CS1800 Day 18

Admin:

- Exam2:
    - next friday Nov 17
    - review next week in recitation (no quiz)
    - you'll get practice problems this Friday Nov 10
- HW7 (induction) is also due next Friday Nov 17
- HW6 due this Friday Nov 10

Content:

Induction (proving a sequence of statements)

- Why prove something?
- Proving a conditional P -> Q
- Weak Induction
- Strong Induction

Why prove something?

In a town there is a barber who says to himself,

"I'm going to cut the hair of everyone who doesn't cut their own hair"

Does the barber cut their own hair?

Punchline:

- Its easy to get "turned around" in our thinking about a situation
- motivate: a proof is a rigorous justification of why something must be true

What kinds of things will I prove?

- Algorithm Correctness: Does Dijkstra's Algorithm really provide the shortest path?
- Algorithm Complexity: This algorithm will need n^2 operations to complete

Proving a conditional

Example: If you don't submit any HW, then you can't pass this course

ASSUME: STUDENT DOESN'T SUBMIT ANY HW

$\Rightarrow$ STUDENT SCORES 0 OF 50% OF TOTAL HW CREDIT

$\Rightarrow$ HIGHEST SCORE STUDENT GETS IS 50%

$\Rightarrow$ STUDENT FAILS

Proving a conditional

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Proving a conditional

$$P \longrightarrow Q$$

1. Assume P
2. Give sequence of implications which end at Q

$$P \; \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \; Q$$

Tip: Use P somewhere in your argument to get to Q
(Otherwise Q true by itself, if so its simpler to drop conditioning on P)

In Class Activity:

We say an integer z is even if there exists some integer a with z = 2a

Prove the following statement:

If an integer z is even, then z^2 is also even.

ASSUME INTEGER z IS EVEN

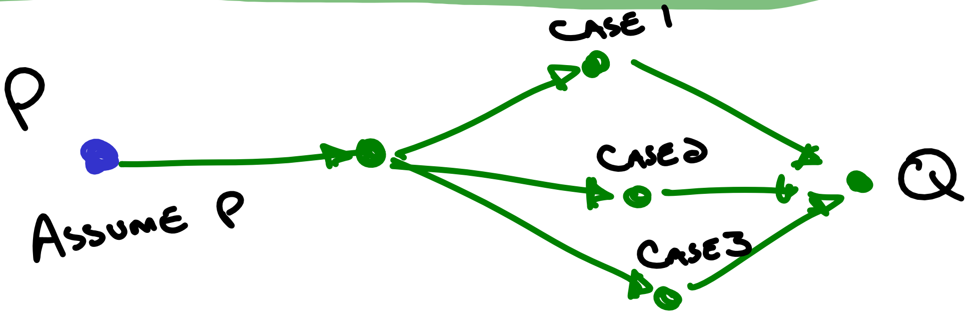
$\exists\, a \in \mathbb{Z}$ WITH $z = 2a$

$$z^2 = (2a)^2 = 4a^2 = 2(2a^2)$$

$\exists\, 2a^2 = a' \in \mathbb{Z}$ WITH $z^2 = 2a'$ SO z^2 IS EVEN

Note to self: Remind students that proofs, like other writing, have an audience. Keeping audience in mind helps them understand what they do and don't need to include.

Conditional Proof Move: Break Your Argument Into Cases



Approach:

Partition all possibilities into cases, argue each will imply Q

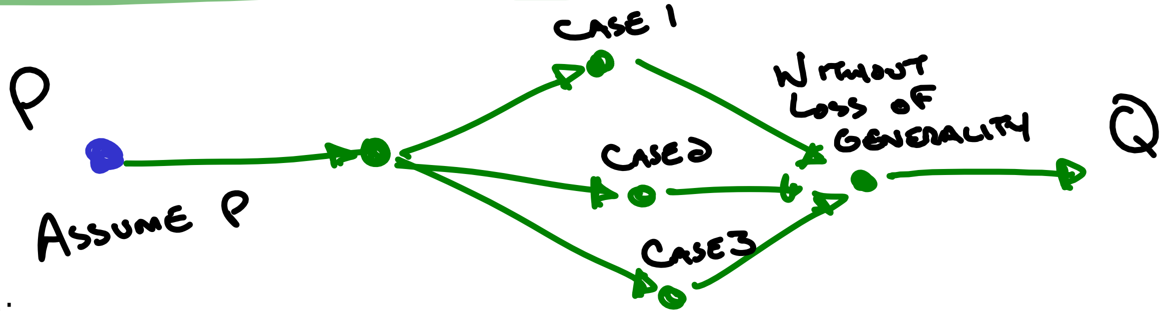
Example: If one drives over the speed limit only when no cops around, they shouldn't get a speeding ticket.

