Day 10:

Admin:
- practice exam on gradescope
- review exam instructions
- hw4 note:
    - please compute (and round) final value in counting problems (as HW instructions indicate)
- hw4 dates:
    - due Friday @ 11:59 PM
    - late due date is Saturday @ 11:59 PM
    - solutions are available Sunday @ 12:10 AM

Content:
- combinations
- leftover principle
- counting partitions of identical objects

$$P(5,3) = \frac{5!}{2!} = 5 \cdot 4 \cdot 3 = 60$$

Over-counting (multiplicative)

How many people are in the room if ...

... there are 100 eyes in the room 50

... there are 90 fingers in the room 9

... there are 400 limbs (legs & arms) in the room 100

Punchline:

If there are n items (eyes, fingers, limbs)
and c items per every item-of-interest (people)

then there are n / c items of interest

Ordering: when does it matter?

Order matters:

How many ways can a student take 3 CS courses from 10 unique courses?

(CS 1800, DS 2000, DS 2500)

$\neq$

(DS 2500, DS 2000, CS 1800)

TUPLE

Order doesn't matter:

How many ways can one take 3 candies from 10 unique candies?

{CHOC, LOLLY, GUMMY}

$=$

{LOLLY, GUMMY, CHOC}

SET

Combination: (intro example)

How many ways can one choose 2 candies from 3 unique candies? $C = \{1, 2, 3\}$
(order doesn't matter)

3 WAYS:

$\{1, 2\}$ $\{1, 3\}$ $\{2, 3\}$

Combination: (intro example)

How many ways can one choose 2 candies from 3 unique candies? C = {1, 2, 3}
(order doesn't matter)

THERE ARE $P(3,2) = \frac{3!}{1!} = 6$ WAYS OF CHOOSING

TWO ORDERED CANDIES:

$(1,2)$ $(1,3)$ $(2,3)$

$(2,1)$ $(3,1)$ $(3,2)$

Combination: (intro example)

How many ways can one choose 2 candies from 3 unique candies? (order doesn't matter) $C = \{1, 2, 3\}$

THERE ARE $P(3,2) = \frac{3!}{1!} = 6$ WAYS OF CHOOSING TWO ORDERED CANDIES:

THERE ARE $2! = 2$ WAYS OF ORDERING 2 CANDIES

$(1,2)$ $(1,3)$ $(2,3)$

$(2,1)$ $(3,1)$ $(3,2)$

OVERCOUNTING (MULTIPLICATION):

WAYS OF CHOOSING 2 FROM 3 (ORDER NOT MATTER) = WAYS OF ORDERING 2 FROM 3 (ORDER MATTERS) / WAYS OF ORDERING 2 (ORDER MATTERS)

Combination: definition & formula

- A combination is a subset of objects (order doesn't matter)
  (how many ways can I choose k items from n possible)

- A permutation is an ordering of objects (order matters)
  (how many ways can I order k items from n possible)

$$C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!\,k!}$$

"n choose k"

$\binom{n}{k}$ AKA Binomial coefficient

In Class Activity

$$\underline{A}\ \underline{Z}\ \underline{E}$$
$$26\ 26\ 26 \qquad 26^3$$

How many ways can the 8 Mario Kart racers form the final podium of 3 winners. The order of the podium matters.

1. TOAD
2. YOSHI
3. DK

PERMUTATION

$$P(8,3) = \frac{8!}{(8-3)!} = 8 \cdot 7 \cdot 6$$
$$= 336$$

How many ways can the teams (mercedes, ferrari, etc) arrange on the podium of 3 winners in a formula 1 race? (assume that each team has at least 3 cars in the race). assume 10 teams

example ordering: (ferrari wins 1st place, ferrari wins 2nd place, mercedes wins 3rd place)

notice we ignore drivers, we're just ordering the teams

$$\underset{1ST}{\underline{10}} \quad \underset{2ND}{\underline{10}} \quad \underset{3RD}{\underline{10}} \qquad 10^3 = 1000$$

How many unique 5 card hands exist in a deck of 52 unique cards? ("hands" are unordered)

ORDER DOESN'T MATTER
NO REPEAT

$$\binom{52}{5} = \frac{52!}{5!\,(52-5)!} = 2.6 \text{ MILLION}$$

How many ways can one select the "remaining" 47 cards after selecting a 5 card hand (as in the problem above)?

