CSU200 Discrete Structures Exam 3 Solutions

Problem 1 [28 pts, (3,3,3,3,4,4,4,4)]: Permutations and Combinations

(a) Express each of the following first as a product of integers or ratio of factorials then evaluate each to give a single integer answer.

i. P(10, 2) $= \frac{10!}{8!} = 10 \cdot 9 = 90$

ii. P(10, 4) $= \frac{10!}{6!} = 10 \cdot 9 \cdot 8 \cdot 7 = 5040$

iii. C(10, 2) $= \frac{10!}{8! \cdot 2!} = \frac{10 \cdot 9}{2} = 45$

iv. C(10, 4) $= \frac{10!}{6! \cdot 4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2} = 10 \cdot 3 \cdot 7 = 5040$

(b) Lon Lee has a collection of twenty photos of friends and family that he keeps to cheer himself up. Each of the following questions refer to this collection.

Give each answer in terms of permutations $P(n, r)$ or combinations $C(n, r)$ then express each as a product of integers or ratio of factorials. DO NOT EVALUATE FURTHER.

i. Lon wants to hang 8 of his 20 photos in a line on the wall above his desk. How many ways can he do this?

Solution: **P(20, 8)** $= \frac{20!}{12!}$

ii. How many ways can Lon mount 4 of his 20 photos in a picture frame (2 rows with 2 pictures each).

Solution: **P(20, 4)** $= \frac{20!}{16!}$

iii. Lon wants to select 9 photos of his 20 to take on a trip. How many ways can he do this?

Solution: **C(20, 9)** $= \frac{20!}{11!9!}$

iv. Lon has decided to give his photo collection to his three children. In how many ways can he partition his photo collection among this three children?

Solution: It is OK for a child to not get any photos. This is really not a balls in bins problem as the photos are distinct. If Lon wanted to dole out 20 hundred-dollar bills to his children, a balls in bins solution would apply. In this case, however, each photo can go to (is assigned to) one of his 3 children so there are 3^{20} ways for him to dole them out.

Problem 2 [24 pts, (4,5,5,5,5)]**: Counting**

Jenna Rator, a freshman CCIS student, heard that a monkey typing random letters and spaces would sooner or later type "THAT␣IS␣THE␣QUESTION" (here we have used the symbol ␣ to explicitly denote the space character). She decided to test this out by writing a Scheme program to generate 20-character strings composed of upper-case letters and spaces.

i. How many 20-character strings composed of upper-case letters and spaces are there?

Solution: There are 26 upper-case letters and one space character. Each of the 20 characters in the string can be chosen from these 27 characters. It is clear from the sample string that she hopes to generate that repetition is allowed so, by the product rule, there are 27^{20} possible strings.

ii. Jenna realized, after an hour of printing random strings that some of the stings started or ended with spaces (e.g., "␣LKF␣MRTU␣IOERHUV␣T␣") and that she really should only print strings that started and ended with letters. How many 20-character strings composed of upper-case letters and spaces are there that start and end with letters?

Solution: This is similar to the last part but there are only 26 choices (the letters) for the first and last characters and 27 choices for the other 18 characters. The total number of strings possible strings is $26^2 \cdot 27^{18}$.

iii. After watching many more pages of random strings pile up, Jenna, being much smarter than a monkey, realized that she could save trees by only printing strings that were of the form "xxxx␣xx␣xxx␣xxxxxxxx", where each "x" is an upper-case letter. How many strings are there of this form?

Solution: This time, the spaces are fixed and the remaining 17 characters must be chosen from the 26 upper-case letters. There are 26^{17} possible strings.

iv. Most of the strings still didn't even look like they were made up of words so Jenna decided to only generate strings that had the right number of vowels. She would first generate 4-letter strings with exactly one vowel (A, E, I, O, U). How many strings are there of this type?

Solution: We solve the problem by breaking down the job into three independent tasks and then use the product rule to get the answer.

i. Choose the position for the vowel. There are 4 places the vowel can go.

ii. Choose the vowel. There are 5 choices here.

iii. Choose the consonants for the remaining 3 places. There are 21 choices for each place so there are 21^3 choices for the 3 consonants.

The total number of 4-letter strings with exactly one vowel is then $4 \cdot 5 \cdot 21^3$.

v. How many 8-letter strings are there with exactly three vowels?

