CS1800

11/10 - Fri :)

Admin

- Hw6 due today 11:59
- Hw7 out, due 11/17 (78 pts)
- Exam #2

  11/17

  9Am-6pm

  2 hour window

  remote

  review in the recitation next week

Agenda

1. Induction overview
2. Example - mathy (inequality)
3. Examples - structural (graphs, sets)

1. Induction overview

- our "main" proof technique in CS1800
- used all the time (algo, s/w engineering)

When to use...

Predicate $P(n)$

we could:

$P(1)$ $P(4)$

$P(2)$ $P(5)$ ∞

$P(3)$ $P(6)$ ....

Induction Shortcut:

- Base case $P(1)$ → specific value
  ↳ smallest value we care about
- Inductive step $P(k) \Rightarrow P(k+1)$

  IH $P(k)$

  use to show $P(k+1)$

  k is arbitrary

A couple mathy reminders:

- $(a+b)^2 = a^2 + 2ba + b^2$
- $(a+1)^2 = a^2 + 2a + 1$
- $a^{-b} = \frac{1}{a^b}$

- $a^b \cdot a^c = a^{b+c}$
- $(a^b)^c = a^{bc}$
- $n! = (n)(n-1)(n-2)\dots(1)$ by def
- $\sum_{i=1}^{n} i = 1+2+3+\dots+n$ by def

$\boxed{\text{2. Induction Example - Inequality}}$

- growth of functions (later in semester)
- ex: $3^n < n!$

3^n grows more slowly than $n!$

Pred: $P(n)$ $3^n < n!$

Logic: $\forall n \in \mathbb{Z}$ $n \geq 7 \Rightarrow P(n)$

Base case: $P(7)$

$3^7 < 7!$

$2187 < 5040$ ✓ done!

Inductive Step

$P(k) \Rightarrow P(k+1)$

Goal:

$P(k+1)$ $3^{k+1} < (k+1)!$

$P(k)$ $3^k < k!$ IH

- one step at a time!
- only do true things

$$3^{k+1} = 3^k \cdot 3^1$$

$$< (k!) \cdot 3 \quad \text{by IH}$$

$$< (k!)(k+1)$$

↳ bigger than 3

$$= (k+1)!$$

done!

2. Induction Examples – graphs, sets

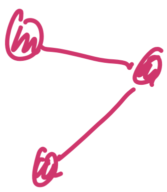
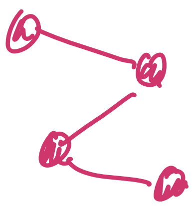
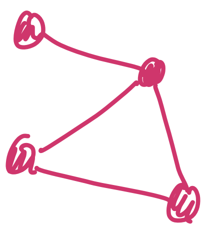
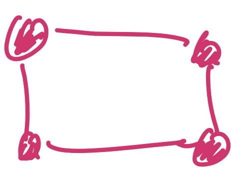
- be clear about what induction is on
  - # edges
  - # vertices
  - # elements in set
  - ...

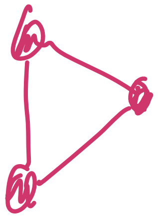
P(n)

is # you chose

- Inductive step: better to start at k+1 and remove something

ex graph example (simple)

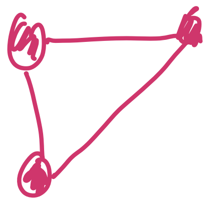
	# vertices	# edges	total degree
	2	1	2
	3	2	4
	4	3	6
	4	4	8
	4	4	8



3 3 6

=

2x # edges



add an edge, add 2 more total degrees

Induction on # edges

$P(n)$ a graph with n edges has total degree $2n$

$\forall n \in \mathbb{Z}^+ \; P(n)$

Base Case: $P(1)$

graph w/ one edge
by def, each vertex has degree one
total degree = 2 ✓

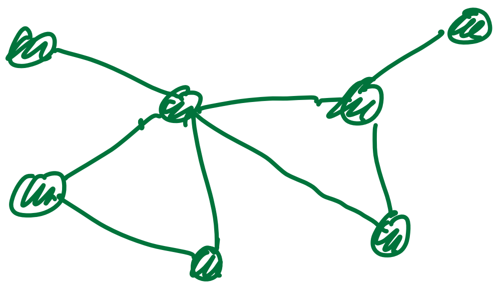
Inductive Step: $P(k) \Rightarrow P(k+1)$