$$\binom{52}{47} = \frac{52!}{47!\,(52-47)!}$$

Combinations: Leftover principle

How many ways can I choose all but 10 student to take out for ice cream from this class of size n?

$$\binom{250}{10} = \binom{250}{240}$$

How many ways can I choose n - 10 students to take out for ice cream from this class of size n?

$$\binom{n}{k} = \binom{n}{n-k}$$

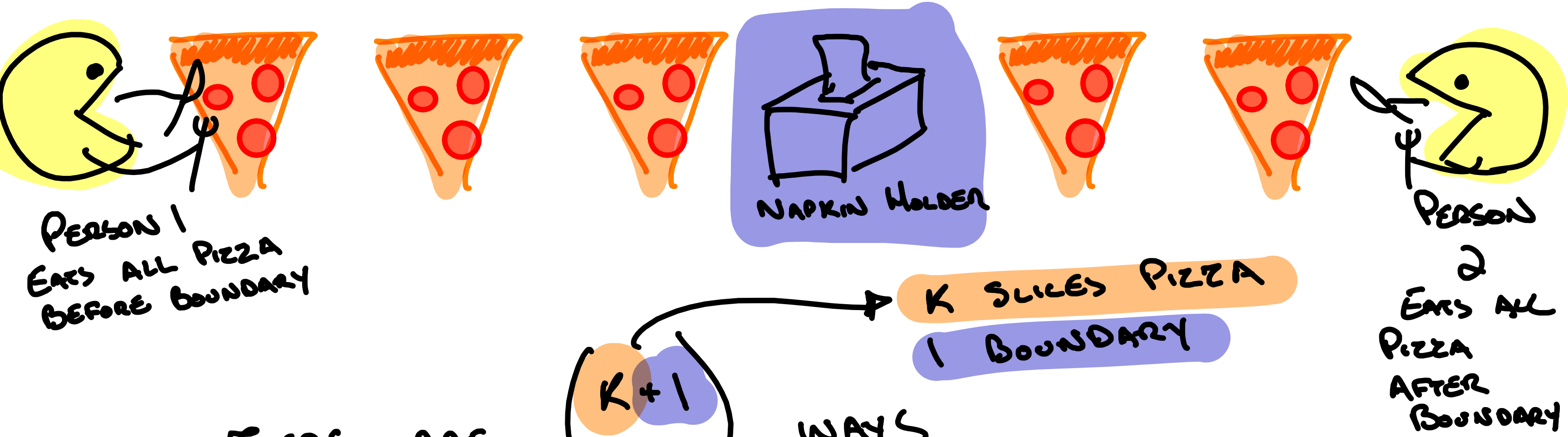
Counting: Putting it together (almost … see later slide for complete version of this table)

HOW TO SELECT K ITEMS FROM N

	NO REPEAT SELECTIONS	REPEAT SELECTIONS
ORDER MATTERS	PERMUTATIONS $P(N,K) = \frac{N!}{(N-K)!}$	PRODUCT RULE N^K
ORDER DOESN'T MATTER	COMBINATIONS $\binom{N}{K} = \frac{N!}{(N-K)!\,K!}$	MYSTERY (FOR NOW)

Counting Partitions of identical objects: (AKA Balls in bins or Stars & bars):

