As you get settled...

now playing: The Postal Service
"Such Great Heights"
"The District Sleeps alone
tonight"

- Get out your notes
- Get out a place to do today's ICA (6)
    - Please remember to write your name, my name, the ICA #, and the date!
- We'll start with a linear perceptron warm-up question!
- (then we'll do lots of matrix math :D)

A: yay! B: good C: meh D: apprehensive
E: not so good :^

$f(\vec{x}) = 1$ if $\vec{x}\cdot\vec{w} >= 0$ else 0

Matrix Math & Manipulations

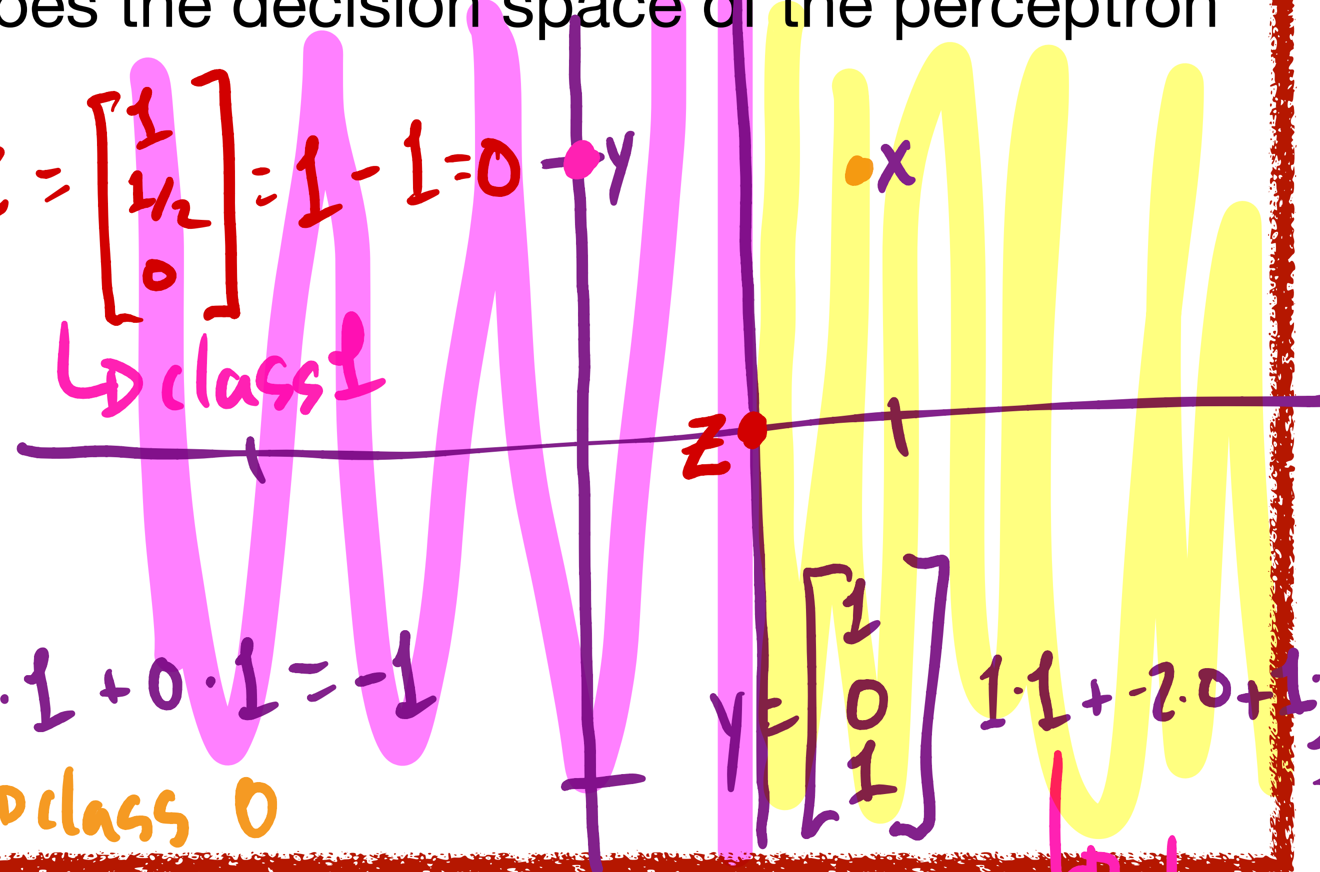
We learned that a linear perceptron is defined by a set of weights. Suppose that I gave you the following weights. What does the decision space of the perceptron look like?

