CS1800 Day 19

Admin:
- HW6 due today
- HW7 released today (due next Friday)
    - slightly shorter than most
        - more time to prep for exam2
        - will only count as 78% of other HWs with 100 points
- practice exam2 problems are out
    - slightly more induction examples
        - I want to make sure you have plenty to pull from in exam2

Content:
- Induction with equalities and inequalities

Induction (Weak):

Induction allows us to prove a never-ending sequence of statements: S(1), S(2), S(3), S(4), ...

Process:

- Prove the first statement, S(n) for some n

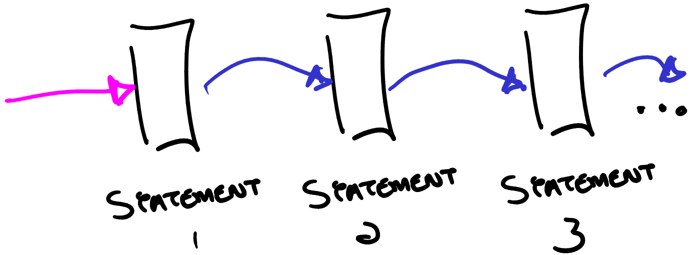
- Show that each statement implies the next statement:

Metaphor (Dominos):

To knock over all the dominos

- Push over the first one

- Place each other domino so that if the one behind it falls, it too will fall



EXAMPLE

SUMMATION NOTATION

$$1+2+4+8+16+32+64$$

$$=2^0+2^1+2^2+2^3+2^4+2^5+2^6$$

LAST VALUE OF K

$$=\sum_{k=0}^{6} 2^k$$

"The sum of 2^k where k goes from 0 to 6"

STARTING VALUE OF K

In Class Activity: Summation Notation

Express each sum below in summation notation

$9 + 11 + 13 + 15 + 17$

$9+2\cdot0$ $\quad$ $9+2\cdot2$ $\quad$ $9+2\cdot4$

$9+2\cdot1$ $\quad$ $9+2\cdot3$

$$\sum_{k=0}^{4} 9+2k$$

$1+2+3+4+5+\ldots+n$

$$\sum_{k=1}^{n} k$$

Compute each sum below (the second one has a pattern and simplifies)

$$\sum_{k=10}^{12} 2k$$

$= 2\cdot10 + 2\cdot11 + 2\cdot12$

$= 66$

$$\sum_{k=0}^{101} (-1)^k = 0$$

$(-1)^0 + (-1)^1 + (-1)^2 + (-1)^3 + \ldots$

$1 + (-1) + 1 + (-1)$

Algebraic Induction:

Show that the $\underbrace{\text{sum of the first n even integers}}$ is n(n+1)

$n=1$	$n=2$	$n=3$
$2 = 1(1+1)$	$2+4 = 2(2+1)$	$2+4+6 = 3(3+1)$

Algebraic Induction: Expressing our sum in summation notation

Show that the sum of the first n even integers is n(n+1)

$$2+4+6+8+\dots+2n$$

$$= 2\cdot 1+2\cdot 2+2\cdot 3+2\cdot 4+\dots+2n$$

STATEMENT n: " $\sum_{k=1}^{n} 2k = n(n+1)$ "

Induction Four Step Recipe: (AKA: how to not get turned around in a big induction proof)

1. Write out the statement for general n

2. Specify the list of statements you're proving
   - your base case need not be n=1
   - you're welcome to skip by more than 1 if you'd like S(3), S(5), S(7), S(9), ...

PREP WORK (NOT NEEDED IN FINAL PROOF)

3. Prove the "Base case" (the smallest n for which your statement is true)

4. Prove the conditional: "If S(n) then S(n+1)"

→ EXPRESS $S(n+1)$

→ START @ MORE COMPLEX SIDE

Algebraic Induction:

Show that the sum of the first n even integers is n(n+1)

STATEMENT n: " $\sum_{k=1}^{n} 2k = n(n+1)$ "

BASE CASE (n=1)

$$\sum_{k=1}^{1} 2k = 2 \cdot 1 = 2$$

$$= 1(1+1)$$

$$= n(n+1)$$

$S(n) \to S(n+1)$:

ASSUME STATEMENT n IS TRUE

STATEMENT n: "$\sum_{k=1}^{n} 2k = n(n+1)$"

$$S(n+1) \equiv \sum_{k=1}^{n+1} 2k = (n+1)(n+2)$$

$$\sum_{k=1}^{n+1} 2k = 2(n+1) + \sum_{k=1}^{n} 2k$$

$$= 2(n+1) + n(n+1)$$

$$= (n+1)(n+2)$$

NOTE TO SELF

- EXPRESS $S(n+1)$
- NEXT SLIDE

Summation Notation in Induction Problems

$$\sum_{k=1}^{n+1} 2k = 2\cdot 1 + 2\cdot 2 + 2\cdot 3 + \ldots + 2\cdot n + 2\cdot(n+1)$$

$$= 2\cdot(n+1) + \underline{\sum_{k=1}^{n} 2k}$$

Induction proofs will ask us to prove S(n) ⟶ S(n+1).

For sums, its helpful to be able to "pull out" the sum relevant in S(n) from the sum relevant for S(n+1) in this way.

