CS1800 Day8

Admin:

- regrade requests (processing)
- tagging pages
  - extra credit question needn't have a page tagged
  - "hw formatting" needn't have first page tagged on HW1 or HW2 (please tag first page for HW3+)
- math1365 overlap

Content:

- pigeonhole principle
- product rule
  - set operation: cartesian product of two sets
- principle of inclusion exclusion
  - sum rule

Floor and Ceiling Functions

Ceiling

$\lceil x \rceil$ = smallest $y \in \mathbb{Z}$ with $x \le y$

Round up to nearest integer

$\lceil 7.1 \rceil = 8$, $\lceil 7.9999 \rceil = 8$, $\lceil 7 \rceil = 7$, $\lceil 8.1 \rceil = 9$

Floor

$\lfloor x \rfloor$ = largest $y \in \mathbb{Z}$ with $y \le x$

Round down to nearest integer

$\lfloor 6.2 \rfloor = 6$, $\lfloor 6.0001 \rfloor = 6$, $\lfloor 6 \rfloor = 6$, $\lfloor 5.9 \rfloor = 5$

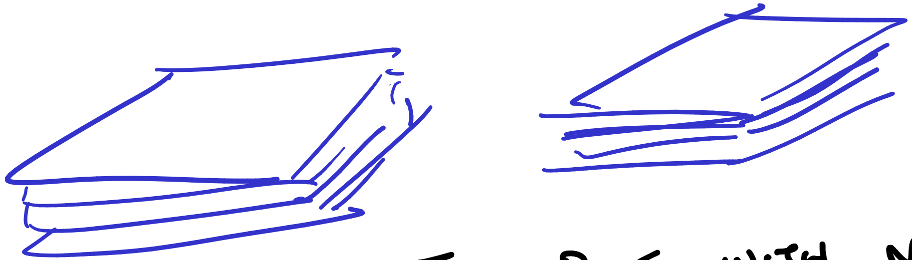
Compute x

$x = \lfloor 1.2 \rfloor$	$x = \lceil 71.2 \rceil$	$x = \lceil 100 \rceil$	$x = \lfloor -1002 \rfloor$
1	72	100	-1002

Split and Pick: Cards

A strategy to split a deck of cards (each player wants most cards)

Player 1 cuts deck:



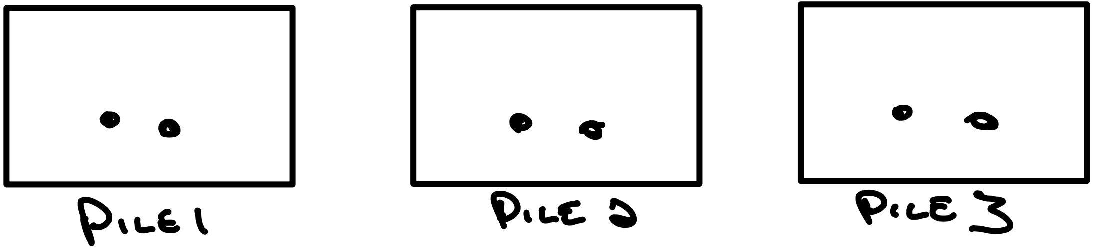
Player 2 chooses the pile with most cards

No matter how player 1 splits deck, player 2 will get, at least, half of cards

Chocolate Pigeonhole

Suppose I divide N chocolates into 3 piles.
You may take (and keep) the pile with the most chocolate.