$$\vec{x} = \begin{bmatrix} b \\ x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ x_0 \\ x_1 \end{bmatrix}$$

$$\vec{w} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

$z = \begin{bmatrix} 1 \\ 1/2 \\ 0 \end{bmatrix} : 1 - 1 = 0$

↳ class 1



$x_2 = -\frac{w_0}{w_2} - \frac{w_1}{w_2} x_1$

$x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $1\cdot 1 + -2\cdot 1 + 0\cdot 1 = -1$

↳ class 0

$y = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $1\cdot 1 + -2\cdot 0 + 1\cdot 0 = 1$

↳ class 1

$x_2 = -\frac{1}{0} - \frac{-2}{0} x_1$

$x_1 = \frac{1}{2}$

if we wanted to define our eq'n in terms of x_1 instead of x_2 ...

$$1w_0 + w_1 x_1 + w_2 x_2 = 0$$

$$w_1 x_1 = -w_2 x_2 - w_0$$

$$x_1 = \frac{-w_2}{w_1} x_2 - \frac{w_0}{w_1}$$

gives us an eq'n w/ a **x**-intercept

Matrix Multiplication

- matrix-matrix multiplication

today!

- Matrix multiplication as a function
- Building matrix functions from linearity
  - scaling
  - rotating
- Composing matrix functions $BAx = y$

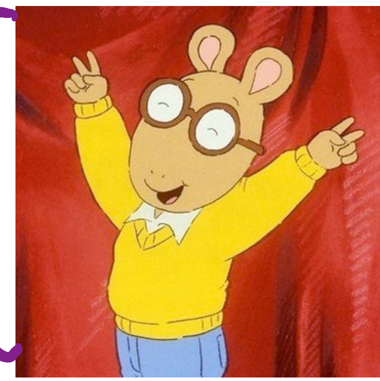
Matrix Multiplication: shape rule

- We'll be working with two matrices today: A (for "Aardvark", aka "Arthur") and B (for "Brontosaurus", aka "Bronty", aka "Brontë")

- Arthur has shape (n, m)

m

n

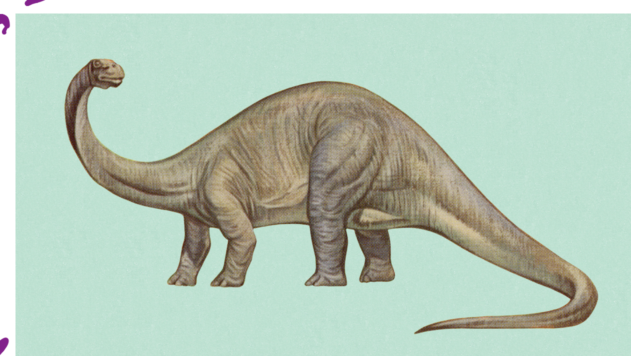


(rows, columns)

q

- Bronty has shape (p, q)

p



Matrix Multiplication: shape rule

- Some shapes are compatible for matrix - matrix multiplication, and **many are not**.

- Key points:

  - **Inner dimensions** must match

  - **Order matters**

AB is not the same as BA

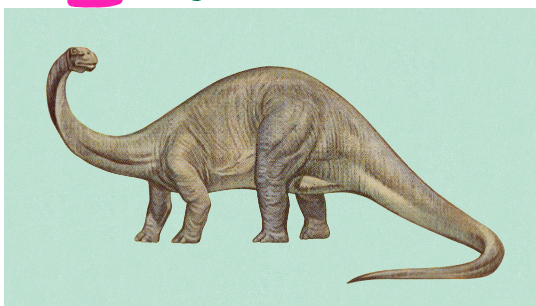
$2*3 = 3*2$

Matrix Multiplication: shape rule

$(3,7)$ → programming

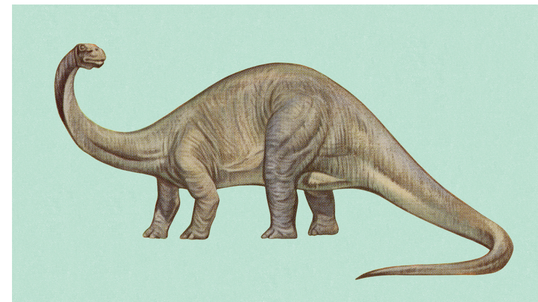
3×3

3×7



$= 3 \times 7$

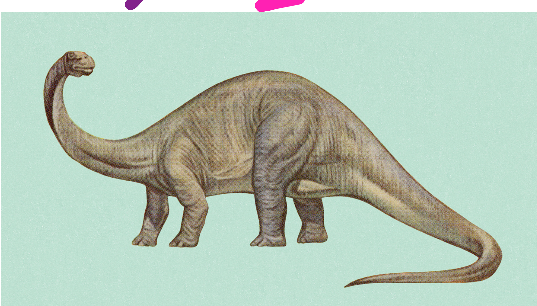
3×7



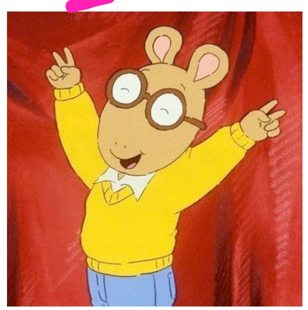
7×2

$= 3 \times 2$

3×7



3×3



$=$ No!

