Day 10:

Admin:
- practice exam on gradescope
- review exam instructions
- hw4 note:
    - please compute (and round) final value in counting problems (as HW instructions indicate)
- hw4 dates:
    - due Friday @ 11:59 PM
    - late due date is Saturday @ 11:59 PM
    - solutions are available Sunday @ 12:10 AM

Content:
- combinations
- leftover principle
- counting partitions of identical objects

$$P(5,3) = \frac{5!}{(5-3)!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{2\cdot1} = 60$$

Over-counting (multiplicative)

How many people are in the room if ...

... there are 100 eyes in the room 50

... there are 90 fingers in the room 9

... there are 400 limbs (legs & arms) in the room 100

Punchline:

If there are n items (eyes, fingers, limbs)
and c items per every item-of-interest (people)

then there are n / c items of interest

Ordering: when does it matter?

Order matters:

How many ways can a student take 3 CS courses from 10 unique courses?

(CS 1800, DS 2000, DS 2500)

≠

(DS 2500, DS 2000, CS 1800)

TUPLE

Order doesn't matter:

How many ways can one take 3 candies from 10 unique candies?

{CHOC, LOLLY, GUMMY}

=

{LOLLY, GUMMY, CHOC}

SET

Combination: (intro example)

How many ways can one choose 2 candies from 3 unique candies? $C = \{1, 2, 3\}$
(order doesn't matter)

3 WAYS:

$\{1, 2\}$ $\{1, 3\}$ $\{2, 3\}$

Combination: (intro example)

How many ways can one choose 2 candies from 3 unique candies? (order doesn't matter)

$C = \{1, 2, 3\}$

THERE ARE $P(3,2) = \frac{3!}{1!} = 6$ WAYS OF CHOOSING

TWO ORDERED CANDIES:

$(1,2)$ $(1,3)$ $(2,3)$

$(2,1)$ $(3,1)$ $(3,2)$

Combination: (intro example)

How many ways can one choose 2 candies from 3 unique candies? C = {1, 2, 3}
(order doesn't matter)

THERE ARE $P(3,2) = \frac{3!}{1!} = 6$ WAYS OF CHOOSING TWO ORDERED CANDIES:

THERE ARE $2! = 2$ WAYS OF ORDERING 2 CANDIES

(1, 2) (1, 3) (2, 3)

(2, 1) (3, 1) (3, 2)

OVERCOUNTING (MULTIPLICATION) : WAYS OF CHOOSING 2 FROM 3 (ORDER NOT MATTER) = WAYS OF ORDERING 2 FROM 3 (ORDER MATTERS) / WAYS OF ORDERING 2 (ORDER MATTERS)

Combination: definition & formula

- A combination is a subset of objects (order doesn't matter)
  (how many ways can I choose k items from n possible)

- A permutation is an ordering of objects (order matters)
  (how many ways can I order k items from n possible)

$$C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!\,k!}$$

"n choose k"

$\binom{n}{k}$ AKA BINOMIAL COEFFICIENT

In Class Activity

$\underset{1}{\underline{8}}\ \underset{2}{\underline{8}}\ \underset{3}{\underline{8}}$ 8^3

How many ways can the 8 Mario Kart racers form the final podium of 3 winners. The order of the podium matters.

NO REPEAT

ORDER MATTERS

$$P(8,3) = \frac{8!}{(8-3)!} = \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1} = 8\cdot7\cdot6$$

How many ways can the teams (mercedes, ferrari, etc) arrange on the podium of 3 winners in a formula 1 race? (assume that each of the 10 teams has at least 3 cars in the race).
An example podium: 1st place: Mercedes, 2nd place: Mercedes, 3rd place: Ferrari

MERC MERC MERC

$\underset{10}{\underline{\ \ }}\ \underset{10}{\underline{\ \ }}\ \underset{10}{\underline{\ \ }}$

10^3

REPEAT

ORDER MATTER

How many unique 5 card hands exist in a deck of 52 unique cards? ("hands" are unordered)

NO REPEAT
ORDER NOT
MATTER

$$\binom{52}{5} = \frac{52!}{5!\,(52-5)!}$$

How many ways can one select the "remaining" 47 cards after selecting a 5 card hand (as in the problem above)?