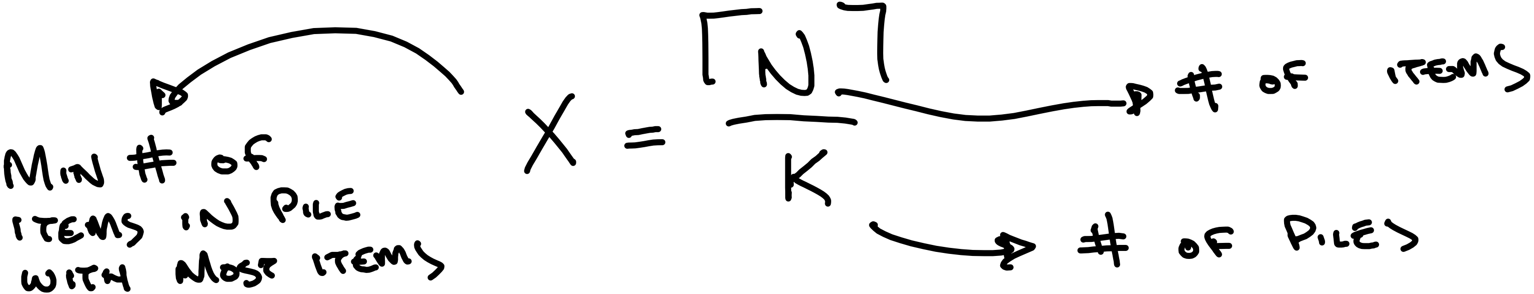
How many chocolates are you gauranteed (at least) to get, no matter how I split?



N chocs	0	1	2	3	4	5	6	7	8	9	10	11
gauranteed min chocs in some pile		1	1	1	2	2	2	3	3	3	4	4

Pigeonhole Principle

If we divide N items into K piles then there exists some pile with at least ceiling(N/k) items.



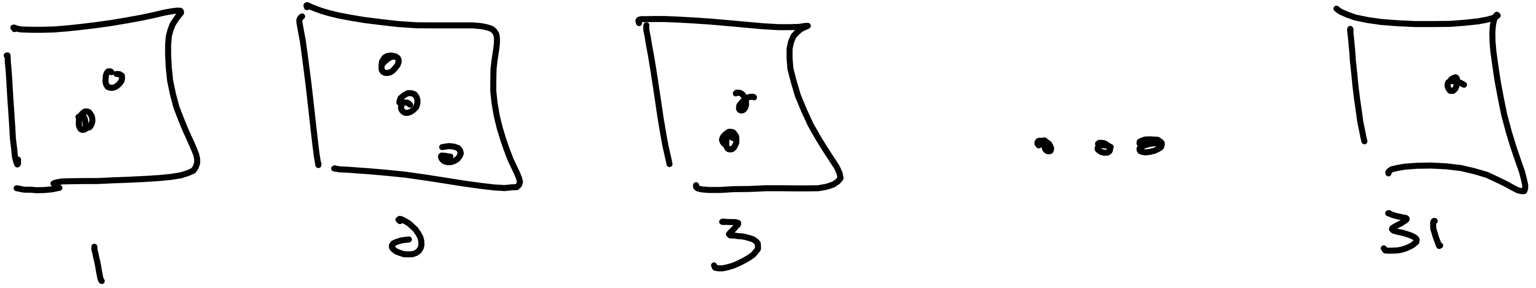
In Class Activity: Pigeonhole

If we group 3 pigeons into 2 nests, how many pigeons, at least, will be in the nest with the most pigeons?

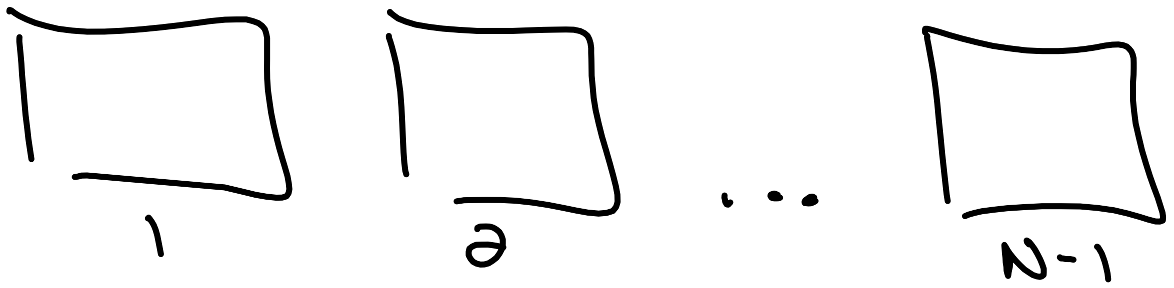
If we group everyone in this room by their day-of-the-month birthday, how many people will be in the largest group (at minimum)?

