Sets

A *set* is an unordered collection of objects. The objects are called *elements* of the set. Many of the sets we talk about in discrete structures are sets of numbers.

Sets of Numbers:

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$,	natural numbers[1]
$\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$	integers
$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$	positive integers
$\mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$	rational numbers
$\mathbb{R}$	real numbers
$S = \{1, 2, 3\}$	the set containing the three numbers: 1, 2, 3

We use braces (also called curly brackets) to show the elements of a set. The elements of a set do not have to be numbers. We can talk about the set of all CCIS freshmen or just the CCIS freshmen who showed up for class today. Here are some other sets:

Some Other Sets:

Letters = {a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z}
Vowels = {a, e, i, o, u}
Nibbles ={0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111}
{George, textbook, a piece of chalk, 6} George is my son. This is a weird set.

1 Set Basics

If x is an element of S, we write $x \in S$ which may also be read as "x is in S."
If x is not in S, we write, $x \notin X$.

Two sets are *equal* if and only if they have the same elements.
1. {a, b, c} = {b, c, a} = {b, a, c, b}
We do not usually write a element twice, as in {b, a, c, b} but sometimes it just happens. We might, for example, compute $\{2x, x^2, x^3 - 4\}$ for some values of *x*. Here's what we get for a few small values of x.

$$x = 1: \{2x, x^2, x^3 - 4\} = \{2, 1, -3\}$$
$$x = 3: \{2x, x^2, x^3 - 4\} = \{6, 9, 23\}$$
$$x = 0: \{2x, x^2, x^3 - 4\} = \{0, 0, -4\} = \{0, -4\}$$
$$x = 2: \{2x, x^2, x^3 - 4\} = \{4, 4, 4\} = \{4\}$$

If a set has a finite number of elements, we say it is a *finite set* and the *cardinality* or *size* of the set is the number of elements it contains. We use the notation |S| to denote the cardinality of S. If S = {1, 2, 5, 7}, then |S| = 4. Cardinality is also defined for *infinite sets*, i.e. sets with infinitely

[1] Mathematicians are split about 50-50 as to whether to use $\mathbb{N}$ for {0, 1, 2, ...} or {1, 2, 3, ...}. We'll use $\mathbb{N}$ for {0, 1, 2, ...} as we have $\mathbb{Z}^+$ for {1, 2, 3, ...}.

many elements. In fact, though $\mathbb{R}$ and $\mathbb{Z}$ both have infinitely many elements, they do not have the same cardinality. In fact, there are an infinite number of infinities. You will learn about this in Theory of Computation, if not sooner.

A set may also have no elements at all. We call a set with no elements the *empty set* and denote it by ∅ or by {}. This may seem silly but the empty set is very important. In a way, it is a place-keeper like the number zero. When you declare a set variable in computing, it is an empty set until you put something in it. If we compute

S = $\left\{x \in \mathbb{R} \mid x^2 + 4x + c = 0\right\}$

for various values of *c,* then when

$c = 0$,	S = {-4, 0}
$c = 4$,	S = {-2}
$c = 6$,	S = {}.

Set-Builder Notation

Sometimes we define a set by listing all the elements of the set inside curly brackets as we did above for S = {1, 2, 3}. We do this too for infinite sets as in $\mathbb{Z}^+$={1, 2, 3, …, } or $\mathbb{Z}$ = {. . . -2, -1, 0, 1, 2, ...}. When we need to describe which elements are in the set instead of just leaving it to the reader to figure out how to continue a list, we use *set-builder notation*. A set description in set-builder notation looks like

{ v | condition on v} or { v ∈ S| condition on v}

where v is a variable and S is a set. The braces '{' and '}' tell us to say "the set of" and the vertical bar '|' is read as "such that." We sometimes use a colon ":" in place of the vertical bar.

{ v | condition on v} is read as "the set of v such that" the condition on v holds.
{ v S| condition on v} is read as "the set of v in S such that" the condition on v holds.

The examples below should explain how set builder notation works.

Examples

$\left\{x \middle| x \in \mathbb{Z} \text{ and } |x| < 5\right\}$ means "*the set of x in* $\mathbb{Z}$ *such that |x| is less than 5*" which is equal to {-4, -3, -2, -1, 0, 1, 2, 3, 4}. We might also write $\left\{x \in \mathbb{Z} \middle| |x| < 5\right\}$.