cannot do this

Matrix Multiplication: shape rule

ICA Question 1: for each of the following matrix multiplications (and dimensions), say

a) whether or not it is defined and

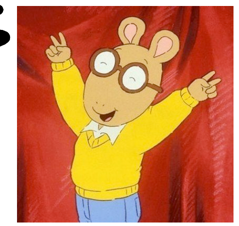
b) what the shape of the resulting matrix would be

A

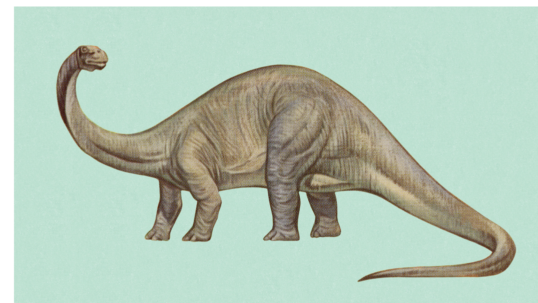
4 x 4

4 x 8

B

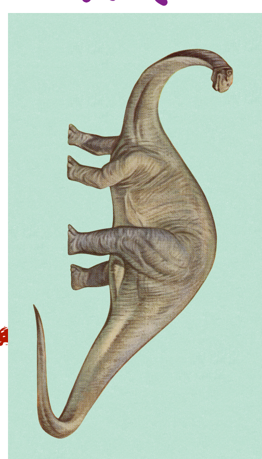


2 x 2



3 x 7

C



7 x 3

3 x 7

Matrix Manipulations: Transposes

- The **transpose** of a matrix is the matrix made by "flipping" the rows to columns

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & -5 & -6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & -4 \\ 2 & -5 \\ 3 & -6 \end{bmatrix}$$

$n \times m$ $\qquad$ $m \times n$

- What is the result of AA^T?

↳ if A is $(n \times m)$ → a $\underline{n \times n}$ matrix

- What is the result of A^TA?

↳ $(m \times n)(n \times m)$ → $m \times m$ matrix

Matrix Multiplication: Computing

- Each element in the product matrix is the dot product of the corresponding **row** from the left matrix and **column** from the right matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & -5 & -6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & -4 \\ 2 & -5 \\ 3 & -6 \end{bmatrix}$$

2×3 $\qquad$ 3×2

$$AA^T = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 & 1 \cdot -4 + 2 \cdot -5 + 3 \cdot -6 \\ -4 \cdot 1 + -5 \cdot 2 + -6 \cdot 3 & -4 \cdot -4 + -5 \cdot -5 + -6 \cdot -6 \end{bmatrix} = \begin{bmatrix} 14 & -32 \\ -32 & 77 \end{bmatrix}$$

2×2

Linear Combinations (weighted sum)

- A **linear combination** of $x_0, x_1, x_2 \ldots$ is $\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 \ldots$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & -5 & -6 \end{bmatrix} \qquad x = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$$

2×3 3×1

$$Ax = \begin{bmatrix} 1 \cdot 2 + 2 \cdot 4 + 3 \cdot -2 \\ -4 \cdot 2 + -5 \cdot 4 + -6 \cdot -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 16 \end{bmatrix}$$

$\hookrightarrow 2 \times 1$

Linear Combinations (matrix-vector)

- A **linear combination** of $x_0, x_1, x_2 \ldots$ is $\alpha x_0 + \alpha_1 x_1 + \alpha_2 x_2 \ldots$

- Matrix-vector multiplication (Ax) is a linear combination of the rows of A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & -5 & -6 \end{bmatrix} \qquad x = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$$

B = 10x3   y = ~~3x1~~ 1x3

- What must be true about the dimensions of the vector that we are multiplying with our matrix?

↳ same # of rows as columns in matrix

↳ in practice → take a transpose if it was switched (1xm) instead of (mx1)

Linear Combinations (Vector-matrix)

- A **linear combination** of $x_0, x_1, x_2 \ldots$ is $\alpha x_0 + \alpha_1 x_1 + \alpha_2 x_2 \ldots$
- Vector-matrix multiplication (xA) is a linear combination of the columns of A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & -5 & -6 \end{bmatrix} \qquad x = \begin{bmatrix} 2 & 4 \end{bmatrix}$$

$xA = (1, 3)$

- What must be true about the dimensions of the vector that we are multiplying with our matrix?