NO REPEAT
ORDER NOT
MATTER

$$\binom{52}{47} = \frac{52!}{(52-47)!\,47!} = \frac{52!}{5!\,(52-5)!}$$

Combinations: Leftover principle

How many ways can I choose all but 10 student to take out for ice cream from this class of size n?

$$\binom{250}{10} = \binom{250}{240}$$

How many ways can I choose n - 10 students to take out for ice cream from this class of size n?

•

$$\binom{n}{k} = \binom{n}{n-k}$$

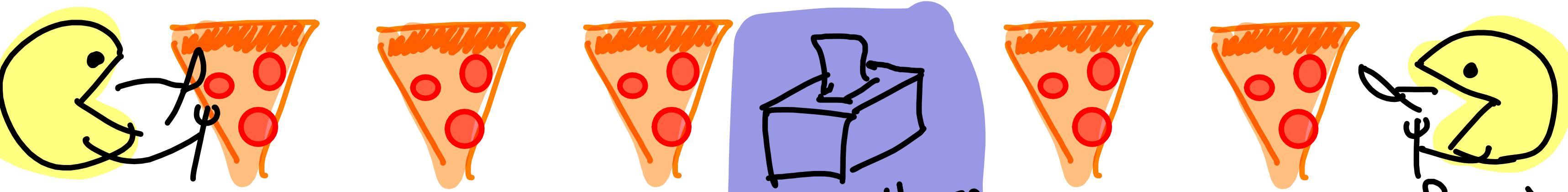
Counting: Putting it together (almost … see later slide for complete version of this table)

HOW TO SELECT K ITEMS FROM N

	NO REPEAT SELECTIONS	REPEAT SELECTIONS
ORDER MATTERS	PERMUTATIONS $P(N,K) = \frac{N!}{(N-K)!}$	PRODUCT RULE N^K
ORDER DOESN'T MATTER	COMBINATIONS $\binom{N}{K} = \frac{N!}{(N-K)!\,K!}$	MYSTERY (FOR NOW)

Counting Partitions of identical objects: (AKA Balls in bins or Stars & bars):

How many different ways can two people split k slices of pizza?



NAPKIN HOLDER

PERSON 1
EATS ALL PIZZA BEFORE BOUNDARY

PERSON 2
EATS ALL PIZZA AFTER BOUNDARY

K SLICES PIZZA
1 BOUNDARY

THERE ARE $\binom{K+1}{1}$ WAYS

Counting Partitions of identical objects: (AKA Balls in bins or Stars & bars):

How many different ways can ~~two~~ THREE people split K slices of pizza?

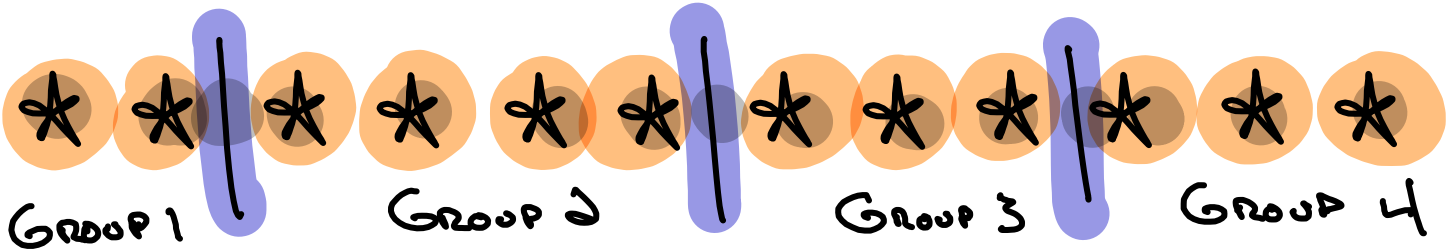
EATS ALL PIZZA BETWEEN BOUNDARY

$$\binom{K+2}{2} \text{ WAYS}$$

Counting Partitions of identical objects: (AKA Balls in bins or Stars & bars):

How many different ways can [redacted]e split k s[redacted]?

