, 2, 3, 4, 5, 6

roll a die ⟿ $\boxed{\text{experiment}}$

$S = \{1, 2, 3, 4, 5, 6\}$

Probability of event E

$$Pr(E) = \frac{|E|}{|S|}$$

$|S| = 6$

$E = \{6\}$

$|E| = 1$

$$Pr(6) = \frac{|E|}{|S|} = \frac{1}{6}$$

$E = \{2, 4, 6\}$

$|E| = 3$

$$Pr(\text{even}) = \frac{|E|}{|S|} = \frac{3}{6} = \frac{1}{2}$$

Rolling two dice

die #1 $\quad S = \{1,2,3,4,5,6\} = |6|$

die #2 $\quad S_2 = \{1,2,3,4,5,6\} = |6|$

Cartesian product — ordered pairs

$\{(1,1), (1,2), (1,3) \ldots, (6,6)\}$

$|S \times S_2| = 36$

sample space

Pr(getting a 3 and 4, in either order) $\rightarrow \frac{|E|}{|S|} = \frac{2}{36} = \frac{1}{18}$

$E = \{(3,4), (4,3)\}$

ex Craps in Old Vegas

First roll ~ sum of 2 dice

- win: 7, 11
- lose: 2, 3, 12

Pr(win)?

$|S| = 36$

$E = \{(1,6), (6,1), (3,4), (4,3), (2,5), (5,2), (6,5), (5,6)\}$

$|E| = 8$

$Pr(\text{win}) = \frac{8}{36} = \frac{2}{9}$

Pr(lose)?

$|S| = 36$

$E = \{(1,1), (1,2), (2,1), (6,6)\}$

$|E| = 4$

$Pr(\text{lose}) = \frac{4}{36} = \frac{1}{9}$

Q: why roll two dice $|S| = 36$

$A = \{1, 2, 3, 4, 5, 6\}$

$B = \{1, 2, 3, 4, 5, 6\}$ $\longrightarrow$ (a, b) to get $A \times B$

$(1, \_)$

$(2, \_)$

$\hookrightarrow \{1, 2, 3, 4, 5, 6\}$

In craps example ~ pr of all possible outcomes sums to 1

lose win 2/9

lose 4/9

$$\boxed{Pr(S_1) + Pr(S_2) + \ldots + Pr(S_n) = 1}$$

$Pr(\text{neither}) = \underline{1 - 3/9}$ → subtraction / complement

$= 6/9$

$= 2/3$

ex Pigeonhole makes guarantees!

200 of us, 12 months/year } at least one month has at least 17 of us

Probability is what's likely, unlikely

- Assume: all birthday months equally likely

$Pr(\text{born in Sept}) = 1/12$

$Pr(\text{share bday with Laney}) = 1/12$

$Pr(\text{born in not sept}) = 11/12$

ex 10 of us in the room

Pr(all 10 born in Sept)

Product rule

Pr(A in Sep) <u>and</u> Pr(B in Sep) <u>and</u> ... Pr(J in Sep)

$Pr(\text{all } 10) = 1/12 \cdot 1/12 \cdot 1/12 \cdots 1/12 = (1/12)^{10}$

$\boxed{10 \text{ people}}$

$= 1.615e^{-11}$

Pr(at least one born in sep)

1 - Pr(none in sept)

$1 - \left(\frac{11}{12}\right)^{10} = \boxed{.58}$

subtraction rule:

Case 1: exactly one

Case 2: exactly two

...

Case 10: exactly 10 omg!

Pr(none born in Sep)

$\left(\frac{11}{12}\right)^{10} = \boxed{.418}$

$\boxed{10:47}$

2. Expected Value

↳ "mean" ⟿ experiment is repeatable

run the experiment ∞ times,
what happens on average?

"not" guarantee about the outcome
of one experiment

Random Variable

- not random, not variable
- X = value associated with outcome of an experiment
- numeric ⟶ some value tied to outcomes
- $E[X]$ — what is avg X over ∞ experiments

(ex) Rolling one die

X is RV associated with number on die

$Pr(X=3)$ — Pr X is value 3 after experiment

What is $E[X]$?

$$E[X] = \sum Pr(S_i) \cdot X_i$$

- every outcome S_i has prob. $Pr(S_i)$
- every outcome S_i has a value X_i

• everything in sample space

ex Larry's bet on Aces to win !!

X = RV associated with outcome of game ($)

2 outcomes: win, lose

Every outcome has prob. and value
$Pr(s_i)$ x_i

$Pr(\text{win}) = .60$ $Pr(\text{lose}) = .40$

$x_{win} = 30$ $x_{lose} = -30$

$E[X] = (.60)(30) + (.40)(-30)$

$= \$6$

ex Flip a coin (fair)

outcomes = H, T

X = RV associated with # tails

$Pr(H) = .5$ $Pr(T) = .5$

$x_H = 0$ $x_T = 1$

$E\{X\} = (.5)(0) + (.5)(1) = .5$

For everyone... flip a coin 3 times (experiment)

- # tails (X)
- outcomes: HHH, HHT, HTH, THH, ...

$E[X] = ?$

$$E[X] = \sum \Pr(S_i) \cdot X_i$$

$\Pr(X=0) = 1/8$ HHH

$\Pr(X=1) = 3/8$ THH, HTH, HHT

$\Pr(X=2) = 3/8$ TTH, THT, HTT

$\Pr(X=3) = 1/8$ TTT

Sanity check:
$1/8 + 3/8 + 3/8 + 1/8 = 1$

$$E[X] = \underbrace{1/8}_{\Pr(X=0)} \cdot \underbrace{0}_{\text{value}} + 3/8 \cdot 1 + 3/8 \cdot 2 + 1/8 \cdot 3$$

$$= 0 + 3/8 + 6/8 + 3/8$$

$$= 12/8 = 1.5$$

Shortcut EV formula
· same experiment } linearity of expectation

$$E\{\#\text{tails in 3 tosses}\} = E\{\#\text{tails in toss 1}\} + E\{\#\text{tails in toss 2}\} + E\{\#\text{tails in toss 3}\}$$

$$\underset{\text{3 flips}}{E\{X\}} = .5 + .5 + .5 = 1.5$$

$$E\{\#\text{ tails in toss 1}\} = 0 \cdot .5 + 1 \cdot .5 = .5$$

↳ indicators

$$\underset{\text{100 flips}}{E\{X\}} = (.5)(100)$$