<u>*Solution:*</u> We solve the problem by breaking down the job into three independent tasks and then use the product rule to get the answer.

i. Choose the positions for the 3 vowels. There are $C(8,3)$ ways to choose places for the 3 vowels.

ii. Choose the vowel. There are 5 choices for each of the 3 vowels so there are 5^3 ways to choose the 3 vowels.

iii. Choose the consonants for the remaining 5 places. There are 21 choices for each place so there are 21^5 choices for the 5 consonants.

The total number of 8-letter strings with exactly 3 vowels is then $C(8,3) \cdot 5^3 \cdot 21^5 = \frac{8!}{3!5!} \cdot 5^3 \cdot 21^5$.

Problem 3 [24 pts, (4 each)]: The Binomial Theorem and Pascal's Triangle

Binomial Theorem

i. State the Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

ii. Show that

$$1 \cdot \binom{n}{0} + 2 \cdot \binom{n}{1} + 4 \cdot \binom{n}{2} + 8 \cdot \binom{n}{3} + \cdots + 2^n \cdot \binom{n}{n} = 3^n.$$

<u>*Solution:*</u> By the Binomial Theorem,

$$3^n = (2+1)^n = \sum_{k=0}^{n} \binom{n}{k} 2^k 1^{n-k} = 1 \cdot \binom{n}{0} + 2 \cdot \binom{n}{1} + 4 \cdot \binom{n}{2} + 8 \cdot \binom{n}{3} + \cdots + 2^n \cdot \binom{n}{n}.$$

iii. Show that

$$\binom{10}{0} - \binom{10}{1} + \binom{10}{2} - \binom{10}{3} + \cdots - \binom{10}{7} + \binom{10}{8} - \binom{10}{9} + \binom{10}{10} = 0.$$

<u>*Solution:*</u> By the Binomial Theorem,

$$\begin{aligned} 0 = (1-1)^{10} &= \sum_{k=0}^{10} \binom{10}{k} 1^k (-1)^{n-k} \\ &= \binom{10}{0} - \binom{10}{1} + \binom{10}{2} - \binom{10}{3} + \cdots - \binom{10}{7} + \binom{10}{8} - \binom{10}{9} + \binom{10}{10}. \end{aligned}$$

Pascal's Triangle

The problem follows this reminder of how Pascal's triangle is formed. The binomial coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n-1}, \binom{n}{n}$ form the n^{th} row of Pascal's triangle. The rows are staggered so that each number inside the triangle lies diagonally below two other numbers.

Theorem: If n is a positive integer and $0 \leq k < n$, then $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Informally, the top entry of Pascal's Triangle is a 1. The first and last element on each row is a 1. Each other element can be obtained by adding the two elements, from the previous row, that are diagonally above it.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

The following four integers are the first four elements in the 61st row of Pascal's Triangle:

1 60 1770 34220

iv. Give the first four elements of the *previous* (60th) row.

Solution:

1 59 1711 32509

1 60 1770 34220

The number 1 is always the first element.

The number 60 is the sum of the fist and second elements of the previous row so $59 = 60 - 1$ is the second element.

The number 1770 is the sum of the second and third elements of the previous row so $1711 = 1770 - 59$ is the third element.

Similarly, the fourth element is $32509 = 34220 - 1711$.

v. Give the first four elements of the *next* (62nd) row.

Solution:

1 60 1770 34220

1 61 1830 35990

1 is always the first element.