N GROUPS STARS



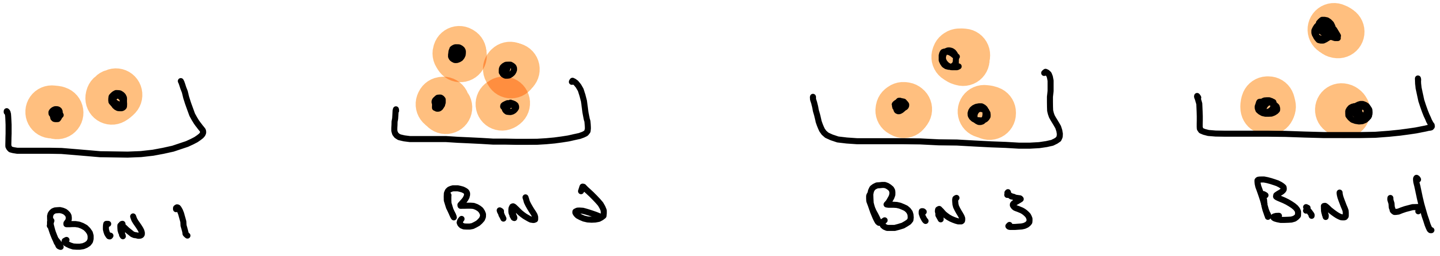
$$\binom{K+N-1}{N-1}$$ WAYS

NEED N-1 BOUNDARIES FOR N GROUPS

Counting Partitions of identical objects: (AKA Balls in bins or Stars & bars):

How many different ways can ~~the people~~ split k ~~slices of pizza~~?

N BINS BALLS



$$\binom{K+N-1}{N-1} \text{ WAYS}$$

Something is still missing in our chart

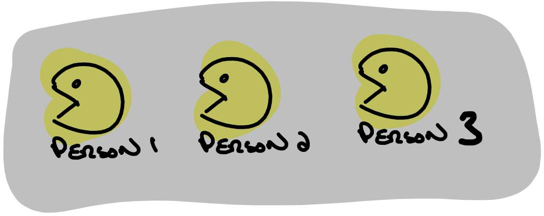
HOW TO SELECT K ITEMS FROM N

	NO REPEAT SELECTIONS	REPEAT SELECTIONS
ORDER MATTERS	PERMUTATIONS $P(N,K)=\frac{N!}{(N-K)!}$	PRODUCT RULE N^K
ORDER DOESN'T MATTER	COMBINATIONS $\binom{N}{K}=\frac{N!}{(N-K)!K!}$	MYSTERY (FOR NOW)

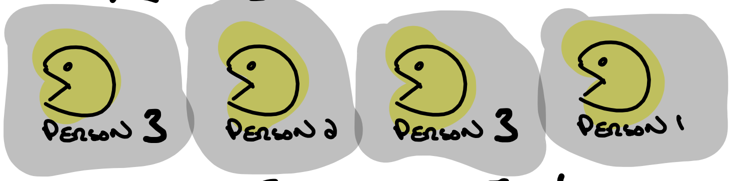
How is the balls-in-bins fit into bottom right box of "putting it together"?

Selecting k items from N items
- repeat selections allowed
- order of selections doesn't matter

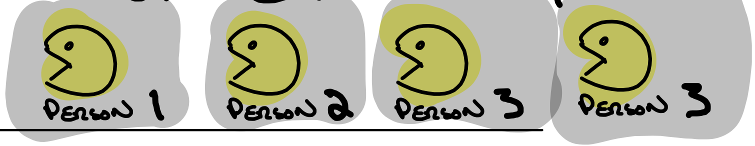
N ITEMS:



K SELECTIONS



OR EQUIVALENTLY



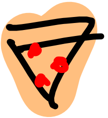
EQUIVALENTLY

How to select k items from N

	No repeat selections	Repeat selections
Order matters	**Permutations** $P(N,k) = \frac{N!}{(N-k)!}$ How many tuples of length k can one make from N items? (no repeats)	**Product rule** N^k How many tuples of length k can one make from N items? (repeats)
Order doesn't matter	**Combinations** $\binom{N}{k} = \frac{N!}{(N-k)!\,k!}$ How many sets with k unique items can one make from N items? (no repeats)	**Partition of identical items (stars + bars / balls in bins)** $\binom{k+N-1}{N-1}$ How many ways can we split k identical items among N groups?