Suppose all of New York City were to have a "hair-party" where they collect into groups of people who have exactly the same number of hairs on their head. How many people are in the largest group (at minimum)? (google search as needed, rounding encouraged)

(++) In a cocktail party with two or more people, is it possible that everyone has a different number of friends at the party? Assume that friendships are symmetric (if A is B's friend then B is A's friend).



$$X = \left\lceil \frac{N}{R} \right\rceil = \left\lceil \frac{8 \cdot 10^6}{1.5 \cdot 10^5} \right\rceil = 6 \cdot 10^1$$


1
2
...
N-1

Pigeonhole & counting Motivation:

Goal: publish everyone's grades publically online, each student's is associated with a "secret code"

- If you knew your code, you could identify your grade
- others don't know your code, they can't identify your grade

SECRET CODE	C1	D0	FF	01	2A	3C	B4
GRADE	A	A-	B-	B	A	A-	B+

Suppose there are 800 students in the class and the secret code is a two-digit hex number. Are there enough secret codes for all students?

$$0 \quad 16^2 - 1 \approx 256$$

Counting Motivation:

If a computer can guess 1000 times a second, how long does it take to guess a password which is:

- 4 lowercase characters? (a, b, c, d, ...) $= \frac{26^4}{1000}$
- meets the requirements to the right

Password must:

- **Have at least one lower case character**
- **Have at least one capital letter**
- Have at least one number
- Your password must not contain more than 2 consecutive identical characters.
- Not be the same as the account name
- Be at least 8 characters
- Not be a common password

NOTATION

SET	TUPLE
$\{a,b,c\}$	(a,b,c,a)
NO REPEATS	MAY REPEAT
UNORDERED	ORDER MATTERS
	$(a,b) \neq (b,a)$

Set Operation: Cartesian Product

The cartesian product of A and B (A x B) is the set of all tuples, one item from A and the next from B

$A = \{1, 2\}$

$B = \{2, 3, 4\}$

$A \times B = \{(1,2)\ (1,3)\ (1,4)\ (2,2)\ (2,3)\ (2,4)\}$

FIRST ITEM FROM A

SECOND ITEM FROM B

$(2,1) \notin A \times B$

$(2,1) \in B \times A$

Set Operation: Cartesian Product (detail)

Example sets:

A = {1, 2} B = {3, 4} C = {5, 6}

You can apply the product to multiple sets at once

$$A \times B \times C = \{(1,3,5), (1,3,6), (1,4,5), (1,4,6), (2,3,5), (2,3,6), (2,4,5), (2,4,6)\}$$

The cartesian product is ordered

$$A \times B \neq B \times A$$

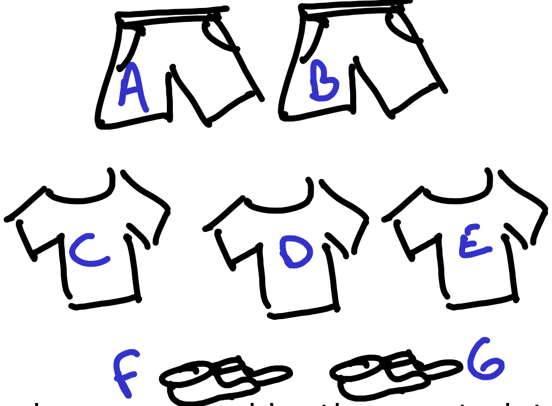
$$A \times B = \{(1,3)(1,4)(2,3)(2,4)\}$$