$61 = 1 + 60$

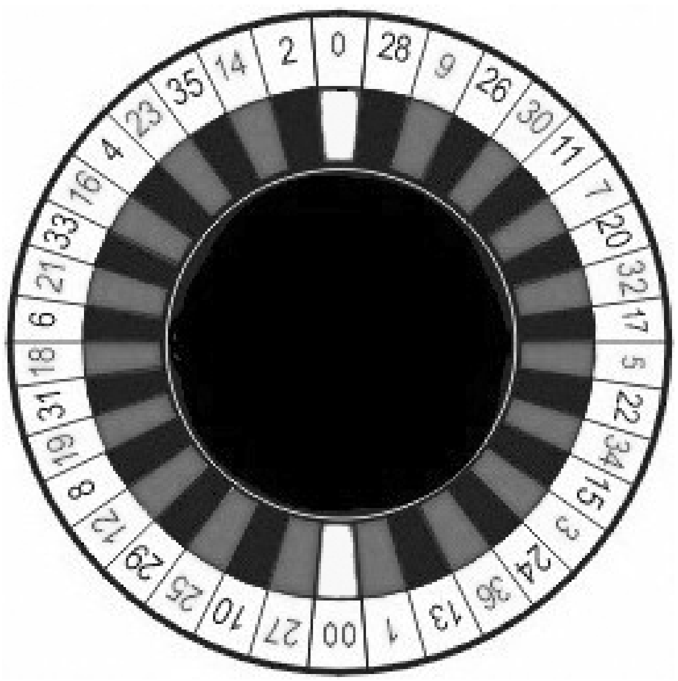
$1830 = 60 + 1770$

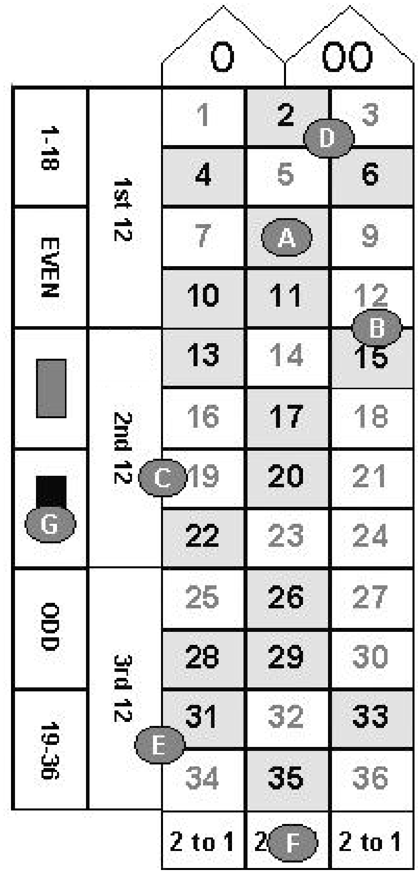
$35990 = 1770 + 34220$

vi. Give the *last* four elements of the 63rd row.

<u>*Solution:*</u> The first four elements in the 63rd row are 1, $62 = 1 + 61$, $1891 = 61 + 1830$, and $37820 = 1830 + 35990$. The row ends with the same four elements in reverse order: 35990, 1891, 62, 1.

Problem 4 [24 pts, (2,2,2,9[3each],9[3each])]**: Roulette**





Note: The number 24 should be gray on this board. You can see from the wheel, the 24 is a black number.

An American roulette wheel has 38 slots labeled with the numbers 1 through 36 plus 0 and 00. Eighteen numbers are red (gray on the wheel to the left and white on the board to the right) and eighteen numbers are black. The numbers 0 and 00 are neither red nor black and neither odd nor even.

A player bets by placing a chip or chips on the board, e.g., in one of the positions labeled Ⓐ through Ⓖ on the board to the right. Then the wheel is spun and there is one winning number.

(a) For each of the following, tell the probability that the bet pays.

i. Straight up Ⓐ: A chip is placed on one number — only that number pays.

<u>*Solution:*</u> There are 38 possible outcomes when the wheel is spun, the numbers 1 through 36 as well as 0 and 00. There is one successful outcome so the bet pays with probability $\frac{1}{38}$.

ii. Columns Ⓕ: A chip placed at the end of a column — any of the 12 numbers in the column pays.

<u>*Solution:*</u> There are 12 successful outcomes, the 12 numbers in the column so the probability that the bet pays is $\frac{12}{38}$.

iii. Black Ⓖ: A chip is placed on "Black" — any black number pays.

<u>*Solution:*</u> There are 18 successful outcomes, the 18 black numbers, so the probability that the bet pays is $\frac{18}{38}$.

(b) Amy and Bert take a trip to Las Vegas together. For each of the following, tell the probability of each of these events:

1. both of their bets pay
2. at least one of their bets pays
3. Amy's bet pays and Bert's bet loses.

Note: Though I did state during the exam that the bets were placed simultaneously (one roll), many answers were correct only if the bets were placed on different rolls (two rolls). I gave credit for both answers. If, however, one answer fit one assumption while another answer fit the other assumption, only one answer received credit. –Prof. Fell).

i. Amy places a chip at Ⓑ, i.e. on the numbers 12 and 15, and Bert places a chip on Black Ⓖ.

1. both of their bets pay

<u>*Solution:*</u>

(one roll) Only 15 pays for both Amy and Bert so the probability is $\frac{1}{38}$.