Proof: Assume one drives over the speed limit only when no cops around:

case1: no cops around $\longrightarrow$ it isn't possible to get ticket $\longrightarrow$ they shouldn't get a ticket

case2: one doesn't drive over speed limit. $\longrightarrow$ they shouldn't get a ticket

Proof Move: Without Loss of Generality (WLOG)



Approach:

Simplify your argument by combining your cases, often by re-labelling or re-orienting how you define things.

Example: If you cut a 100g wheel of cheese into 2 pieces, one side will be at least 50g

Proof: Assume we cut a 100g wheel of cheese into two pieces.

WLOG, let us call the mass of larger piece A and the smaller mass B (if equal, either can be A)

Then $100 = A + B \leq A + A = 2A$

so that $50 <= A$

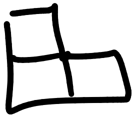
TILING A BATHROOM FLOOR

$n \in \mathbb{N}$
$n \neq 0$

GIVEN A BATHROOM FLOOR IS AN ARRAY OF $2^n \times 2^n$

MAY WE ALWAYS BE ABLE TO TILE ALL BUT ONE SPACE WITH AN "L SHAPE"?

→ NO MATTER WHERE SPOT IS

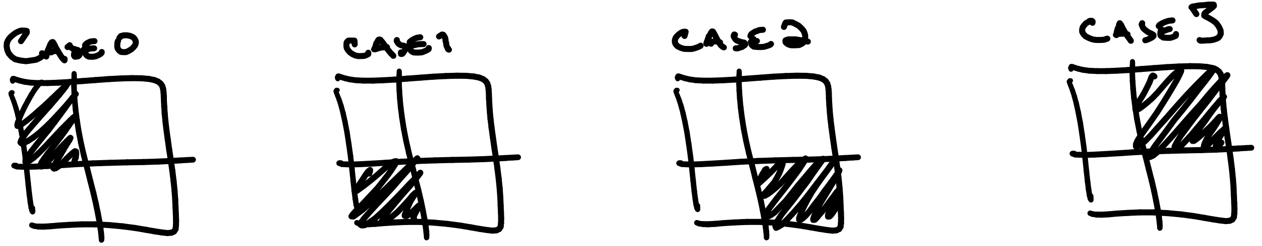


LET'S SOLVE SIMPLER PROBLEM...

STATEMENT ($n=1$)

A $2^1 \times 2^1$ ARRAY WITH ONE MISSING SPOT MAY BE TILED WITH L SHAPES

PROOF



IN ANY CASE, WE MAY TILE ARRAY

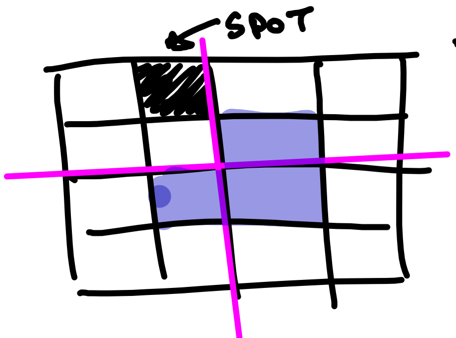
Generalizing from $n=1$ to $n=2$

Statement ($n=2$)

A $2^2 \times 2^2$ array with one missing spot may be tiled with L shapes

Proof

WLOG: Assume missing spot in top left quadrant (rotate array until this is true)



- Add L tile to induce "spot" in other quadrants
- $n=1$ case (previous slide) tells us each quadrant can be tiled

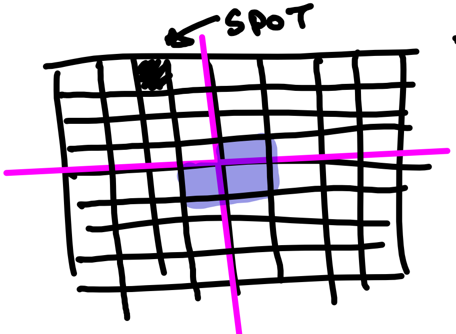
GENERALIZING FROM $n=2$ TO $n=3$

STATEMENT ($n=3$)

A $2^3 \times 2^3$ ARRAY WITH ONE MISSING SPOT MAY BE TILED WITH L SHAPES

PROOF

WLOG: ASSUME MISSING SPOT IN TOP LEFT QUADRANT (ROTATE ARRAY UNTIL THIS IS TRUE)