↳ same # of columns as rows in matrix

⚠ prefer matrix-vector over vector matrix ⚠

Matrix Multiplication: matrix-vector

ICA Question 2: which matrix-vector multiplications are defined given the below matrices and vectors? Do those matrix-vector multiplications

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} -5 & -6 \\ 0 & 1 \end{bmatrix} \quad x = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad y = \begin{bmatrix} -2 \\ -3 \\ -4 \end{bmatrix}$$

Ay = a 3x1 matrix

Bx = a 2x1 matrix

please do show your work!

Another way to write matrix-vector multiplications

A: Great

E: Bad!

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \quad y = \begin{bmatrix} -2 \\ -3 \\ -4 \end{bmatrix}$$

$$Ay = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ -4 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + -3 \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + -4 \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$Cz = \begin{bmatrix} | & | & & | \\ c_0 & c_1 & \cdots & c_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_n \end{bmatrix} = v_0 \begin{bmatrix} c_0 \end{bmatrix} + v_1 \begin{bmatrix} c_1 \end{bmatrix} + \cdots + v_n \begin{bmatrix} c_n \end{bmatrix}$$

Another way to write matrix-vector multiplications

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \quad y = \begin{bmatrix} -2 \\ -3 \\ -4 \end{bmatrix}$$

$$Ay = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \cdot -2 + 2 \cdot -3 + 3 \cdot -4 \\ 1 \cdot -2 + 2 \cdot -3 + 3 \cdot -4 \\ 1 \cdot -2 + 2 \cdot -3 + 3 \cdot -4 \end{bmatrix} = \begin{bmatrix} -20 \\ -20 \\ -20 \end{bmatrix}$$

Break: until 12:50

HW2 → released tomorrow → Prof. Higger and I are finalizing wording this afternoon!!

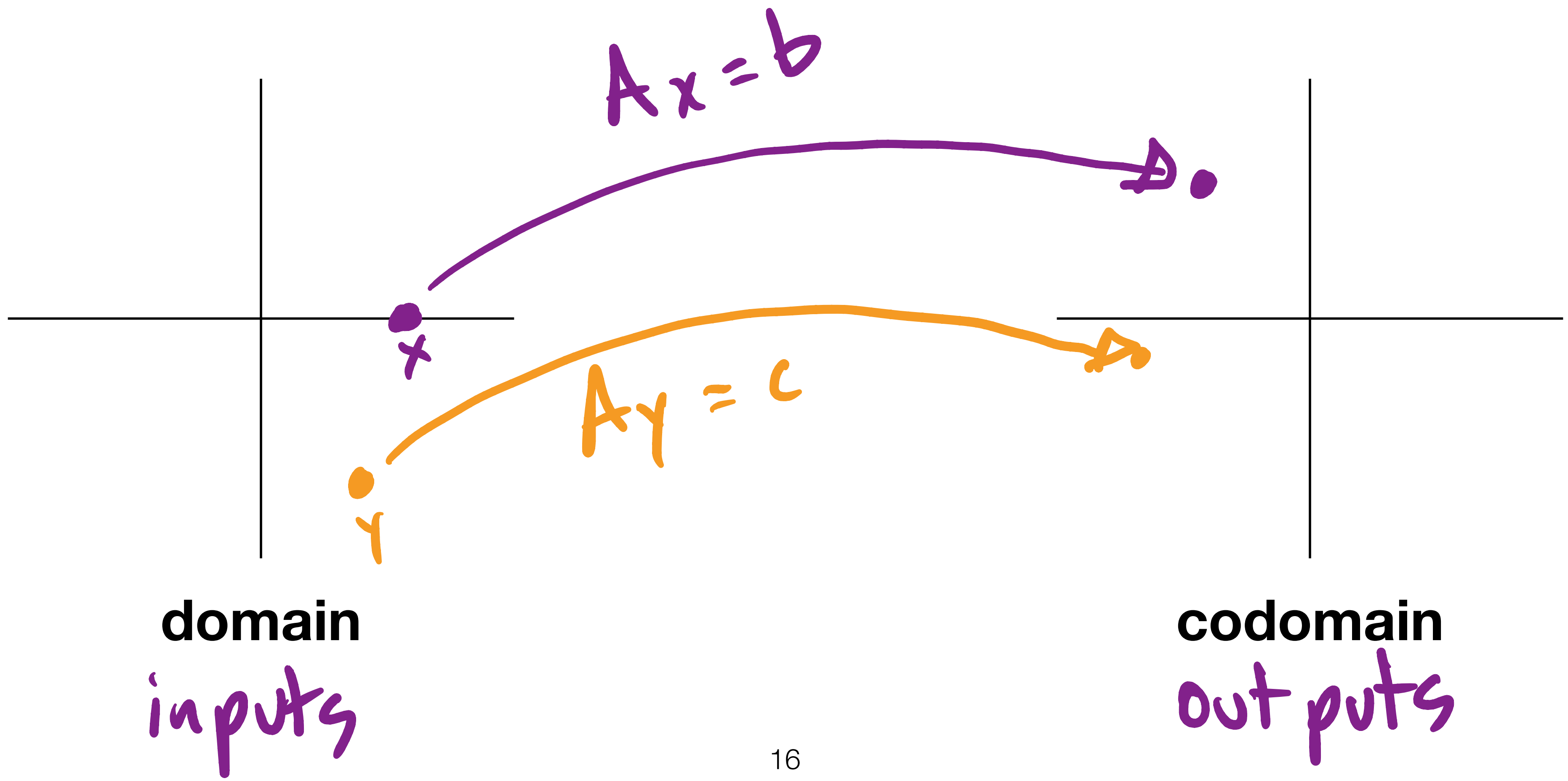
Matrix-Vector multiplication as a function