(two rolls) The product rule applies. The probability that Amy's bet pays is $\frac{2}{38}$ and the probability that Bert's bet pays is $\frac{18}{38}$ so the probability that they both win is $\frac{2}{38} \cdot \frac{18}{38} = \frac{1}{19} \cdot \frac{9}{19} = \frac{9}{19^2} = \frac{9}{361}$.

2. at least one of their bets pays

<u>*Solution:*</u>

The Inclusion/Exclusion Principle applies in both cases.

(one roll) Amy's bet pays if the roll is 12 or 15. Bert's bet pays if the roll is black, i.e. any of these 18 numbers $\{2, 4, 6, 8, 10, 11, 13, 15, 17, 20, 22, 24, 26, 28, 29, 31, 33, 35\}$. As 15 wins for both Amy and Bert, we must exclude it from our count of numbers that pay for Amy OR for Bert. That leaves 19 numbers that pay for Amy OR for Bert and the probability that at least one of their bets pays is $\frac{19}{38} = \frac{1}{2}$.

(two rolls) By the Inclusion/Exclusion Principle, the probability that Amy's bet pays OR Bert's bet pays is

P(Amy's bet pays) + P(Bert's bet pays) − P(Amy's bet pays) · P(Bert's bet pays) =

$$\frac{1}{19} + \frac{9}{19} - \frac{1}{19} \cdot \frac{9}{19} = \frac{10}{19} - \frac{9}{361} = \frac{190 - 9}{361} = \frac{181}{361}.$$

3. Amy's bet pays and Bert's bet loses.

<u>*Solution:*</u>

(one roll) Only 12 pays for Amy and loses for Bert so the probability is $\frac{1}{38}$.

(two rolls) The product rule applies. The probability that Amy's bet pays is $\frac{2}{38}$ and the probability that Bert's bet loses is $\frac{20}{38}$ so the probability that Amy wins and Bert loses is $\frac{2}{38} \cdot \frac{20}{38} = \frac{1}{19} \cdot \frac{10}{19} = \frac{10}{19^2} = \frac{10}{361}$.

ii. Amy places a chip on Black Ⓖ and Bert places a chip on the middle column Ⓕ.

1. both of their bets pay

<u>*Solution:*</u> (one roll) Only the black numbers in the middle column pay for both Amy and Bert so the probability is $\frac{8}{38} = \frac{4}{19}$.

(two rolls) The product rule applies. The probability that Amy's bet pays is $\frac{18}{38}$ and the probability that Bert's bet pays is $\frac{12}{38}$ so the probability that they both win is $\frac{18}{38} \cdot \frac{12}{38} = \frac{9}{19} \cdot \frac{6}{19} = \frac{54}{19^2} = \frac{54}{361}$.

2. at least one of their bets pays

<u>*Solution:*</u>

The Inclusion/Exclusion Principle applies in both cases.

(one roll) Amy's bet pays if the roll is any of these 18 numbers {2, 4, 6, 8, 10, 11, 13, 15, 17, 20, 22, 24, 26, 28, 29, 31, 33, 35}. Bert's bet pays if the roll is any of these 12 numbers {2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35}. The 8 numbers that are black and in the middle column pay for both Amy and Bert, so we must subtract them from our sum to prevent double counting. Thus, we obtain $18 + 12 - 8 = 22$ numbers that pay for Amy OR for Bert and the probability that at least one of their bets pays is $\frac{22}{38} = \frac{11}{19}$.

(two rolls) By the Inclusion/Exclusion Principle, the probability that Amy's bet pays OR Bert's bet pays is

P(Amy's bet pays) + P(Bert's bet pays) − P(Amy's bet pays) · P(Bert's bet pays) =

$$\frac{9}{19} + \frac{6}{19} - \frac{9}{19} \cdot \frac{6}{19} = \frac{15}{19} - \frac{54}{361} = \frac{285 - 54}{361} = \frac{231}{361}.$$

3. Amy's bet pays and Bert's bet loses.

<u>*Solution:*</u>

(one roll) Only black numbers not in the middle row pay for Amy and lose for Bert so the probability is $\frac{18-8}{38} = \frac{10}{38} = \frac{5}{19}$.

(two rolls) The product rule applies. The probability that Amy's bet pays is $\frac{18}{38} = \frac{9}{19}$ and the probability that Bert's bet loses is $\frac{38-12}{38} = \frac{13}{19}$ so the probability that Amy wins and Bert loses is $\frac{9}{19} \cdot \frac{13}{19} = \frac{117}{361}$.