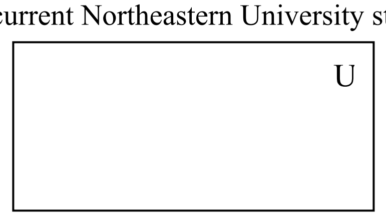
$\left\{x \in \mathbb{Z} \middle| x^2 < 10\right\} =$ the set of integers x such that $x^2 < 10$ $= \left\{-3, -2, -1, 0, 1, 2, 3\right\}$.

$\left\{n \in \mathbb{N} \middle| n^2 < 10\right\} =$ the set of natural numbers n such that $n^2 < 10$ $= \left\{0, 1, 2, 3\right\}$.

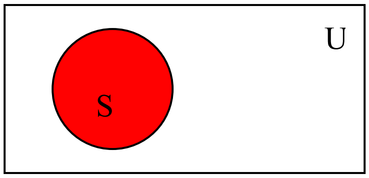
$\left\{z = 2k \textit{ and } k \in \mathbb{N}\right\} =$ the set of z such that $z = 2k$ where k is a natural number $= \left\{0, 2, 4, 6, \cdots\right\} =$ the positive even integers.

2 Venn Diagrams

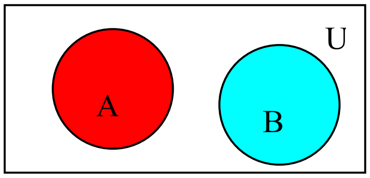
Venn Diagrams are a way of using pictures to describe sets and set operations. The first thing we do is draw a *universe* or ***universal set***, U. U contains all the objects that we might want to be in the sets we are talking about, for example, U might be all real numbers or all people living today or all current Northeastern University students. We use a rectangle to show U.



We then show a single set S with elements in the universe U like this.



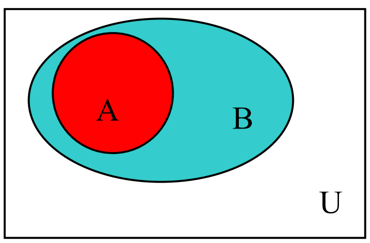
If two sets A and B with elements from the universe U have no elements in common, we say A and B are *disjoint*. Here is a Venn diagram showing the relationship between A and B.



We say A is a *subset* of B (or A *is included in* B or B *includes* A) if every element of A is also an element of B. We write

2. $A \subseteq B$ if A is a subset of B, In this case, A might be equal to B.
3. $A \subset B$ if A is a subset of B but A is not equal to B. This means that every element of A is an element of B but there is at least one element of B that is not an element of A. We say A is ***a proper subset*** of B.

Here is a Venn diagram that shows $A \subseteq B$ or $A \subset B$.

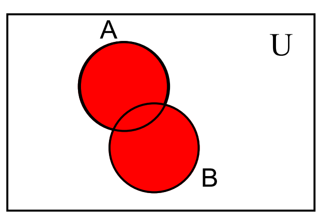


3 Set Operations

Just like we use arithmetic operations, e.g. +, -, *, /, to combine numbers to yield new numbers, we use set operations to combine sets to form new sets. The basic set operations follow with definitions, corresponding Venn diagrams, and examples.

$\cup$ ***Union***

$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$



Let $A = \{1,2,3,4\}$, $B = \{2,4,3,6,7,9\}$, and $C = \{1,2,9\}$. Then

$A \cup B = \{1,2,3,4,6,7,9\}$

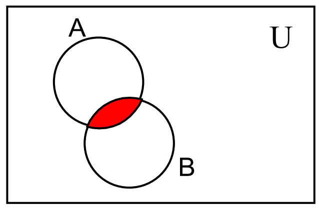
$B \cup C = \{1,2,4,3,6,7,9\}$

$A \cup C = \{1,2,3,4,9\}$.

For any set S, $S \cup \varnothing = S$ and $S \cup U = U$ where U is the universe.

$\cap$ ***Intersection***

$A \cap B = \{\ x \mid x \in A \ and\ x \in B\ \}$



Let $A = \{1,2,3,4\}$, $B = \{2,4,3,6,7,9\}$, and $C = \{1,2,9\}$. Then

$A \cap B = \{2,3,4\}$

$B \cap C = \{2,9\}$

$A \cap C = \{1,2\}$.