In Class Activity: $2^0+2^1+2^2+2^3+\ldots+2^{n-1}=\sum_{k=0}^{n-1}2^k=2^n-1$

Show that the sum of the first n powers of 2 is equal to 2^n-1

$n=1$

$2^0=1=2^1-1=2^n-1$

$n=2$

$2^0+2^1=1+2=3$

$2^2-1=4-1=3$

→ EXPRESS $S(n+1)$

→ START @ MORE COMPLEX SIDE

In Class Activity: $2^0+2^1+2^2+2^3+\dots+2^{n-1}=\sum_{k=0}^{n-1}2^k=2^n-1$

Show that the sum of the first n powers of 2 is equal to 2^n-1

$$S(n)\equiv\sum_{k=0}^{n-1}2^k=2^n-1 \qquad S(n+1)=\sum_{k=0}^{n}2^k=2^{n+1}-1$$

INDUCTIVE STEP $S(n)\to S(n+1)$

BASE CASE $(n=1)$

$$\sum_{k=0}^{n-1}2^k=2^0=1=2^1-1=2^n-1$$

ASSUME $\sum_{k=0}^{n-1}2^k=2^n-1$

$$\sum_{k=0}^{n}2^k=2^n+\sum_{k=0}^{n-1}2^k$$
$$=2^n+2^n-1$$
$$=2^{n+1}-1$$

$$S(n) = \sum_{k=0}^{n-1} 2^k = 2^n - 1$$

$$S(n+1) = \sum_{k=0}^{n} 2^k = 2^{n+1} - 1$$

BASE CASE $(n=1)$

$$\sum_{k=0}^{n-1} 2^k = 2^0 = 1 = 2^1 - 1$$
$$= 2^n - 1$$

INDUCTIVE STEP $S(n) \to S(n+1)$

ASSUME $S(n)$

$$\sum_{k=0}^{n} 2^k = 2^n + \sum_{k=0}^{n-1} 2^k$$
$$= 2^n + 2^n - 1$$
$$= 2^{n+1} - 1$$

Induction with inequalities: why? (preview a bit ahead, not necessary for exam2)

Suppose two algorithms both accomplish the same task but take a different number of "computes" to do so. For a list of size N

Algorithm 1 takes: 2^N computes
Algorithm 2 takes: N! computes

List Size	1	2	3	4	5	6	7
Algorithm 1 Computes	2	4	8	16	32	64	128
Algorithm 2 Computes	1	2	6	24	120	720	5040

Goal: We want to show that algorithm 1 is faster for all lists sufficiently large (N is greater than some threshold)

Induction with inequalities:

Prove that 2^N < N! for all N above some threshold.

$S(n) = \text{"} 2^N < N! \text{"}$

$S(n+1) = 2^{N+1} < (N+1)!$

BASE CASE $n=4$

$2^N = 2^4 = 16 < 24 = 4! = N!$

INDUCTIVE STEP $S(n) \to S(n+1)$

ASSUME $2^N < N!$

$$
\begin{aligned}
(N+1)! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot N \cdot (N+1) \\
&= N! \cdot (N+1) \\
&> 2^N \cdot (N+1) \\
&> 2^N \cdot 2 \\
&= 2^{N+1}
\end{aligned}
$$

Algebra: Working with inequalities (1 of 3)

Move 1: add the same things to both sides, it preserves the inequality

IF $3 < 4$ THEN $3+10 < 4+10$

$$x < y \Rightarrow x+c < y+c \quad \forall c \in \mathbb{R}$$

Algebra: Working with inequalities (2 of 3)

Move 2: multiply by a positive value, it preserves the inequality

IF $3 < 4$ THEN $3 \cdot 10 < 4 \cdot 10$

IF $x < y$ THEN $xc < yc$ $\forall$ $c \in \mathbb{R}$ WITH $c > 0$

Move 3: multiply by a negative value, it swaps the inequality

IF $3 < 4$ THEN $3 \cdot -1 > 4 \cdot -1$

IF $x < y$ THEN $xc > yc$ $\forall$ $c \in \mathbb{R}$ WITH $c < 0$

Algebra: Working with inequalities (3 of 3)

Move 4: sum two inequalities (large side together & small side together)

IF $3 < 4$ AND $5 < 6$ THEN $3 + 5 < 4 + 6$

IF $x < y$ AND $w < z$ THEN $x + w < y + z$

In Class Activity:

Show that n^2 > n + 10 for all n above some threshold

BASE CASE

n	1	2	3	4	5	6
n^2	1	4	9	16	25	36
$n+10$	11	12	13	14	15	16

In Class Activity:

STATEMENT N = $N^2 > N+10$

Show that n^2 > n + 10 for all n above some threshold

$S(N+1) \equiv (N+1)^2 > N+11$

BASE CASE $N=4$

$N^2 = 4^2 = 16 > 14 = 4+10$
$= N+10$

INDUCTIVE STEP $S(n) \rightarrow S(n+1)$

ASSUME $N^2 > N+10$

$(N+1)^2 = N^2 + 2N + 1$
$> N+10 + 2N + 1$
$= N + 11 + 2N$

$> N + 11$