

Exponentials and Logs

1 Exponentials

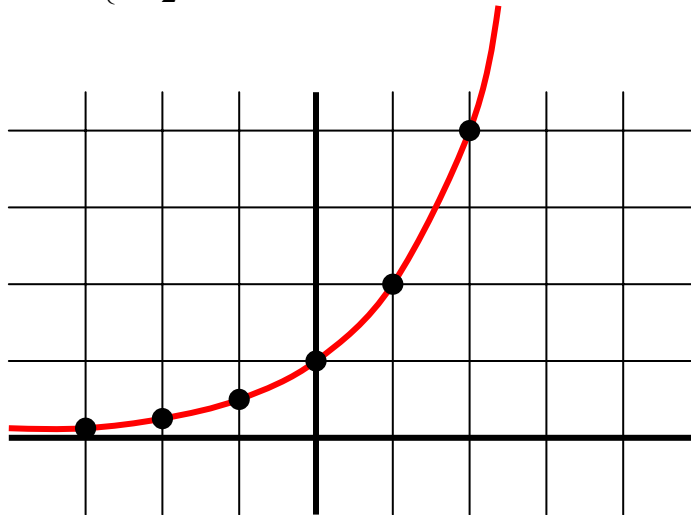
An *exponential* function has an equation of the form

$$y = a \cdot b^x \quad b > 0$$

where a is the constant of proportionality and b is the *base*.

An exponential function $y = a \cdot b^x$ is defined for all real x . Let's look at the example, $y = 2^x$, before we state the general rules. (You have probably seen the function $y = (10)^x$ in high school and $y = e^x$ if you did AP calculus, but $y = 2^x$ is very important in computer science.) It is easy to define 2^x for integer values of x .

$$2^x = \begin{cases} \underbrace{2 \cdot 2 \cdots 2}_{x \text{ twos}} & \text{if } x \text{ is a positive integer} \\ 1 & \text{if } x = 0 \\ \frac{1}{2^{-x}} & \text{if } x \text{ is a negative integer} \end{cases}$$



$y=2^x$ plotted at integer values of x , $-3 \leq x \leq 2$ and extended smoothly. Grid lines are at integer values.

We define $y = 2^x$ at all rational values of x as follows. Set $x = \frac{m}{n}$ where m and n are integers and n is positive, so the sign of x is the same as the sign of m .

$$2^{\frac{m}{n}} = \left(\sqrt[n]{2}\right)^m$$

This agrees with the definition above when x is an integer. These values fit along the graph we have already drawn.

Values of $y = 2^x$ can be defined rigorously at irrational values of x using limits of values at rational approximations to x . We won't be that rigorous here (It is material for a mathematical analysis course.) but if we want to compute 2^π , for example, we know that we could approximate it as closely as we like by computing

$$2^3, 2^{3.1}, 2^{3.14}, 2^{3.1415}, \text{ and so on.}$$

Properties of Exponentiation

Product of two powers with the same base

$$b^x \cdot b^y = b^{x+y}$$

To multiply powers of the same base, add the exponents.

Quotient of two powers with the same base

$$\frac{b^x}{b^y} = b^{x-y}$$

To divide powers of the same base, subtract the exponent of the numerator from the exponent of the denominator.

Power of a Power

$$(b^x)^y = b^{xy}$$

To raise a power to a power, multiply the exponents.

Power of a Product

$$(xy)^a = x^a y^a$$

To raise a product to a power, raise each factor to that power.

Power of a Quotient

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$$

To raise a quotient to a power, raise the numerator and denominator to that power.

Things to Prove about Exponents

If b is a positive real number and n is a positive integer then, as we did for $b = 2$ above, we define

$$b^n = \underbrace{2 \cdot 2 \cdots 2}_{n \text{ twos}}.$$

Note: Later this semester, you will learn about recursion in CSU211 and in CSU200. We can define b^n recursively for positive integers n by

$$b^n = \begin{cases} b & \text{if } n = 1 \\ b \cdot b^{n-1} & \text{if } n > 1 \end{cases}.$$

Assume that b is a positive real number and n is a positive integer. Try using the first definition and only the properties of exponentiation listed above to prove:

$$b^0 = 1$$

$$b^{-n} = \frac{1}{b^n}$$

$$b^{\frac{1}{n}} = \sqrt[n]{b}$$

$$b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m$$

2 Logarithms

The *logarithm base b* is defined by

$$y = \log_b x \quad \text{means} \quad b^y = x.$$

The functions $\log_b x$ and b^x are *inverse* functions which means:

$$\text{If } y = b^x, \text{ then } \log_b y = x.$$

$$\text{If } y = \log_b x, \text{ then } b^y = x.$$

We will talk more generally about relations, functions, and inverse functions later in the semester.

Properties of Logarithms

Logarithm of a product

$$\log_b (xy) = \log_b (x) + \log_b (y)$$

To compute the logarithm of a product, add the logarithms of the factors.

Logarithm of a quotient

$$\log_b \frac{x}{y} = \log_b (x) - \log_b (y)$$

To compute the logarithm of a quotient, subtract the logarithm of the denominator from the logarithm of the numerator..

Logarithm of a Power

$$\log_b (x^c) = c (\log_b x)$$

To compute the logarithm of a power multiply the logarithm of the base of the argument by the exponent of the argument. (x^c is the argument.)