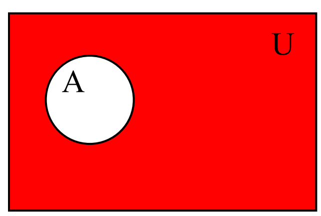
For any set S, $S \cap \varnothing = \varnothing$ and $S \cap U = S$ where U is the universe.

Sets, A and B are disjoint if and only if $A \cap B = \varnothing$.

If A and B are finite sets, the cardinality of $A \cup B$, is given by $|A \cup B| = |A| + |B| - |A \cap B|$. This is the **Principle of Inclusion-Exclusion**. When we add up $|A| + |B|$, we have counted all the elements of A and all the elements of B but we have counted the elements in $A \cap B$ twice so we must subtract that number to get the correct result. The statement seems obvious but often proves to be a stumbling block in the counting problems we will get to later in the semester.

$\overline{A}$ ***Complement***

$\overline{A} = \{\ x \mid x \in U \ and\ x \notin A\ \}$



Let $A = \{1,2,3,4\}$, $B = \{2,3,4,6,7,9\}$, $C = \{1,2,9\}$, and $U = \{0,1,2,3,4,5,6,7,8,9\}$. Then

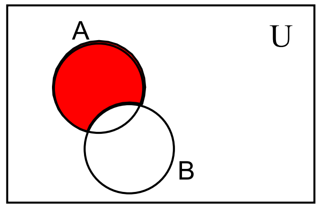
$\overline{A} = \{0,5,6,7,8,9\}$

$\overline{B} = \{0,1,5,8\}$

$\overline{C} = \{0,3,4,5,6,7,8\}$.

$\overline{\varnothing} = U$ and $\overline{U} = \varnothing$.

Difference



$A - B = \{ x \mid x \in A \text{ and } x \notin B \}$
$A - B = A \cap \overline{B}$

Let $A = \{1,2,3,4\}$, $B = \{2,3,4,6,7,9\}$, and $C = \{1,2,9\}$. Then

$A - B = \{1\}$
$B - A = \{6,7,9\}$
$B - C = \{3,4,6,7\}$
$C - B = \{1\}$
$A - C = \{3,4\}$
$C - A = \{9\}$.

For any set S, $S - \varnothing = S$ and $S - U = \varnothing$ where U is the universe.

All of the set operations introduced above yield sets in the same universe as the original sets. We now look at two ways of building sets from a given set or sets that result in a set with elements from a different universe.

Power Set

If A is a set, the *power set* $\wp(A)$ is the set of all subsets of A. We often need to use subsets of a set when we model a real problem, e.g. if the universe is all Northeastern students, we may want to consider possible subsets of students in particular classes, or dorms, or teams.

If $A = \{1, 2\}$ then $\wp(A) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$. The elements of $\wp(A)$ are sets, not numbers. In general, the cardinality of the power set $|\wp(A)| = 2^{|A|}$ and sometimes we use 2^A instead of $\wp(A)$ to denote the power set of A.

More Examples

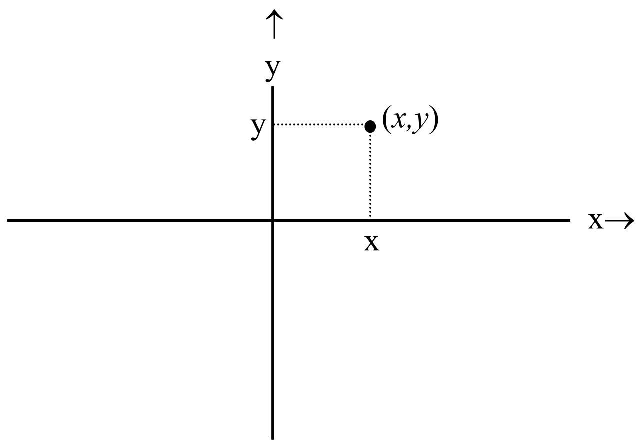
If $A = \{a, b, c\}$ then $\wp(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

$\wp(\varnothing) = \{\varnothing\}$ which is not the same as $\varnothing$. The empty set $\varnothing$ has no elements but it does have one subset, $\varnothing$.

If $S = \{x\}$, $\wp(S) = \{\varnothing, \{x\}\}$ and $\wp(\wp(S)) = \{\varnothing, \{\varnothing\}, \{\{x\}\}, \{\varnothing, \{x\}\}\}$. Note that $|S| = 1$, $|\wp(S)| = 2 = 2^{|S|}$, and $|\wp(\wp(S))| = 4 = 2^{|\wp(S)|}$.