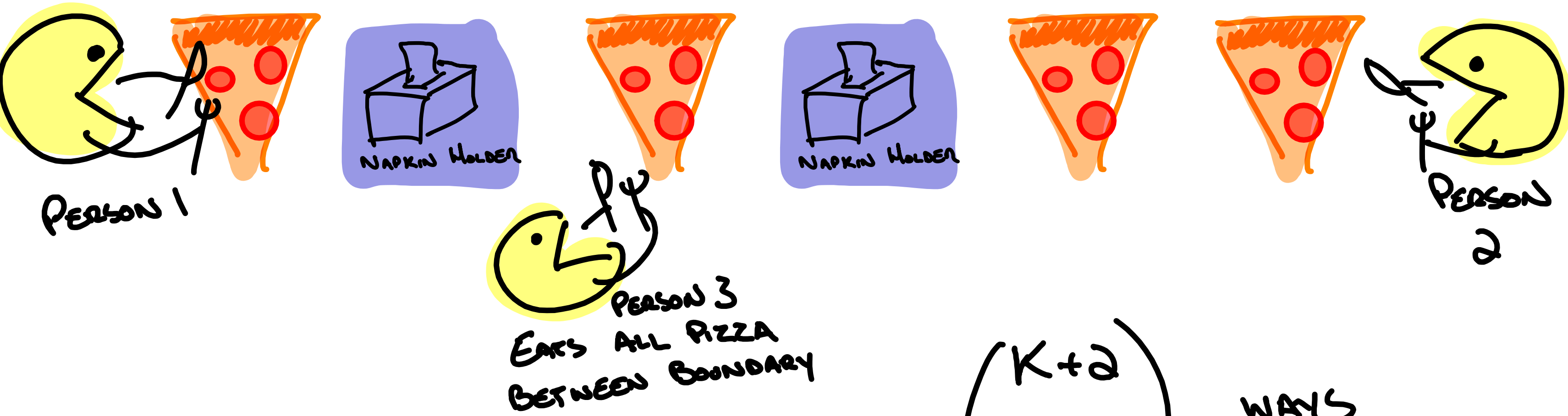
How many different ways can two people split k slices of pizza?



THERE ARE $\binom{k+1}{1}$ WAYS

Counting Partitions of identical objects: (AKA Balls in bins or Stars & bars):

How many different ways can ~~two~~ THREE people split K slices of pizza?

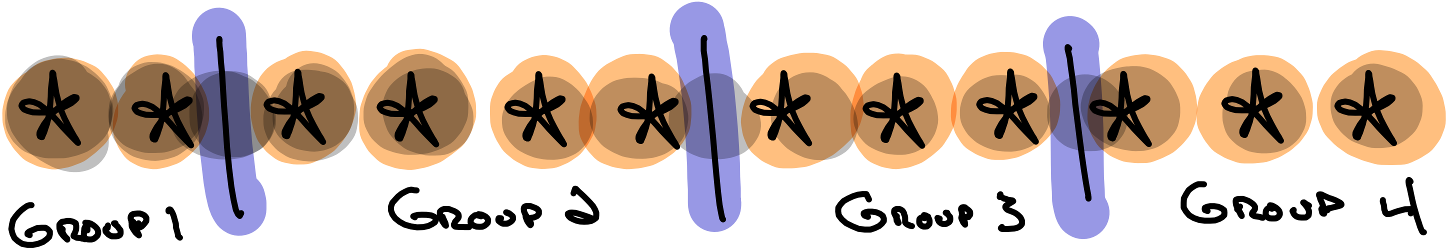


$$\binom{K+2}{2}$$ WAYS

Counting Partitions of identical objects: (AKA Balls in bins or Stars & bars):

How many different ways can [redacted]e split k s[redacted]?

N GROUPS STARS



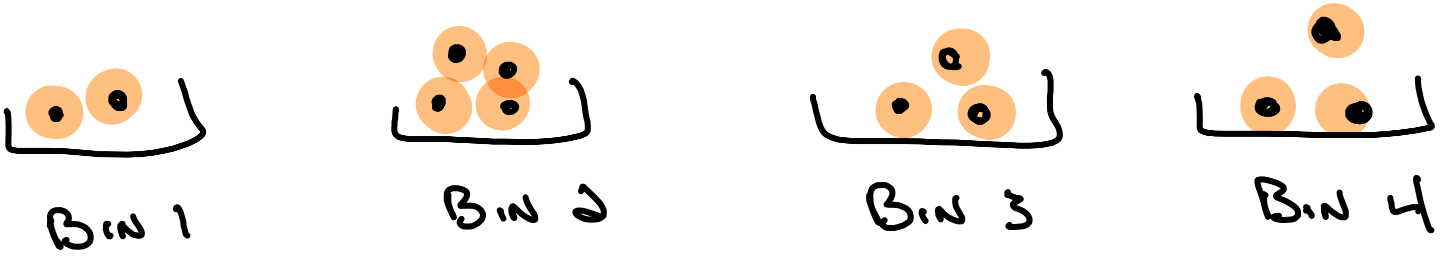
$$\binom{K+N-1}{N-1} \text{ WAYS}$$

NEED N-1 BOUNDARIES FOR N GROUPS

Counting Partitions of identical objects: (AKA Balls in bins or Stars & bars):

How many different ways can [redacted]e split k s[redacted]?