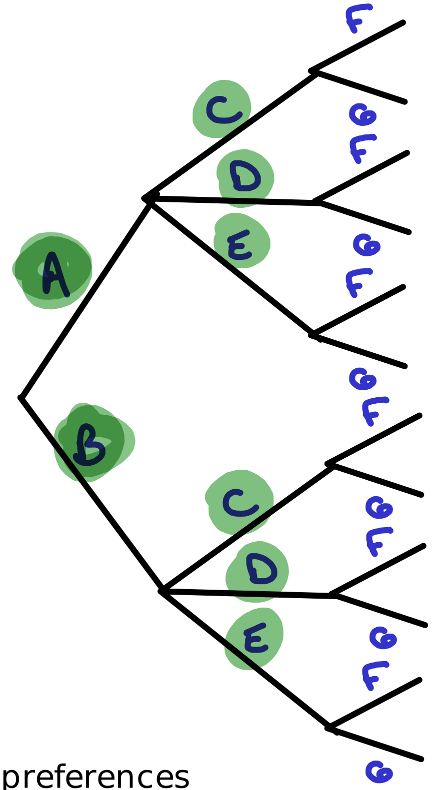
$$B \times A = \{(3,1)(4,1)(3,2)(4,2)\}$$

GETTING DRESSED

My daughter has*:
- 2 pants
- 3 shirts
- 2 socks

How unique outfits can she wear?





*when you consider the constraints of toddler preferences
our laundry situation, these are optimstic estimates!

GETTING DRESSED

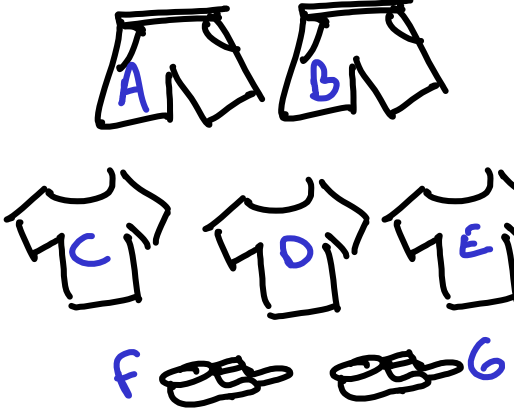
My daughter has*:
- 2 pants
- 3 shirts
- 2 socks

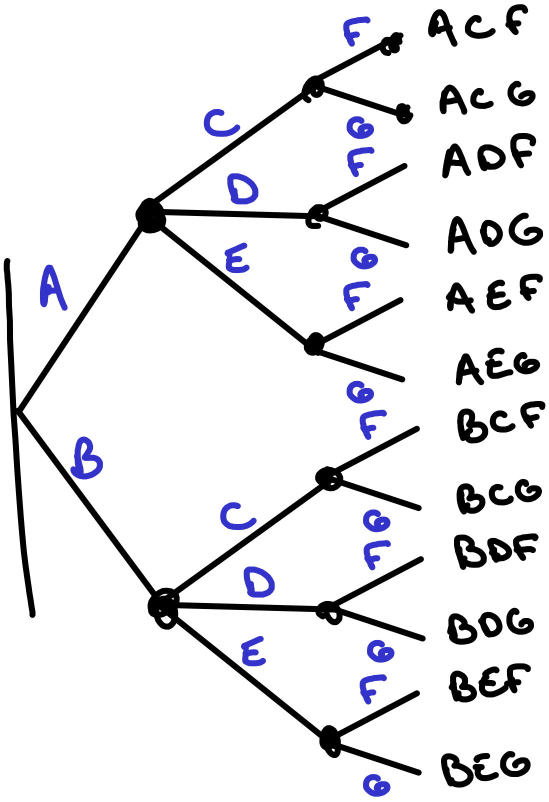
$P = \{A, B\}$

$H = \{C\ D\ E\}$

$O = \{F\ G\}$

How unique outfits can she wear?





