CS1800 Day 19

Admin:
- HW6 due today
- HW7 released today (due next Friday)
  - slightly shorter than most
    - more time to prep for exam2
    - will only count as 78% of other HWs with 100 points
- practice exam2 problems are out
  - slightly more induction examples
    - I want to make sure you have plenty to pull from in exam2

Content:
- Induction with equalities and inequalities

Induction (Weak):

Induction allows us to prove a never-ending sequence of statements: S(1), S(2), S(3), S(4), ...

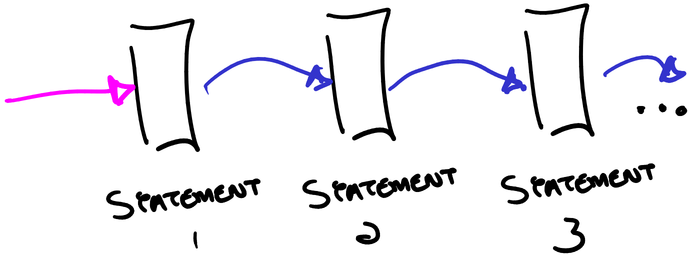
Process:

- Prove the first statement, S(n) for some n
- Show that each statement implies the next statement:

Metaphor (Dominos):

To knock over all the dominos

- Push over the first one
- Place each other domino so that if the one behind it falls, it too will fall



EXAMPLE

Summation Notation

$$1+2+4+8+16+32+64$$

$$=2^0+2^1+2^2+2^3+2^4+2^5+2^6$$

LAST VALUE OF K

$$=\sum_{k=0}^{6} 2^k$$

"The sum of 2^k where k goes from 0 to 6"

STARTING VALUE OF K

In Class Activity: Summation Notation

Express each sum below in summation notation

$9 + 11 + 13 + 15 + 17$

$9+2\cdot 0$ $\quad 9+2\cdot 2$ $\quad 9+2\cdot 4$

$9+2\cdot 1$ $\quad 9+2\cdot 3$

$$\sum_{k=0}^{4} 9+2k$$

$1 + 2 + 3 + 4 + 5 + \ldots + n$

$$\sum_{k=1}^{n} k$$

Compute each sum below (the second one has a pattern and simplifies)

$$\sum_{k=10}^{12} 2k$$

$= 2\cdot 10 + 2\cdot 11 + 2\cdot 12$

$$\sum_{k=0}^{101} (-1)^k = 0$$

$-1^0 + -1^1 + -1^2 + -1^3 + \ldots$

$1 + -1 + 1 + -1 + 1 + -1 + \ldots$

Algebraic Induction:

Show that the sum of the first n even integers is n(n+1)

$n=1$

$$2 = 1(1+1)$$

$n=2$

$$2+4 = 2(2+1)$$

$n=3$

$$2+4+6 = 3(3+1)$$

Algebraic Induction: Expressing our sum in summation notation

Show that the sum of the first n even integers is n(n+1)

$$2+4+6+8+\dots+2n$$

$$= 2\cdot 1+2\cdot 2+2\cdot 3+2\cdot 4+\dots+2n$$

STATEMENT n: " $\sum_{k=1}^{n} 2k = n(n+1)$ "

Induction Four Step Recipe: (AKA: how to not get turned around in a big induction proof)

1. Write out the statement for general n → PREP WORK

2. Specify the list of statements you're proving ↗ (NOT NEEDED IN FINAL PROOF)
   - your base case need not be n=1
   - you're welcome to skip by more than 1 if you'd like S(3), S(5), S(7), S(9), ...

3. Prove the "Base case" (the smallest n for which your statement is true)

4. Prove the conditional: "If S(n) then S(n+1)"

✓ — EXPRESS $S(n+1)$ IN MARGIN

✓ — WORK FROM SUM SIDE $\sum_{k=0}$

→ REMOVE LAST ITEM FROM SUM

→ APPLY $S(n)$

Algebraic Induction:

Show that the sum of the first n even integers is n(n+1)

STATEMENT n: "$\sum_{k=1}^{n} 2k = n(n+1)$"

BASE CASE ($n=1$)

$$\sum_{k=1}^{1} 2k = 2\cdot 1 = 2 = 1(1+1) = n(n+1)$$

$S(n) \to S(n+1)$:

ASSUME STATEMENT n IS TRUE

"$S(n) = \sum_{k=1}^{n} 2k = n(n+1)$"

"$S(n+1) = \sum_{k=1}^{n+1} 2k = (n+1)(n+2)$"

$$\sum_{k=1}^{n+1} 2k = 2(n+1) + \sum_{k=1}^{n} 2k$$

$$= 2(n+1) + n(n+1)$$

$$= (n+1)(n+2)$$

NOTE TO SELF

- EXPRESS $S(n+1)$
- NEXT SLIDE

SUMMATION NOTATION IN INDUCTION PROBLEMS

$$\sum_{k=1}^{n+1} 2k = 2\cdot 1 + 2\cdot 2 + 2\cdot 3 + \ldots + 2\cdot n + 2\cdot(n+1)$$

$$= 2\cdot(n+1) + \sum_{k=1}^{n} 2k$$

Induction proofs will ask us to prove $S(n) \longrightarrow S(n+1)$.