- We can write any function as a mapping from one domain to another:
- $f(x) = x + 1$ $\quad f : \mathbb{R} \to \mathbb{R}$
- Now, say that we have $A \in \mathbb{R}^{2x2}$
  - consider $f : \mathbb{R}^{2x1} \to \mathbb{R}^{2x1}$ outputs $\begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$

input $\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$

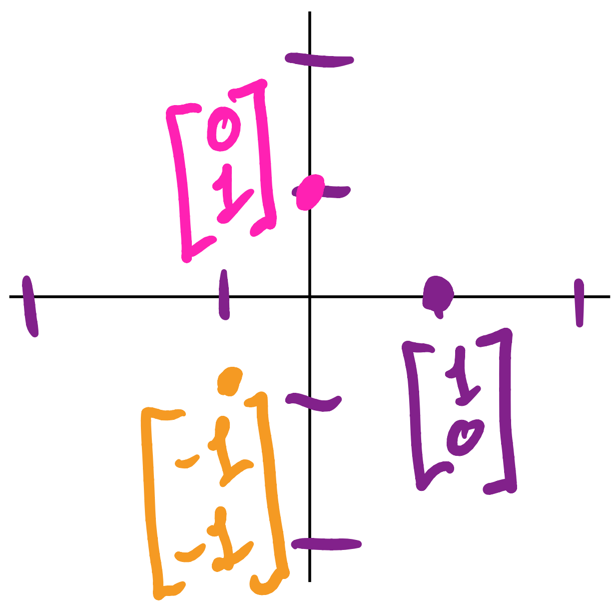
Matrix-Vector multiplication as a function

- Now, say that we have $A \in \mathbb{R}^{2x2}$
  - consider $f : \mathbb{R}^{2x1} \to \mathbb{R}^{2x1}$