$P \times H \times O$

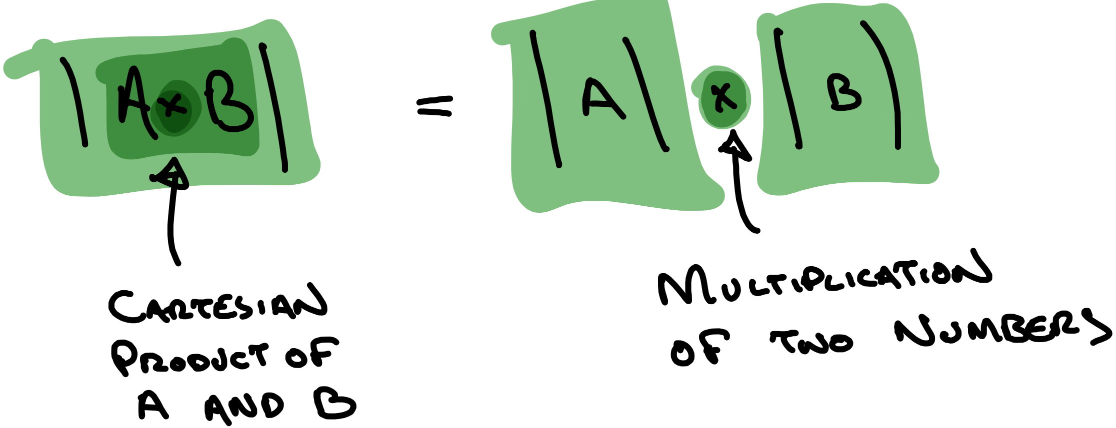
HAS

$2 \times 3 \times 2$

ITEMS

Product Rule

The number of items in a cartesian-product is the product (multiplication) of items in each set:



Multi-Set Products too! $|A \times B \times C| = |A| \times |B| \times |C|$

In Class Activity: Return of Password Counting

HINT: THERE ARE 26 LETTERS
A, B, C, D, ..., X, Y, Z

How many passwords of length 4 can be made from lowercase letters?

26^4

How many passwords of length 4 can be made from lower or upper case letters?

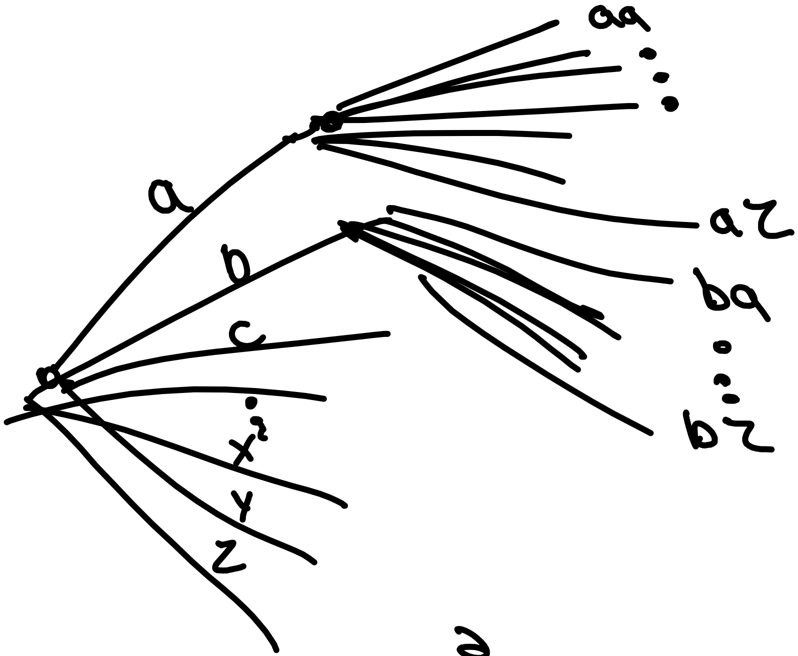
$52^4 = 52 \times 52 \times 52 \times 52$

How many passwords of length 4 can be made from lowercase letters if the first letter must be 'a'?