- ADD L TILE TO INDUCE "SPOT" IN OTHER QUADRANTS
- $n=2$ CASE (PREVIOUS SLIDE) TELLS US EACH QUADRANT CAN BE TILED

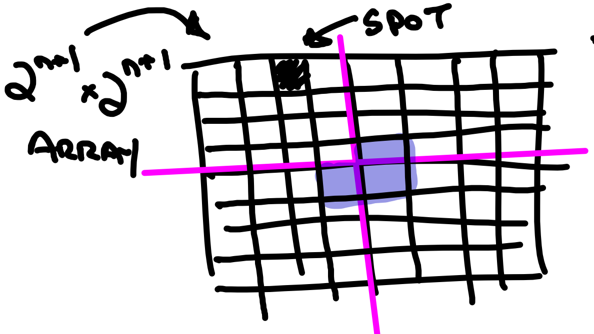
GENERALIZING FROM n TO $n+1$

STATEMENT $(n+1)$

A $2^{n+1} \times 2^{n+1}$ ARRAY WITH ONE MISSING SPOT MAY BE TILED WITH L SHAPES

PROOF

WLOG: ASSUME MISSING SPOT IN TOP LEFT QUADRANT (ROTATE ARRAY UNTIL THIS IS TRUE)



- ADD L TILE TO INDUCE "SPOT" IN OTHER QUADRANTS
- n CASE (PREVIOUS SLIDE) TELLS US EACH QUADRANT CAN BE TILED

Induction (Weak):

Induction allows us to prove a never-ending sequence of statements: S(1), S(2), S(3), S(4), ...

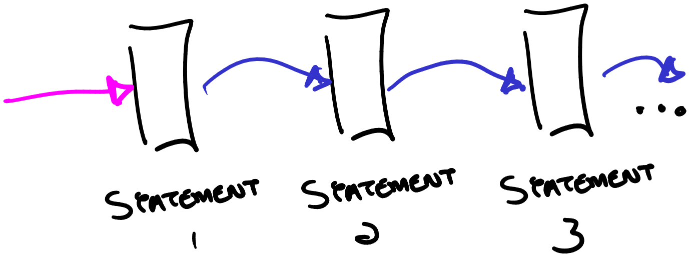
Process:

- Prove the first statement, S(n) for some n
- Show that each statement implies the next statement:

Metaphor (Dominos):

To knock over all the dominos

- Push over the first one
- Place each other domino so that if the one behind it falls, it too will fall



Induction Four Step Recipe: (AKA: how to not get turned around in a big induction proof)

1. Write out the statement for general n

2. Specify the list of statements you're proving
   - your base case need not be n=1
   - you're welcome to skip by more than 1 if you'd like S(3), S(5), S(7), S(9), ...

3. Prove the "Base case" (the smallest n for which your statement is true)

4. Prove the conditional: "If S(n) then S(n+1)"

In Class Activity

Using induction, show that if a set has N items, then the numbers of subsets which can be formed from that set is 2^N.

$$A = \{1, 4\} \qquad |A| = 2$$

$$|P(A)| = 4 = 2^2$$

$$P(A) = \{\{\}, \{1\}, \{4\}, \{1, 4\}\}$$

In Class Activity

Using induction, show that if a set has N items, then the numbers of subsets which can be formed from that set is 2^N.

step1: Statment for N:
S(N) = If a set has N items, then the number of subsets which can be formed from that set is 2^N

step 2:
S(-1) = If a set has -1 items, then the number of subsets which can be formed from that set is 2^-1
S(0) = If a set has 0 items, then the number of subsets which can be formed from that set is 2^0

$A = \{\}$ $\quad P(A) = \{\{\}\}$ $\quad |P(A)| = 1$

S(1) = If a set has 1 items, then the number of subsets which can be formed from that set is 2^1
S(2) = If a set has 2 items, then the number of subsets which can be formed from that set is 2^2

I'd like to prove this statement is true for S(0), S(1), S(2), ...

S(N) = If a set has N items, then the number of subsets which can be formed from that set is 2^N

Base Case (N=0):

Assume a set A has 0 items in it. We may only form 1 subset from items in A, the empty set. So it has 2^0 = 1 subset possible.

Inductive Step: Goal If S(N) then S(N+1)

Assume S(N), if a set has N items, then the number of subsets which can be formed from that set is 2^N.

Some other set B has N + 1 items. Pick any item in B, call it x. We can form 2^N subsets of B which don't have an x in them (by assumption above). Adding an x to each yields another unique 2^N subsets. There are 2^N + 2^N = 2^{N+1} subsets of B.

Induction (Strong):

Induction allows us to prove a never-ending sequence of statements: S(1), S(2), S(3), S(4), ...