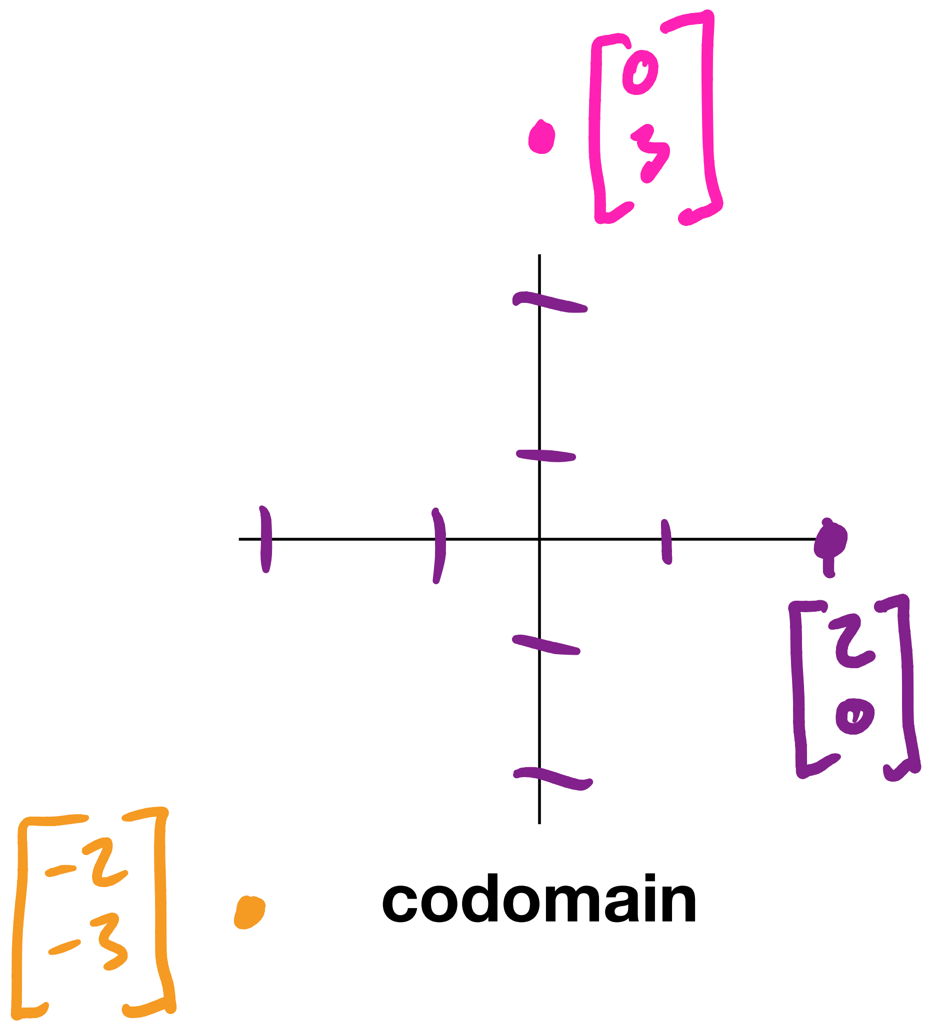


Building transforms

- consider $f : \mathbb{R}^{2x1} \rightarrow \mathbb{R}^{2x1}$
  - double x_0 magnitude
  - triple x_1 magnitude



domain



codomain

Building transforms

$$Ax = \begin{bmatrix} 2x_0 \\ 3x_1 \end{bmatrix}$$

- Let a_0, a_1 be column vectors of A $\rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$
  - $A = \begin{bmatrix} | & | \\ a_0 & a_1 \\ | & | \end{bmatrix}$
- Let:
  - $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} | & | \\ a_0 & a_1 \\ | & | \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} a_0 \end{bmatrix} + 0\begin{bmatrix} a_1 \end{bmatrix} = a_0$
  - $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} a_0 & a_1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0a_0 + 1a_1 = a_1$

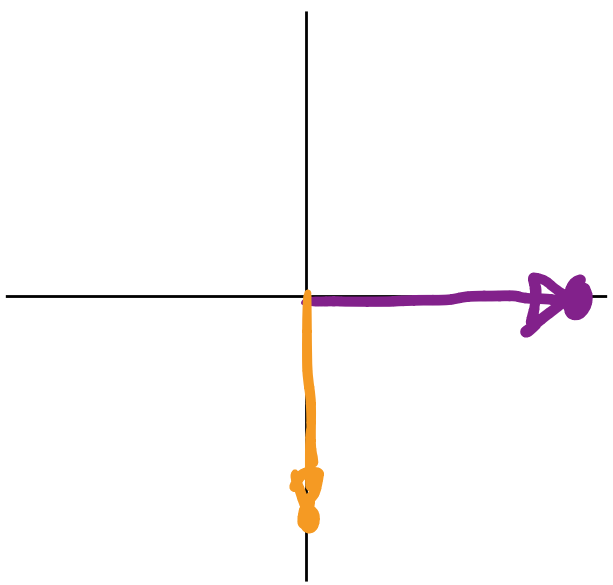
Matrix multiplication: it's linear!

- $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$
- All matrix-matrix multiplications are linear
- $\underline{A}(\alpha x + \beta y) = \alpha \underline{A} x + \beta \underline{A} y$
- This is how we know that our transform matrix works even though we only built it on two examples!

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \qquad A \begin{bmatrix} 7 \\ 21 \end{bmatrix} = \begin{bmatrix} 14 \\ 63 \end{bmatrix}$$

Building transforms

- consider $f : \mathbb{R}^{2x1} \rightarrow \mathbb{R}^{2x1}$
  - rotate counter clockwise by θ