P(k): a graph with k edges has degree 2k

IH

Goal: P(k+1)
graph w/ k+1 edges has degree 2(k+1)

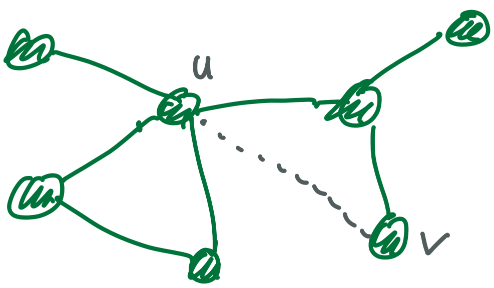
Graph with k+1 edges



(not real)

total degree: unknown

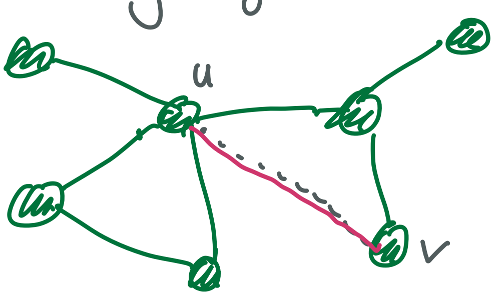
Remove an edge



Graph with k edges (not real)

- has total degree $\boxed{2k}$
- removed edge (u, v)

Add missing edge back in



- added edge (u, v)
- Graph with k+1 edges
- added one degree for u
- added one degree for v

total degree = 2k + 2

$\boxed{= 2(k+1)}$

done!

ex Induction on a set

- Subsets of size 2

$A = \{a, b\}$

Subsets of A w/ cardinality 2: $\{a,b\}$ 1

$A = \{a, b, c\}$

Subsets of A w/ cardinality 2: $\{a,b\}, \{a,c\}, \{b,c\}$ 3

$A = \{a, b, c, d\}$

Subsets of A w/ cardinality 2: $\{a,b\}, \{a,c\}, \{b,c\}, \{a,d\}, \{b,d\}, \{c,d\}$ 6

$\lvert A\rvert$	#subsets
2	1
3	3
4	6

$P(2)$ $\frac{2\cdot 1}{2} = 1$

$P(3)$ $\frac{3\cdot 2}{2} = 3$

$P(4)$ $\frac{4\cdot 3}{2} = 6$

$P(n)$ Set with cardinality n has $\boxed{\frac{(n)(n-1)}{2}}$ subsets of size 2

$\forall n \in \mathbb{Z}\ n \geq 2 \Rightarrow P(n)$

Base case $P(2)$

defn ✓

$A = \{x_1, x_2\}$ has exactly $\boxed{\text{one}}$ subset of size 2, A

formula ✓

$\frac{(2)(1)}{2} = 1$

Inductive Step $P(k) \Rightarrow P(k+1)$

IH

$P(k)$

$|A| = k$, then # subsets of size 2 is $\frac{(k)(k-1)}{2}$

Goal $P(k+1)$

$|A| = k+1$, then $\frac{(k+1)(k)}{2}$

Let A be a set with $|A| = k+1$

$\{a_1, a_2, a_3, \ldots, a_{k+1}\}$

- remove one element from A
- $A' = A - \{a_1\}$

  $= \{a_2, a_3, a_4 \ldots, a_{k+1}\}$

$|A'| = k$

- # subsets of size 2 in A' is $\frac{(k)(k-1)}{2}$
- add a_1 back in

now our set has cardinality $k+1$

- a_1 gets paired with every existing element to create a subset of size 2

total # subsets of size 2:

$$\frac{(k)(k-1)}{2} + k$$

IH

everything z_4 gets paired with

old subsets + new subsets

$$\frac{(k)(k-1)}{2} + k = \frac{(k)(k-1)}{2} + \frac{2k}{2}$$

$$= \frac{(k)(k-1) + 2k}{2}$$

$$= \frac{k^2 - k + 2k}{2}$$

$$= \frac{k^2 + k}{2}$$

$$= \frac{(k+1)(k)}{2}$$

done!