⚠ Two conventions for stars and bars

In class now:

K items 

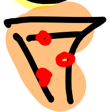
N groups 

$$\binom{K+N-1}{N-1}$$

Consistent in summary chart

More common:

K groups 

N items 

$$\binom{N+K-1}{K-1}$$

While we're making counting review materials:

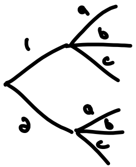
Counting Fundamentals:

- Sum Rule: If two sets, A and B, don't share any common items

$$|A \cup B| = |A| + |B|$$

- Product Rule: How many tuples can be made pulling first item from A and next from B?

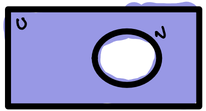
$$|A \times B| = |A| \times |B|$$



Counting moves:

- Count-by-partition: Partition items we want to count into subsets which are more easily counted

- Count-by-complement: Count items not-of-interest, subtract it from "everything"



$$|U - N| = |U| - |N|$$

- Count-by-simplification: Be on the lookout for simpler, equivilent problems

Counting advice:

1. Clearly document your thinking on the paper
   (you'll clarify your thinking and find errors)

2. If you're stuck:
   - head back to the materials of the past few slides
   - try solving a simpler "sub-problem", the experience may provide fresh insight
     - (often useful for count-by-partition)

In Class Activity

5 10

How many passwords of length 5 can be made from vowels (upper and lowercase)? let's say y is not a vowel

ORDER MATTERS

REPEAT ALLOWED

$\underline{10}\ \underline{10}\ \underline{10}\ \underline{\ \ }\ \underline{\ \ }$

10^5

How many ways can I select 10 students in this room to give a million extra credit points to?
(assume: 250 students in room)

ORDER NOT MATTER

REPEATS NOT ALLOWED

$$\binom{250}{10} = \frac{250 \cdot 249 \cdot \ldots \cdot 241}{10!}$$

10 countries each have one woman swimming in the women's 200m freestyle. How many ways might the podium's nationality be arranged?
(e.g. in tokyo 2020 it was Australia, Hong Kong (China) & Canada)