For sums, its helpful to be able to "pull out" the sum relevant in $S(n)$ from the sum relevant for $S(n+1)$ in this way.

In Class Activity:

Show that the sum of the first n powers of 2 is equal to $2^{n+1}-1$

$n=0$	$n=1$	$n=2$
$2^0 = 2^{0+1}-1$	$2^0+2^1=2^{1+1}-1$	$2^0+2^1+2^2=2^{2+1}-1$
$1 = 2-1$	$1+2=4-1$	$1+2+4=8-1$
$1=1$	$3=3$	$7=7$

$$2^0+2^1+2^2+2^3+\dots+2^n = \sum_{k=0}^{n} 2^k$$

$$S(n) = \text{“} \sum_{k=0}^{n} 2^k = 2^{n+1} - 1 \text{”}$$

$$S(n+1) = \sum_{k=0}^{n+1} 2^k = 2^{n+2} - 1$$

BASE CASE $(n=0)$

$$\sum_{k=0}^{0} 2^k = 2^0 = 1 = 2^{0+1} - 1$$

$$= 2 - 1$$

$$= 2^{n+1} - 1$$

INDUCTIVE STEP $S(n) \to S(n+1)$

Assume $S(n)$

$$\sum_{k=0}^{n+1} 2^k = 2^{n+1} + \sum_{k=0}^{n} 2^k$$

$$= 2^{n+1} + 2^{n+1} - 1$$

$$= 2^{n+2} - 1$$

Induction with inequalities: why? (preview a bit ahead, not necessary for exam2)

Suppose two algorithms both accomplish the same task but take a different number of "computes" to do so. For a list of size N

Base Case

1st Time $2^N < N!$

Algorithm 1 takes: 2^N computes
Algorithm 2 takes: N! computes

List Size	1	2	3	4	5	6	7
Algorithm 1 Computes	2	4	8	16	32	64	128
Algorithm 2 Computes	1	2	6	24	120	720	5040

Goal: We want to show that algorithm 1 is faster for all lists sufficiently large (N is greater than some threshold)

Induction with inequalities:

Prove that 2^N < N! for all N above some threshold.

$S(n) =$ " $2^N < N!$ "

$S(n+1) = 2^{N+1} < (N+1)!$

BASE CASE $(n=4)$

$2^N = 2^4 = 16 < 24 = 4!$

$= N!$

INDUCTIVE STEP $S(n) \rightarrow S(n+1)$

ASSUME $2^N < N!$

$$(N+1)! = (N+1) \cdot N!$$
$$> (N+1)\, 2^N$$
$$> 2 \cdot 2^N$$
$$= 2^{N+1}$$

Algebra: Working with inequalities (1 of 3)

Move 1: add the same things to both sides, it preserves the inequality

IF $3 < 4$ THEN $3+10 < 4+10$

$$x < y \Rightarrow x+c < y+c \quad \forall c \in \mathbb{R}$$

Algebra: Working with inequalities (2 of 3)

Move 2: multiply by a positive value, it preserves the inequality

IF $3 < 4$ THEN $3 \cdot 10 < 4 \cdot 10$

IF $x < y$ THEN $xc < yc$ $\forall c \in \mathbb{R}$ WITH $c > 0$

Move 3: multiply by a negative value, it swaps the inequality

IF $3 < 4$ THEN $3 \cdot -1 > 4 \cdot -1$

IF $x < y$ THEN $xc > yc$ $\forall c \in \mathbb{R}$ WITH $c < 0$

Algebra: Working with inequalities (3 of 3)

Move 4: sum two inequalities (large side together & small side together)

$3 < 4$

IF AND THEN $3+5 < 4+6$

$5 < 6$

IF $x < y$ AND $w < z$ THEN $x + w < y + z$

In Class Activity:

Show that n^2 > n + 10 for all n above some threshold

Base Case $n=4$

n	1	2	3	4	5	6
n^2	1	4	9	16	25	36
$n+10$	11	12	13	14	15	16
$n^2 > n+10$	F	F	F	T	T	T

In Class Activity:

Show that n^2 > n + 10 for all n above some threshold

STATE n+1 = " $(n+1)^2 > n+11$ "

STATEMENT n = " $n^2 > n+10$ "

BASE CASE n=4

$n^2 = 4^2 = 16 > 14 = 4+10 = n+10$

INDUCTIVE STEP $S(n) \to S(n+1)$

ASSUME $S(n)$

$$(n+1)^2 = n^2 + 2n + 1$$
$$> n + 10 + 2n + 1$$
$$> n + 11$$

$$-n^2 - n + 10 < 0$$

$$10 < n^2 + n$$

$$10 < n(n+1)$$

n is 4 @ SMALLEST