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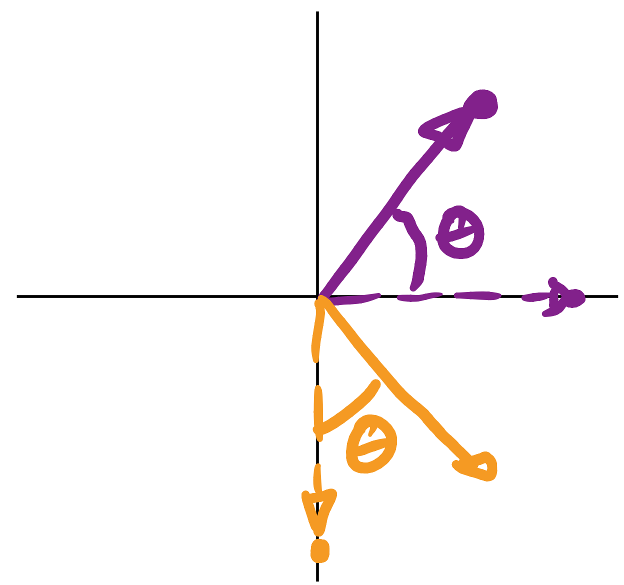

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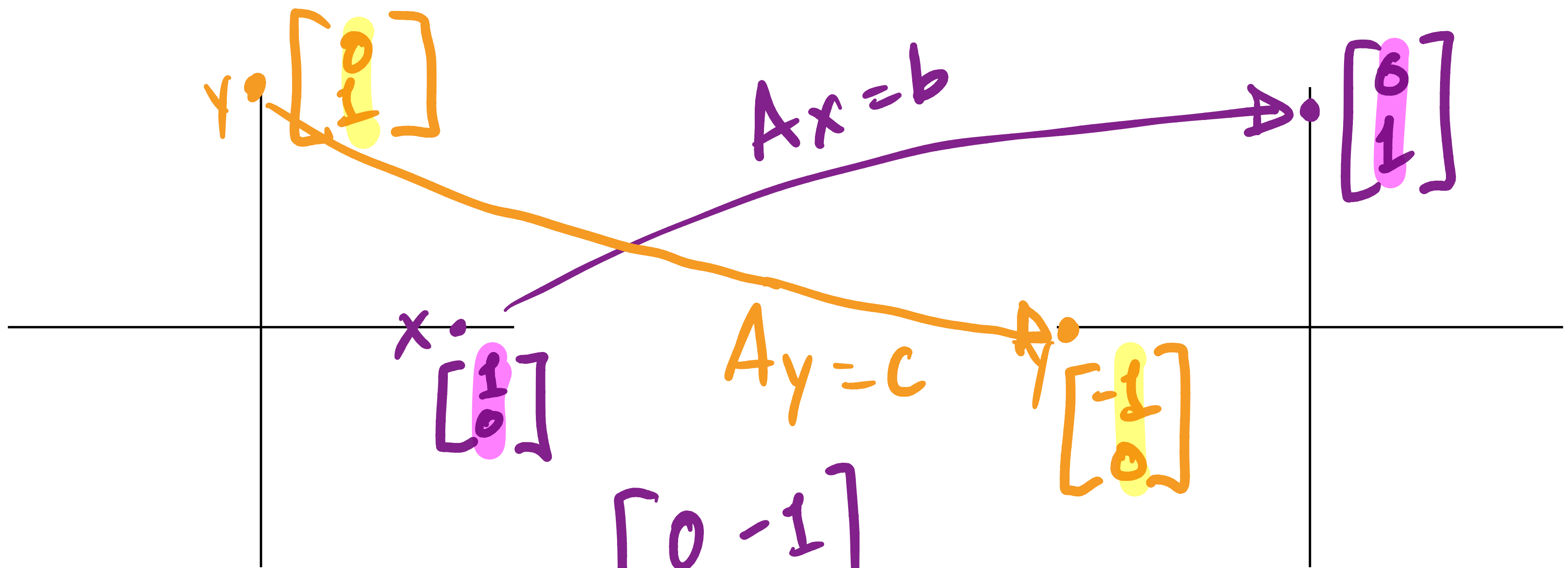
Building transforms

- consider $f : \mathbb{R}^{2x1} \rightarrow \mathbb{R}^{2x1}$
- rotate counter clockwise by $\frac{\pi}{2}$, ~~now expressed generally speaking~~

90° rotation



domain **codomain**

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}| & | \\ a_0 & a_1 \\ | & |\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = 1\begin{bmatrix}a_0\end{bmatrix} + 0\begin{bmatrix}a_1\end{bmatrix} = a_0$$

