Sequences and Sums

A sequence is a function t from a subset of the integers (usually N or $Z+$) to a set S. In this discussion, the set S will be a set of numbers but we could have a sequence of colors, musical notes, or even computer programs.

We often write $a_n = t(n)$. Each a_n is a *term* of the sequence. The sequence $a_1, a_2, a_3, \ldots$ may be denoted by $\{a_n\}$ or $\{a_n\}_{n=1\cdots\infty}$. The numbers, $1, 2, 3, \ldots$ are the term indices and $a_1, a_2, a_3, \ldots$ are the term values

1 Sequence Examples

Can you give the next term of each of these sequences? Can you give the function $t(n)$ that defines the sequence? Assume the first term is a_1.

$1, 1/2, 1/3, 1/4, 1/5, 1/6, \ldots$
$2, 4, 6, 8, 10, 12, \ldots$
$2, 4, 8, 16, 32, 64, \ldots$
$7, 8, 9, 10, 11, 12, \ldots$
$1/2, 1, 2, 4, 8, 16, \ldots$
$1, 3, 6, 10, 15, 21, \ldots$
$1, 1, 1, 1, 1, 1, 1, \ldots$
$1, 1, 2, 3, 5, 8, \ldots$
$1/2, -2/3, 3/4, -4/5, 5/6, -6/7, \ldots$
$2, 3, 5, 7, 11, 13, \ldots$

2 Some Special Sequences

2.1 Arithmetic Sequences

An *arithmetic sequence* is a sequence in which each term equals the preceding term plus a constant. A general arithmetic sequence looks like this $a, a + d, a + 2d, a + 3d, \ldots$ The first term is a and the constant difference between terms is d.

What is the n^{th} term of the sequence $a, a + d, a + 2d, a + 3d, \ldots$?

What is the next term and the nth term of each of these sequences?

$4, 7, 10, 13, 16, 19, \ldots$
$-7, -1, 5, 11, 17, 23, \ldots$

2.2 Geometric Sequences

A *geometric sequence* is a sequence in which each term equals the preceding term times a constant. A general geometric sequence looks like this $a, ar, ar2, ar3, ar4, \ldots$ the first term is a and the constant ratio between successive terms is r.

What is the n^{th} term of the sequence $a, ar, ar^2, ar^3, ar^4, \ldots$?

What is the next term and the nth term of each of these sequences?

$2, 6, 18, 54, 162, \ldots$
$1, -4, 16, -64, 256, \ldots$

2.3 Quadratic Sequences

A *quadratic sequence* is a sequence whose n^{th} term is given by a quadratic function, $a_n = an^2 + b^n + c$. Here are some quadratic sequences. Can you find the quadratic function that generates them?

$1, 4, 9, 16, 25, 36, \ldots$
$6, 15, 28, 45, 66, 91, \ldots$

An arithmetic sequence is given by a linear function and the difference between successive terms is a constant. In a quadratic sequence the differences between successive terms are given by a linear function and the second differences are constant. For example:

sequence	6	15	28	45	66	91	$\cdots$
differences		9	13	17	21	25	$\cdots$
second differences			4	4	4	4	$\cdots$

3 Series and Partial Sums

A *series* is a sum of the terms of a sequence. Since a sequence has infinitely many terms, a series is the sum of infinitely many terms. We often sum only the first n terms of a sequence. The sum of the first n terms is called the nth partial sum. The following is a sum of an infinite number of terms.

$$\sum_{k=1}^{\infty} a_k = a - 1 + a_2 + a_3 + \cdots$$

Below is the sum of the first n terms. This is the n^{th} *partial sum.* k is the *index of summation*; 1 is the *lower limit*; n is the *upper limit.*

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

This is also a partial sum. j is the index of summation; m is the lower limit; n is the upper limit.

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

3.1 Arithmetic Sums

An *arithmetic sum* is a sum of terms of an arithmetic sequence.

$$\sum_{k=1}^{n} a + dk = (a+d) + (a+2d) + (a+3d) + \cdots + (a+nd) = na + (1 + 2 + 3 \cdots + n)d$$

$$na + \frac{n(n+1)}{2} = n\left(\frac{a+1+a+n}{2}\right) = num\_terms\left(\frac{first\_term + last\_term}{2}\right)$$

3.1.1 Examples

$1+2+3+n=\frac{n(n+1)}{2}\dots$
$1+2+\cdots+100=\frac{100(1+100)}{2}\dots=5050$
$2+5++8+11+14+17+20+23+26+29=\frac{10(2+29)}{2}\dots=155$

3.2 Geometric Sums and Series

A *geometric sum* is a sum of terms of a geometric sequence.

$$\sum_{k=1}^{n} ar^k = ar + ar^2 + ar^3 + \cdots + ar^n$$

or, summing from 0

$$\sum_{k=0}^{n} ar^k = a + ar + ar^2 + ar^3 + \cdots + ar^n$$

With geometric sequences, summing the infinite sequence may be meaningful. Infinite sums are called *series*.

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots$$

Let's find a formula fort the n^th partial sum $\sum_{k=1}^{n} ar^k = ar + ar^2 + ar^3 + \cdots + ar^n$.

Let $S = \sum_{k=0}^{n} ar^k = a + ar + ar^2 + ar^3 + \cdots + ar^n$.

Then $rS = \sum_{k=0}^{n} ar^{k+1} = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^n + ar^{n+1}$.

$S - rS = a - ar^{n+1}$ as all the other terms cancel out when we subtract rS from S.

So $S = \frac{a-ar^{n+1}}{1-r} = \frac{a(1-r^{n+1})}{1-r}$.

3.2.1 Examples

$\sum_{k=0}^{6} \frac{1}{3^k} = 1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} + \frac{1}{3^6} = \frac{1\left(1-\frac{1}{\frac{1}{3^7}}\right)}{1-\frac{1}{3}} = \frac{3^7-1}{3^7-3^6} = \frac{2187-1}{2197-729} = \frac{2186}{1458} \approx 1.499.$

$\sum_{k=0}^{6} 3^k = 1 + 3 + 3^2 + 3^3 + 3^4 + 3^5 + 3^6 = \frac{1(1-3^7)}{1-3} = \frac{2^7-1}{2} = \frac{2186}{2} = 1093.$

$\sum_{k=0}^{6} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} = \frac{1\left(1-\frac{1}{\frac{1}{2^7}}\right)}{1-\frac{1}{2}} = \frac{2^7-1}{2^7-2^6} = \frac{128-1}{128-64} = \frac{127}{64} \approx 1.984.$

$\sum_{k=0}^{n} \frac{1}{2^k} = \frac{1\left(1-\frac{1}{\frac{1}{2^{n+1}}}\right)}{1-\frac{1}{2}} = \frac{1-\frac{1}{\frac{1}{2^{n+1}}}}{\frac{1}{2}} = 2 - \frac{1}{2^n}$. What happens when n gets very large?

$\sum_{k=0}^{\infty} \frac{1}{2^k} = ?$

3.3 Some Special Sums

$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.