N BINS BALLS



$$\binom{K+N-1}{N-1} \text{ WAYS}$$

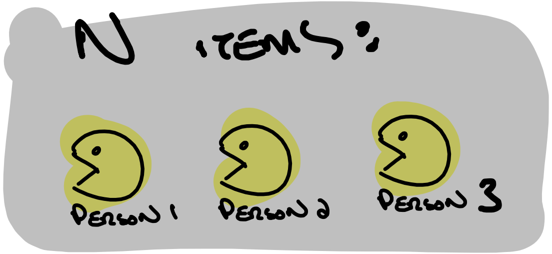
Something is still missing in our chart

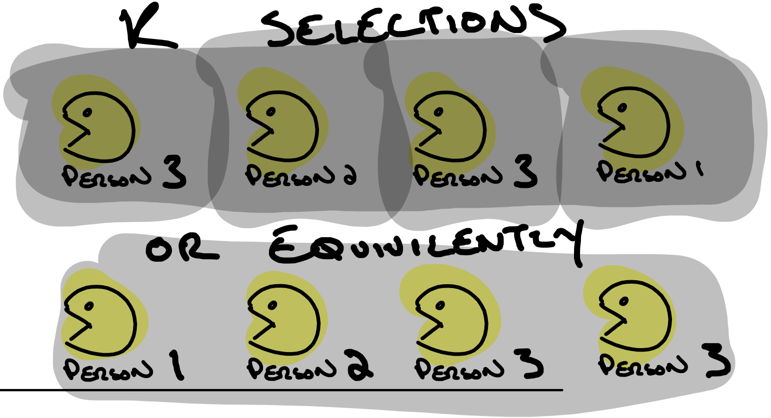
HOW TO SELECT K ITEMS FROM N

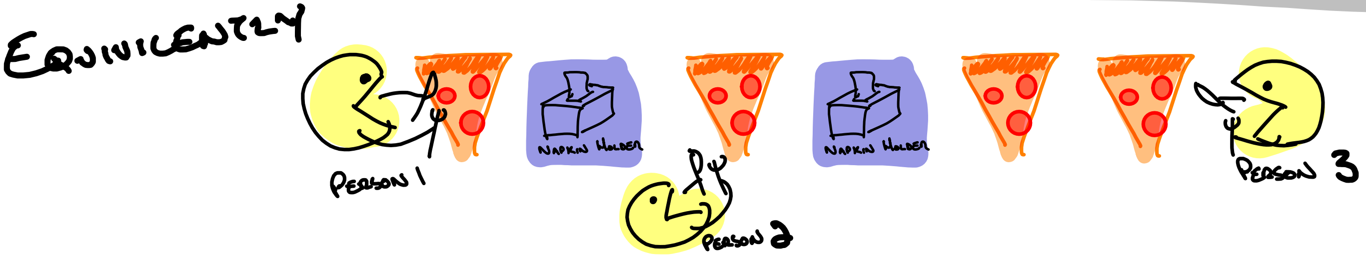
	NO REPEAT SELECTIONS	REPEAT SELECTIONS
ORDER MATTERS	PERMUTATIONS $P(N,K) = \frac{N!}{(N-K)!}$	PRODUCT RULE N^K
ORDER DOESN'T MATTER	COMBINATIONS $\binom{N}{K} = \frac{N!}{(N-K)!K!}$	MYSTERY (FOR NOW)

How is the balls-in-bins fit into bottom right box of "putting it together"?

Selecting k items from N items
- repeat selections allowed
- order of selections doesn't matter





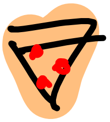


HOW TO SELECT K ITEMS FROM N

	NO REPEAT SELECTIONS	REPEAT SELECTIONS
ORDER MATTERS	PERMUTATIONS $P(N,K) = \frac{N!}{(N-K)!}$ How many tuples of length k can one make from N items? (no repeats)	PRODUCT RULE N^K How many tuples of length k can one make from N items? (repeats)
ORDER DOESN'T MATTER	COMBINATIONS $\binom{N}{K} = \frac{N!}{(N-K)!\,K!}$ How many sets with k unique items can one make from N items? (no repeats)	PARTITION OF IDENTICAL ITEMS (STARS + BARS / BALLS IN BINS) $\binom{K+N-1}{N-1}$ How many ways can we split k identical items among N groups?