Some questions about logarithms

Without a calculator, evaluate

$$\log_2(1) =$$

$$\log_2(2) =$$

$$\log_2(4) =$$

$$\log_2(32) =$$

$$\log_2(256) =$$

$$\log_2(1024) =$$

$$\log_2(.5) =$$

$$\frac{\log_2(9)}{\log_2\left(\frac{1}{9}\right)} =$$

Without a calculator, give approximate values for

$$\log_2(1000) \approx$$

$$\log_2(1,000,000) \approx$$

What does the graph of $y = \log_2 x$ look like?

Things to Prove about Logarithms

Changing the base

$$\log_a x = \frac{\log_b x}{\log_b a}$$

If a and b are both positive numbers then

$$(\log_a b)(\log_b a) = 1$$

References:

"Algebra and Trigonometry: Functions and Applications" by Paul A. Foerster, Chapter 6, Addison-Wesley, 1980.

"Discrete Mathematics, second edition" by James L. Hein, pages 85-87, Jones and Bartlett Mathematics, 2003.

3 Proofs and Answers

Proofs of statements about exponents

$$b^0 = 1 \quad \frac{b^x}{b^y} = b^{x-y} \text{ so } b^0 = b^{n-n} = \frac{b^n}{b^n} = 1.$$

$$b^{-n} = \frac{1}{b^n} \quad \text{Since } b^x \cdot b^y = b^{x+y} \text{ we have } b^n \cdot b^{-n} = b^{n+(-n)} = b^0 = 1 \text{ which means } b^{-n} = \frac{1}{b^n}.$$

$$b^{\frac{1}{n}} = \sqrt[n]{b} \quad \text{Since } (b^x)^y = b^{xy} \text{ we have } \left(b^{\frac{1}{n}}\right)^n = b^{\frac{1}{n} \cdot n} = b^1 = b.$$

So by definition of the nth root, $b^{\frac{1}{n}} = \sqrt[n]{b}$.

$$b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m \quad \text{Since } b^{\frac{1}{n}} = \sqrt[n]{b}, \left(\sqrt[n]{b}\right)^m = \left(b^{\frac{1}{n}}\right)^m = b^{\frac{m}{n}} \text{ by the Power of a Power rule.}$$

Answers to some questions about logarithms

Without a calculator, evaluate

$$\log_2(1) = 0$$

$$\log_2(256) = 8$$

$$\log_2(2) = 1$$

$$\log_2(1024) = 10$$

$$\log_2(4) = 2$$

$$\log_2(.5) = -1$$

$$\log_2(32) = 5$$

$$\frac{\log_2(9)}{\log_2\left(\frac{1}{9}\right)} = \frac{\log_2(9)}{-\log_2(9)} = -1$$

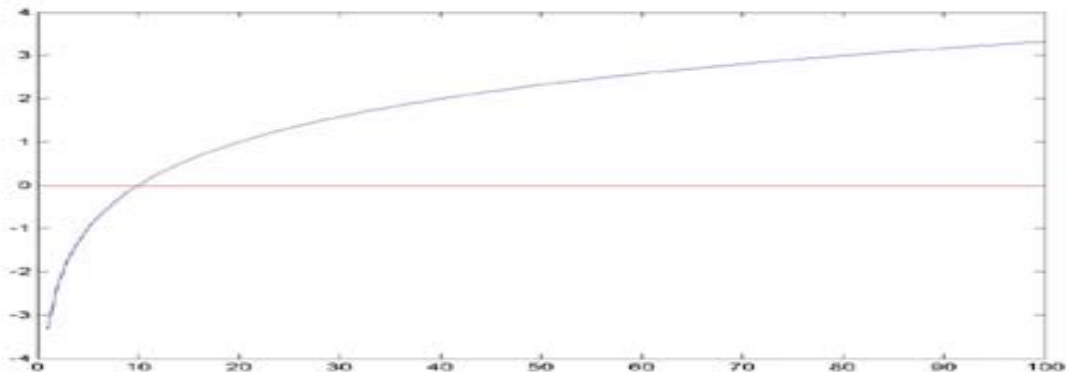
Without a calculator, give approximate values for $\log_2(1,000)$ and $\log_2(1,000,000)$.

$$\log_2(1000) \approx \log_2(1024) = 10$$

According to my PC calculator, $\log_2(1000) \approx 9.9657842846620870436109582884682$
so 10 really is a good approximation.

$$\log_2(1,000,000) = \log_2(1,000 * 1,000) = \log_2(1,000) + \log_2(1,000) \approx 20$$

What does the graph of $y = \log_2 x$ look like?



Proofs of Statements about Logarithms

Changing the base

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Proof:

By the definition, $\log_a x$ is the unique number such that, $a^{\log_a x} = x$.

Similarly, $b^{\log_b x} = x$ and $b^{\log_b a} = a$.

Hence $a^{\frac{\log_b x}{\log_b a}} = (b^{\log_b a})^{\frac{\log_b x}{\log_b a}} = b^{(\log_b a) \frac{\log_b x}{\log_b a}} = b^{\log_b x} = x$.

So $\frac{\log_b x}{\log_b a} = \log_a x$.

If a and b are both positive numbers then

$$(\log_a b)(\log_b a) = 1$$

Proof:

$a^{(\log_a b)(\log_b a)} = (a^{\log_a b})^{\log_b a} = (b)^{\log_b a} = a$.

Hence $(\log_a b)(\log_b a) = \log_a a = 1$.