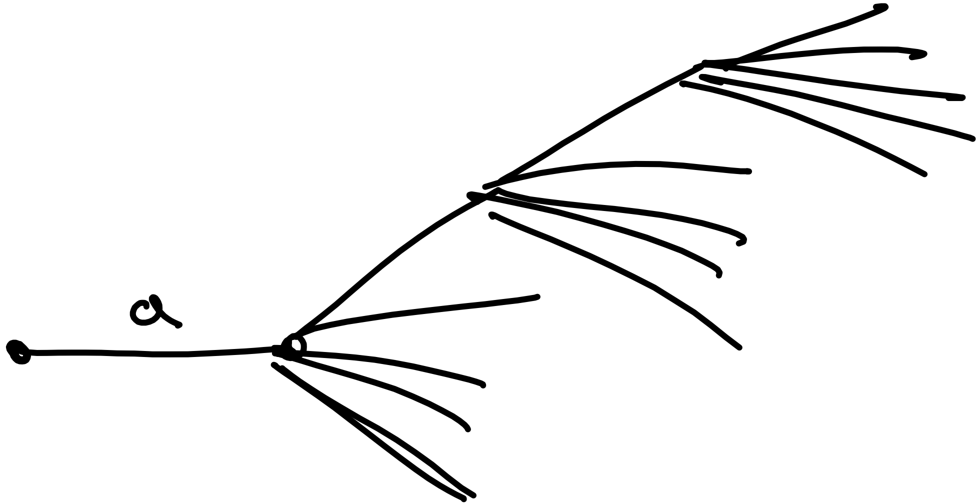
26^3

How many passwords of length 4 can be made from lowercase letters if the first letter must be 'a', 'b', or 'c'?

$3 \cdot 26^3$



26^2


a

$L = \{a, b, c, d, \ldots x y z\}$

$A = \{a, b, c\}$

$|A \times L \times L \times L| = |A| \times |L| \times |L| \times |L|$

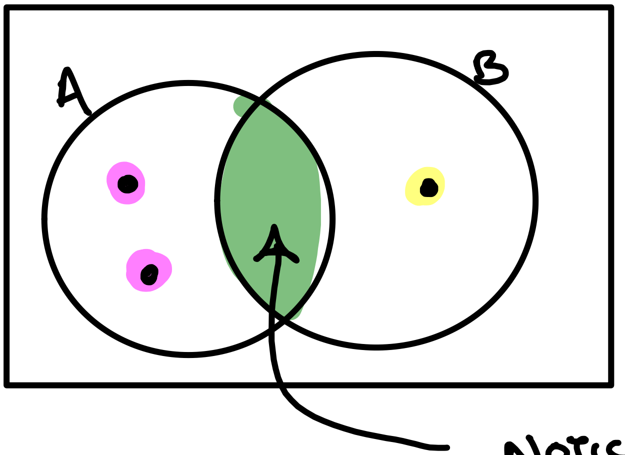
(b, c, x, y)

$= 3 \times 26 \times 26 \times 26$

$= 3 \cdot 26^3$

Sum Rule: counting disjoint unions

If sets A and B are disjoint (no item is in both) then items in A union B is items in A plus items in B:



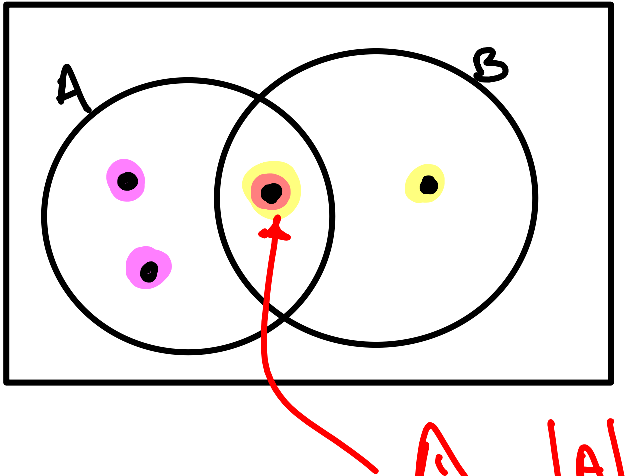
$$|A \cup B| = |A| + |B|$$

$$= 2 + 1 = 3$$

NOTICE: No item is in both A and B $\longleftrightarrow$ $A \cap B = \emptyset$

Sum Rule: counting disjoint unions (what goes wrong when sets aren't disjoint)

If sets A and B are disjoint (no item is in both) then items in A union B is items in A plus items in B:



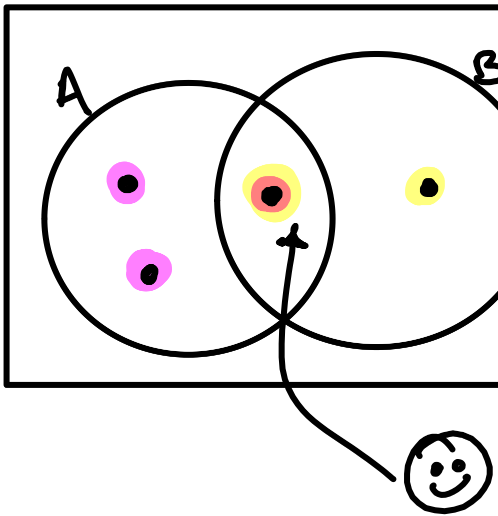
$$|A \cup B| \neq |A| + |B|$$

$$= 3 + 2 = 5$$

⚠ $|A| + |B|$ COUNTS ITEMS IN $|A \cap B|$ TWICE

Principle of Inclusion & Exclusion (PIE) (2 sets): Counting unions which may or may not share an item

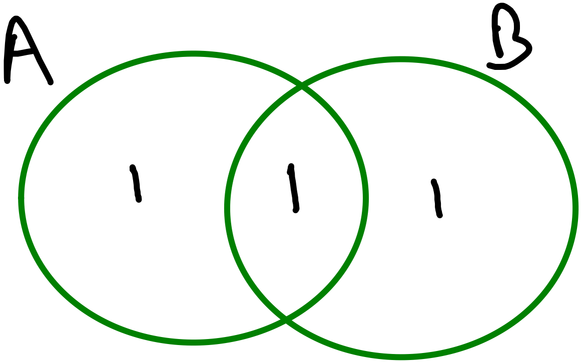
~~If sets A and B are disjoint (no item is in both) then~~ items in A union B is items in A plus items in B minus items in A intersect B



$$|A \cup B| = |A| + |B| - |A \cap B|$$

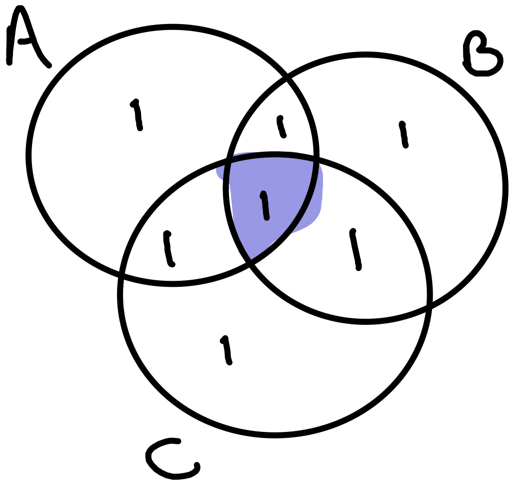
$$= 3 + 2 - 1 = 4$$

$|A| + |B|$ COUNTS ITEMS IN $|A \cap B|$ TWICE SO WE CORRECT



$$|A \cup B| = |A| + |B| - |A \cap B|$$

Principle of Inclusion & Exclusion (PIE) (3 sets): Counting unions which may or may not share an item



$$|A \cup B \cup C| = |A| + |B| + |C|$$
$$- |A \cap C| - |A \cap B| - |B \cap C|$$
$$+ |A \cap B \cap C|$$

Practice together: 3 set PIE problem

M S C

A grocery store has 17 total employees who perform 3 roles (manage, stock and checkout).
The following is a list of the training the 17 employees have.

- 3 are trained as managers $|M| = 3$
- 10 are trained to stock groceries $|S| = 10$
- 7 are trained to work the cash register $|C| = 7$
- 1 employee has 'double-training' in every pair of jobs

How many employees are trained to manage, stock and work the register?