Two conventions for Stars and Bars

In class now:

K items 

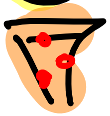
N groups 

$$\binom{K+N-1}{N-1}$$

Consistent in summary chart

More common:

K groups 

N items 

$$\binom{N+K-1}{K-1}$$

While we're making counting review materials:

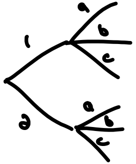
Counting Fundamentals:

- Sum Rule: If two sets, A and B, don't share any common items

$$|A \cup B| = |A| + |B|$$

- Product Rule: How many tuples can be made pulling first item from A and next from B?

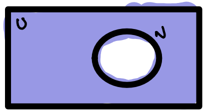
$$|A \times B| = |A| \times |B|$$



Counting moves:

- Count-by-partition: Partition items we want to count into subsets which are more easily counted

- Count-by-complement: Count items not-of-interest, subtract it from "everything"



$$|U - N| = |U| - |N|$$

- Count-by-simplification: Be on the lookout for simpler, equivilent problems

Counting advice:

1. Clearly document your thinking on the paper
   (you'll clarify your thinking and find errors)

2. If you're stuck:
   - head back to the materials of the past few slides
   - try solving a simpler "sub-problem", the experience may provide fresh insight
     - (often useful for count-by-partition)

In Class Activity

A E I O U Y

How many passwords of length 5 can be made from vowels (upper and lowercase)?

$12^5 \qquad 10^5$

How many ways can I select 10 students in this room to give a million extra credit points to?

$$\binom{250}{10} = \frac{250!}{240!\,10!} =$$

10 countries each have one woman swimming in the women's 200m freestyle. How many ways might the podium's nationality be arranged?
(e.g. in tokyo 2020 it was Australia, Hong Kong (China) & Canada)

$$P(10,3) = \frac{10!}{(10-3)!} = \frac{10\cdot 9\cdot 8\cdot \cancel{7\cdots}}{\cancel{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}} = 10\cdot 9\cdot 8$$

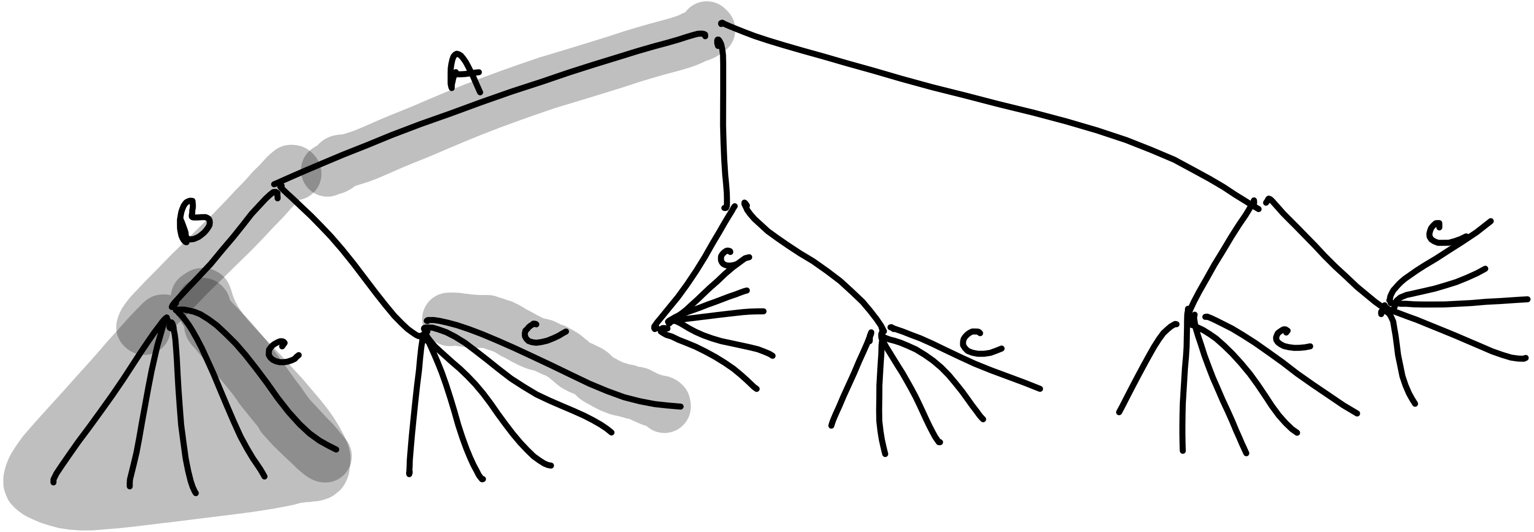
How many ways can we order 14 pizza for our TAs from a pizza place which serves 3 types of pizza (cheese, pepperoni, veggie)? Assume a whole pizza may only be of one type.