Cartesian Product

You learned about one particular Cartesian product $\mathbb{R}\times\mathbb{R}$ back in high school when you drew graphs of functions. $\mathbb{R}\times\mathbb{R}=\{(x,y)|x,y\in\mathbb{R}\}$ and it is usually visualized as a plane.



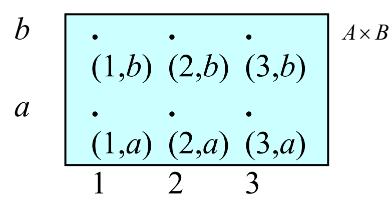
Points correspond to *order pairs* (x,y). Unlike sets, order matters when we write an ordered pair. The ordered pairs (1, 2) and (2, 1) are not equal whereas the sets {1, 2} and {2, 1} are equal. The Cartesian product is named after Rene Descartes.

We can define the *Cartesian product* $A\times B$ of any two sets, *A* and *B,* in a similar way.

$$A\times B=\{(x,y)|x\in A \text{ and } y\in B\}.$$

Example

Let *A* = {1, 2, 3} and *B* = {*a*, *b*}, then $A\times B=\{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$. We can visualize $A\times B$ similarly to the way we visualized $\mathbb{R}\times\mathbb{R}$.



In general, the cardinality of $A\times B=|A\times B|=|A|\times|B|$.

We often need the Cartesian product of many sets, e.g. A × B × C. The elements of the Cartesian product A × B × C are similar to ordered pairs but they have three components instead of two, e.g. (a, b, c). As with ordered pairs, the order matters. We call such an ordered triple a *3-tuple*. An *n-tuple* has *n* components.

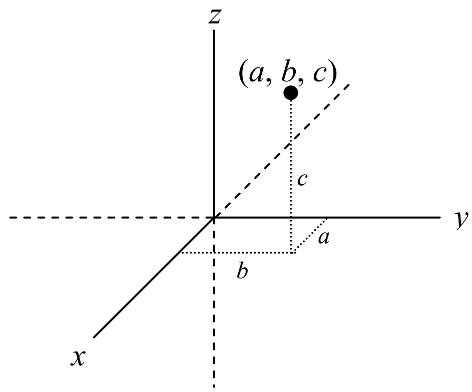
Examples

Tuples

(7, 5, 0,-3, 11, 4) is a 6-tuple of integers.
(Fell, Felleisen, Aslam) is a 3-tuple of CCIS professors.
(Aslam, Felleisen, Fell) is a different 3-tuple of CCIS professors.

Cartesian Products

$\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) | x, y, z \in \mathbb{R}\}$ is usually used to represent 3- dimensional space.



If $A = \{1, 2\}$, $B = \{a, b\}$, and $C = \{X, Y\}$ then
$A \times B \times C = \{(1, a, X), (1, a, Y), (1, b, X), (1, b, Y), (2, a, X), (2, a, Y), (2, b, X), (2, b, Y)\}$.

In general, the cardinality of $|A_1 \times A_2 \cdots \times A_n| = |A_1| \times |A_2| \cdots \times |A_n|$.

Computer Representation of Sets:

Just like numbers, sets can be represented on a computer by 0^s and 1^s. First, we order the elements of the universe. We use bit-strings whose length is the cardinality of the universe *U* to represent the subsets of *U*. Each position in the bit-string corresponds to an element of *U*. A 1 in some position means the corresponding element is in the set while a zero means the element is not in the set.

Example:
Let $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Here are the representations of some subsets of *U*.

$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ 1111111111 All elements are in *U*.

$S = \{1, 2, 5, 6, 7, 9\}$ 0110011101
$\overline{S} = \{0, 3, 4, 8\}$ 1001100010
This is the bit-wise complement of the bit-string for *S*.

$A = \{2, 5, 6, 7, 9\}$ 0010011101
$B = \{1, 4, 6, 8, 9\}$ 0100101011

$A \cap B$ 0000001001
This is the bit-wise **and** of the bit-strings for *A* and *B*.

$A \cup B$ 0110111111
This is the bit-wise **or** of the bit-strings for *A* and *B*.

Reference

Kenneth H. Rosen, *Discrete Mathematics and Its Applications, Fifth Edition*, McGraw Hill, 2003, New York, ISBN 0-07-242434-6, 82-85, 93-94.