$|M \cap S| = 1$

$|S \cap C| = 1$

$|M \cap C| = 1$

$|M \cap S \cap C| = 0$

$$|M \cup S \cup C| = |M| + |S| + |C| - |M \cap S| - |S \cap C| - |M \cap C| + |M \cap S \cap C|$$

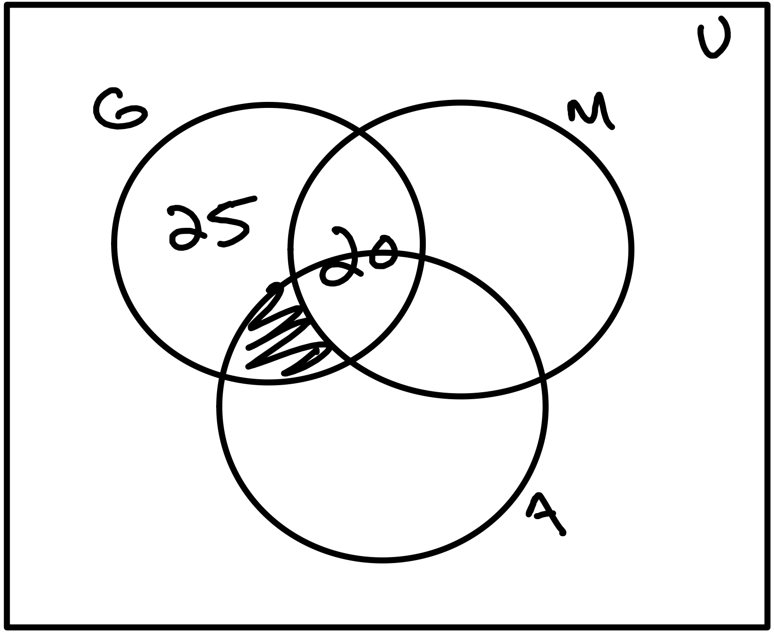
$$17 = 3 + 10 + 7 - 1 - 1 - 1 + |M \cap S \cap C|$$

In Class Assignment: 3 set PIE

Of the 196 kindergarden students which like gym or music or art:

45 like gym class
90 like music class
100 like art class
20 like both gym and music
13 like both gym and art
7 like both art and music

- how many students like gym or music?
- how many students like all 3 subjects?
- how many students like gym but nothing else?



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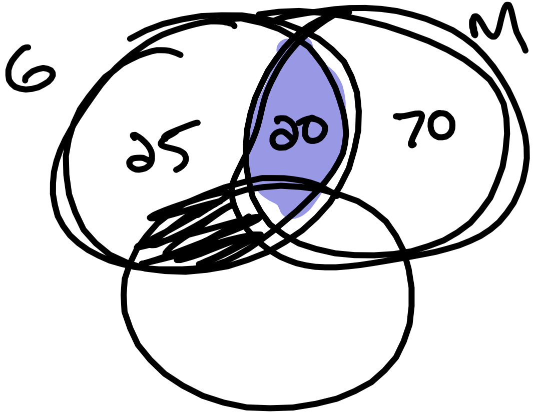


$$|G| = 45$$

$$|M| = 90$$

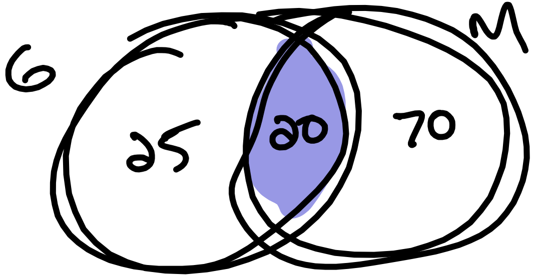
$$|G \cup M| = |G| + |M| - |G \cap M|$$

$$= 45 + 90 - 20 = 115$$

Of the 196 kindergarden students which like
gym or music or art:

45 like gym class
90 like music class
100 like art class
20 like both gym and music
13 like both gym and art
7 like both art and music

- how many students like gym or music?
- how many students like all 3 subjects?



$$|G \cup M \cup A| = |G| + |M| + |A| - |G \cap M| - |G \cap A| - |A \cap M| + |G \cap M \cap A|$$

$$196 = 45 + 90 + 100 - 20 - 13 - 7 + |G \cap M \cap A|$$

$$1 = |G \cap M \cap A|$$