.

$$\underbrace{\circ\ \circ\ \circ}_{\text{CHEESE}} \;\Big|\; \underbrace{\circ\ \circ\ \circ}_{\text{PEPPER}} \;\Big|\; \underbrace{\circ\ \circ\ \circ\ \circ\ \circ\ \circ\ \circ\ \circ}_{\text{VEGGIE}}$$

$$\binom{14+2}{2} = \binom{16}{14}$$

I've got 3 pairs of pants, 2 shirts and 5 hats. How many outfits (pants, shirt & hat) can I wear if I won't wear one pair of pants with either 1 shirt or 1 hat?
(i.e. I can't wear pants A with shirt B and also I can't wear pants A with hat C)



24

I've got 3 pairs of pants, 2 shirts and 5 hats. How many outfits (pants, shirt & hat) can I wear if I won't wear one pair of pants with either 1 shirt or 1 hat?
(i.e. I can't wear pants A with shirt B and also I can't wear pants A with hat C)

$$3 \cdot 2 \cdot 5 - 1 \cdot 1 \cdot 5 - 1 \cdot 2 \cdot 1 + 1$$

$$30 - 5 - 2 + 1$$

$$24$$

How many ways can we order 14 pizza for our TAs from a pizza place which serves 3 types of pizza (cheese, pepperoni, veggie)? Assume a whole pizza may only be of one type.

(++) redo the pizza problem, relaxing our assumption that the whole pizza may only be of one type. Instead, assume each half of the pizza may only be of one type.

~~10~~ 5 countries each have ~~one~~ TWO ~~woman~~ women swimming in the women's 200m freestyle. How many ways might the podium's nationality be arranged?
(e.g. in tokyo 2020 it was Australia, Hong Kong (China) & Canada)

(++) redo the swimming problem, but assume that 5 countries each have 2 swimmers each

NO NATION REPEATED

$P(5,3)$

NATION REPEATED

$\underline{1}\ \underline{1}\ \underline{0}$ $\quad 5 \cdot 4$

$\underline{1}\ \underline{0}\ \underline{1}$ $\quad 5 \cdot 4$

$\underline{\ \ }\ \underline{1}\ \underline{1}$ $\quad 5 \cdot 4$

$P(5,3) + 5 \cdot 4 \cdot 3$

~~10~~ 5 countries each have ~~one~~ 3 ~~woman~~ women swimming in the women's 200m freestyle. How many ways might the podium's nationality be arranged?
(e.g. in tokyo 2020 it was Australia, Hong Kong (China) & Canada)

(++) redo the swimming problem, but assume that 5 countries each have 2 swimmers each

NO NATION REPEATED

$P(5,3)$

NATION REPEATED 2 TIMES

$\underline{1}\ \underline{1}\ \underline{0}$ $\quad 5 \cdot 4$

$\underline{1}\ \underline{0}\ \underline{1}$ $\quad 5 \cdot 4$

$\underline{\ \ }\ \underline{1}\ \underline{1}$ $\quad 5 \cdot 4$

3 TIMES

$\underline{1}\ \underline{1}\ \underline{1}$

5

$P(5,3) + 5 \cdot 4 \cdot 3 + 5$

_ _ _ 5^3

```
>>> from math import perm
>>> perm(5, 3) + 5*4*3 + 5
125
>>> 5 ** 3
125
>>> 5 * 5 * 5
125
>>>
```

(same as previous slide, but this is easier way of counting)