Process:

- Prove the first statement, S(n) for some n
- Show that S(1), S(2), ... S(n) implies S(n+1)

Metaphor (Dominos):

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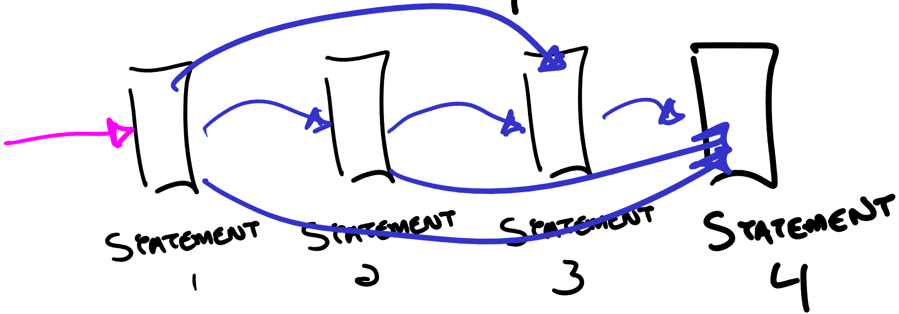
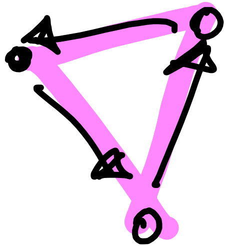
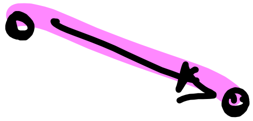
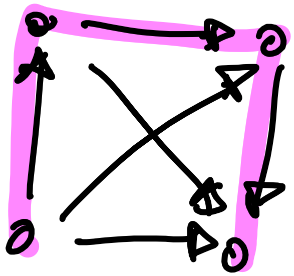


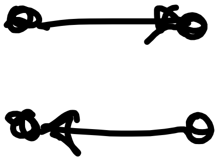
Strong Induction Example: (Tournament contains a Hamiltonian Path)

Given N cities with a one way road between every pair of cities, there is a path (with direction of each edge) which visits all cities exactly once.









Proof by induction:

step 1: define statement n:

Given N cities with a one way road between every pair of cities, there is a path (with direction of each edge) which visits all cities exactly once.

step 2: specify the list of statements we're proving

- It doesn't quite make sense to talk about any N less than 2 (a path between 0 or 1 cities?)
- Anecdotally, we saw this works for N=2

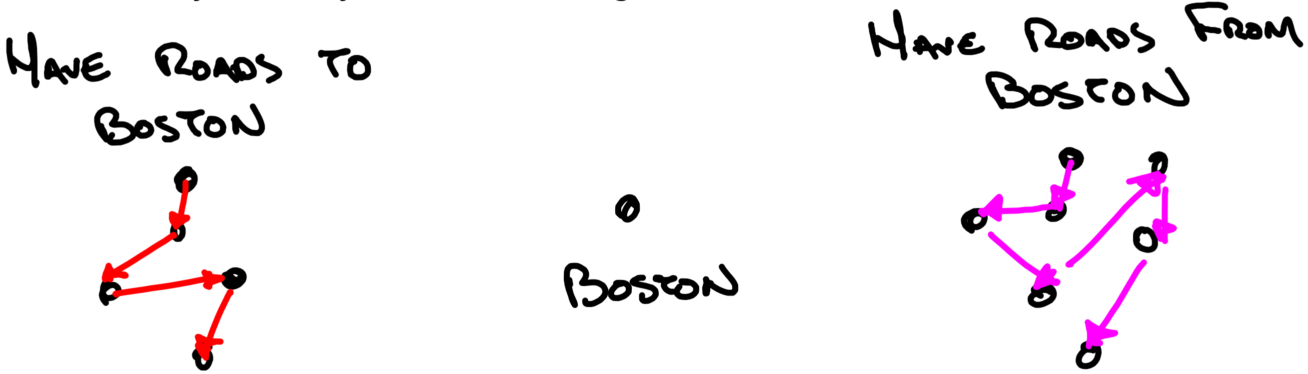
step 3: Show base case

Given two cities with one way roads between every pair of cities, then the cities must look like one of the following graphs:

In either case, there is clearly a path, with the direction of each edge, which visits all the cities once step 4 (strong induction version): Show that $\underline{S(2) \wedge S(3) \wedge \ldots \wedge S(N)} \longrightarrow S(N+1)$

Assume that any 2, 3, 4, ..., N cities (with directed edge between every pair) has a path which visits all cities

Given N+1 cities (with directed edge between every pair) we may select any city (lets call it "Boston" so that we can identify it clearly). The remaining N cities either have a road to or from Boston:



case0: there are between 2 and N cities in subgraph to boston (left) or from boston (right).
By assumption, there is a path for each (red and pink) which visits each city within the group.
To form the desired path, just glue together the red path, the edge from the end of the red path to Boston, the edge from Boston to the start of the pink path, and the pink path itself.

case1: if either group (to Boston or from Boston) has only 1 city, then the argument of case0 is still valid, but there is no "red" or "pink" path, but a single city

case2: if either group (to Boston or from Boston) has no cities, then the argument of case0 is still valid, but we may tag Boston onto the end of the red path (if from Boston is empty) or the beggining of the pink path (if to Boston is empty)

When should I use weak vs strong induction?

Both are always available to you, you may find one method produces a simpler proof (usually weak induction, where it works).