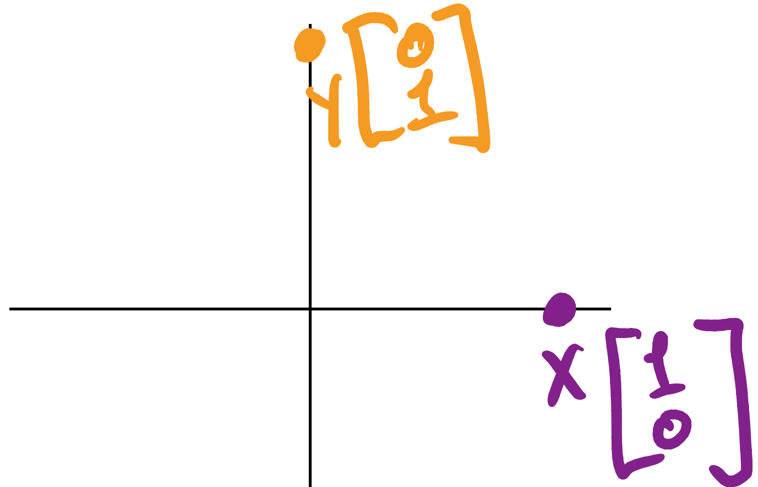
$$A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}| & | \\ a_0 & a_1 \\ | & |\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = 0\begin{bmatrix}a_0\end{bmatrix} + 1\begin{bmatrix}a_1\end{bmatrix} = a_1$$

$$A = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$$

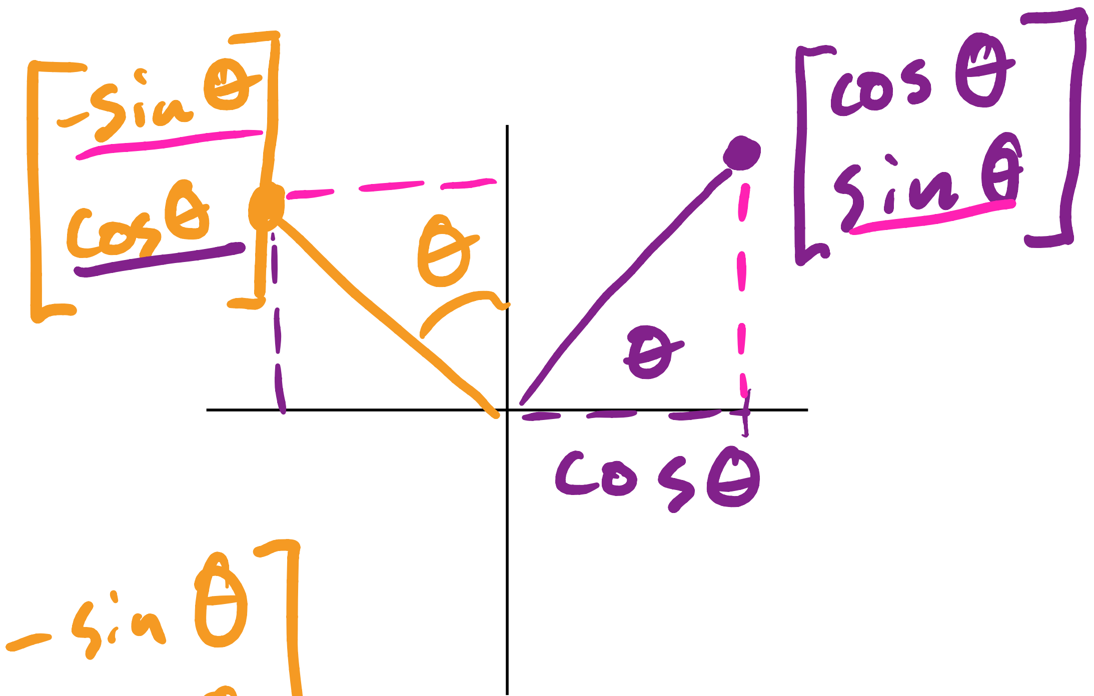
Building transforms

- consider $f : \mathbb{R}^{2x1} \to \mathbb{R}^{2x1}$

- rotate counter clockwise by θ, now expressed generally speaking



domain



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$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Some final matrix multiplication notes

- $AAx = A^2x$
- $ABx = A(Bx)$ (first apply B, then apply A)

Schedule

Turn in ICA 6 on Gradescope

HW 2 will be released tomorrow!

On Monday we'll be in person in Snell Engineering 108. I'll send an announcement with all the details (we'll dial you in if you are sick).

Felix's scheduled office hours will now be entirely on Calendly (currently T, R). Sign up with whatever quandaries you have at least an hour in advance!

They'll also appear on khouryofficehours from time to time.

Mon	Tue	Wed	Thu	Fri	Sat	Sun
January 31st Lecture 5 - Linear Perceptron	**Felix OH Calendly**	**HW 1 due @ 11:59pm**	Lecture 6 - matrix multiplication, transforms **Felix OH Calendly**	**HW 2 released**		
February 7th Lecture 7 - Vector spaces **in Snell Engineering 108**	**Felix OH Calendly**		Lecture 8 - line of best fit **Felix OH Calendly**			**HW 2 due @ 11:59pm**