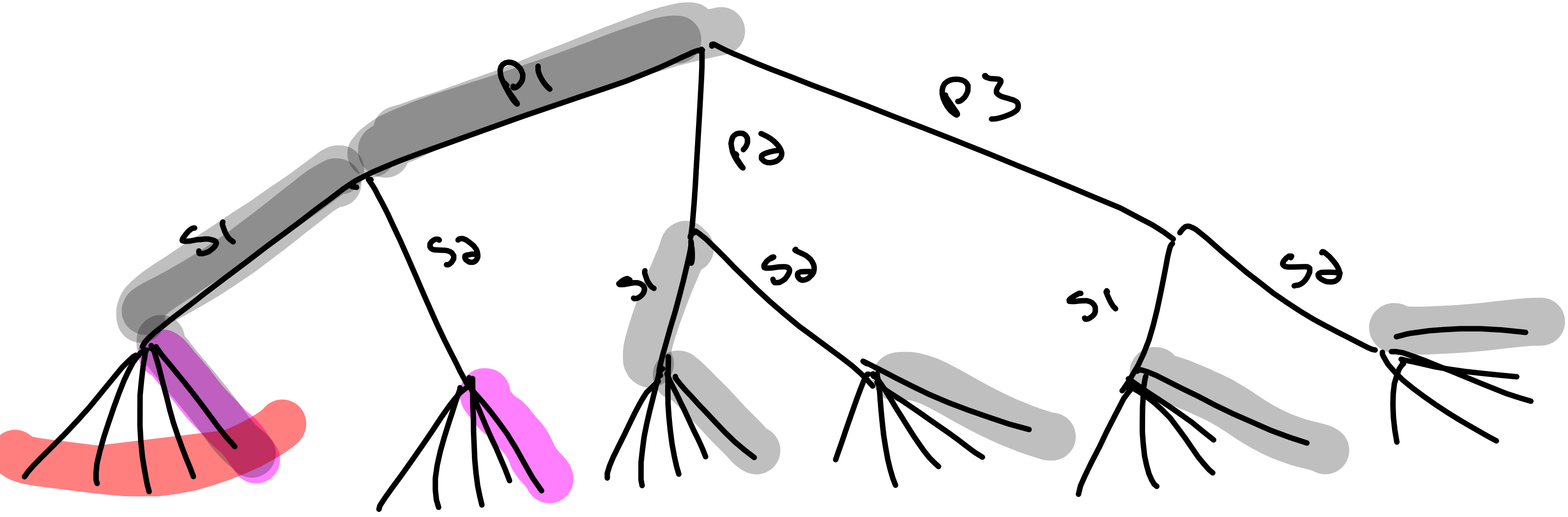
ORDER MATTER
NO REPEATS

$$P(10,3) = 10 \cdot 9 \cdot 8$$

How many ways can we order 14 pizza for our TAs from a pizza place which serves 3 types of pizza (cheese, pepperoni, veggie)? Assume a whole pizza may only be of one type.

$$\binom{14+2}{2}$$

I've got 3 pairs of pants, 2 shirts and 5 hats. How many outfits (pants, shirt & hat) can I wear if I won't wear one pair of pants with either 1 shirt or 1 hat?



$4 + 5 \cdot 4 = 24$

I've got 3 pairs of pants, 2 shirts and 5 hats. How many outfits (pants, shirt & hat) can I wear if I won't wear one pair of pants with either 1 shirt or 1 hat?

$$3 \cdot 2 \cdot 5 - \underline{1 \cdot 1 \cdot 5} - \underline{1} \cdot \underline{2} \cdot \underline{1} + 1$$

$$30 - 5 - 2 + 1$$

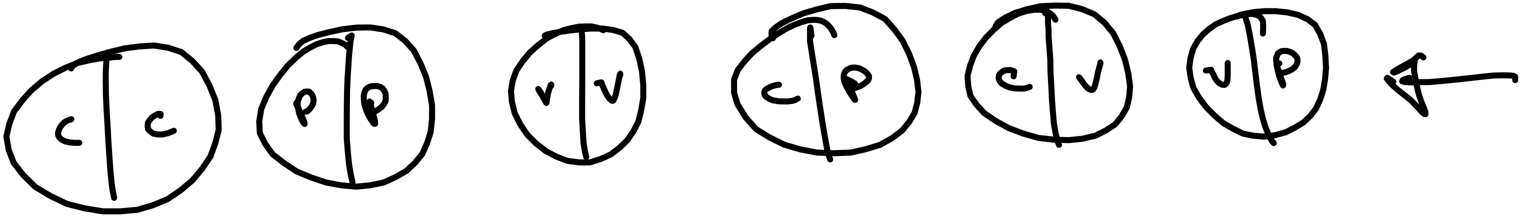
$$23 + 1 = 24$$

How many ways can we order 14 pizza for our TAs from a pizza place which serves 3 types of pizza (cheese, pepperoni, veggie)? Assume a whole pizza may only be of one type.

(++) redo the pizza problem, relaxing our assumption that the whole pizza may only be of one type. Instead, assume each half of the pizza may only be of one type.

$$\binom{14+5}{5}$$

ooo|oo o|o oo|oo o|o o|o



3 TYPES OF PIZZA

2 PIZZA HALFS

2^3	2^2	2^1	2^0
1	1	1	1

$$8 + 4 + 2 + 1 + 1$$

13 17 14

14 17 13

$$\{13, 17, 14\} = \{14, 17, 13\}$$

$$(13, 17, 14) \neq (14, 17, 13)$$

5 COUNTRIES EACH w/ 2 SWIMMERS

HOW MANY PODIUMS OF TOP 3?

UNIQUE PODIUM	1 COUNTRY REPEATS
$P(5,3)$	$\underline{1}\ \underline{1}\ \underline{0}\ 5\cdot 4$
	$\underline{0}\ \underline{1}\ \underline{1}\ 5\cdot 4$
	$\underline{1}\ \underline{\ }\ \underline{1}\ 5\cdot 4$

$$P(5,3) + 5\cdot 4\cdot 3$$

5 PIZZAS OF 3 VARITIES

C C C V V

VV CCC

PEP VEG

$$\binom{2+2}{2} = \binom{4}{2} = \frac{4!}{2!\,2!} = \frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 2} = \frac{24}{4} = 6$$