Composite Expressions

A mathematical function has one *independent* variable and one or more *dependent* variable(s). For example, if

$$y = f(x) = x^3$$

then x is the independent variable and y is the dependent variable. Once we choose a value for x, the value of y is determined. A function may have more than one independent variable, for example, if

$$z = G(u, v) = \frac{3u^2}{1 + v^2}$$

then z is the independent variable and u and v are the dependent variables.

Often, the "independent" variable(s) actually depends on other variable(s). The cost C of a 100 mile car trip depends on the number of gallons of gas consumed N and the cost of a gallon of gas g.

$$C = N \times g.$$

But N in turn depends on miles per gallon, mpg.

$$C = N \times g = \frac{100}{mpg} \times g.$$

We might also observe that the number of miles per gallon mpg is inversely proportional to the wind resistance r which is directly proportional to the square of the speed s or the car.

$$mpg = \frac{K_1}{r} = \frac{K_1}{K_2 \times s^2} = \frac{K}{s^2},$$

where K_1 and K_2 are constants and $K = K_1/K_2$. So

$$C = N \times g = \frac{100}{mpg} \times g = \frac{100}{\frac{K}{s^2}} \times g = \frac{100s^2g}{K}.$$

We also know that cost of a gallon of gas is dependent on events in Iraq and New Orleans but these dependencies are difficult to express mathematically.[1]

Now, let's return to the mathematical functions we introduced above. With our first function, $y = x^3$, we might have $x = 2t + 1$. Then

$$y = x^3 = (2t + 1)^3 = 8t^3 + 12t^2 + 6t + 1$$

We can now think of y as a dependent on x or dependent on t.

[1] Paul A. Foerster, *Algebra and Trigonometry: Functions and Applications*, Addison Wesley, Menlo Park, 1980, Exercise 9, page 242.

When $x = 2$,

$$y = x^3 = 2^3 = 8.$$

When $t = 2$,

$$y = (2t + 1)^3 = (2 * 2 + 1)^3 = 5^3 = 125.$$

Similarly, the variables u and v might each depend on other variables. If $u = r^5$ and $v = s^3$ then

$$z = \frac{3u^2}{1 + v^2} = \frac{3(r^5)^2}{1 + (s^3)^2} = \frac{3r^{10}}{1 + s^6}.$$

It is also possible that u and v both depend on the same variable, for example, they might both vary with a time variable t. If $u = t - 1$ and $v = 1/t$ then

$$z = 3u^2/(1 + v^2) = 3(t - 1)^2/(1 + (1/t)^2) = 3(t^2 - 2t + 1)t^2/(t^2 + 1) = 3(t^4 - 2t^3 + t^2)/(t^2 + 1)$$

We can now think of z as a function of u and v or as a function of r and s or as a function of a single variable t.

When $u = 2$ and $v = 3$,

$$z = \frac{3u^2}{1 + v^2} = \frac{3 * 2^2}{1 + 3^2} = \frac{12}{10} = \frac{6}{5}.$$

When $r = 2$ and $s = 3$,

$$z = \frac{3r^{10}}{1 + s^6} = \frac{3 * 2^{10}}{1 + 3^6} = \frac{300}{1 + 729} = \frac{300}{730} == \frac{30}{73}.$$

When $t = 2$,

$$z = \frac{3(t^4 - 2t^3 + t^2)}{t^2 + 1} = \frac{3(2^4 - 2 * 2^3 + 2^2)}{2^2 + 1} = \frac{3(16 - 16 + 4)}{5} == \frac{12}{5}.$$