Line of best fit → will be a video in the future

↳ so you get everything you need for HW 3 now

Admin

- For my ICAs for my lectures, we are moving to the following format:
- Every lecture, you will answer *the same* three questions:

1. What did you learn from this lecture?
2. What are you confused about?
3. (a question about either an ICA or a homework problem)

- I will stop lecture 10 minutes early for you to do this. You are expected to do this during class time.

determinants, inverses, change of basis

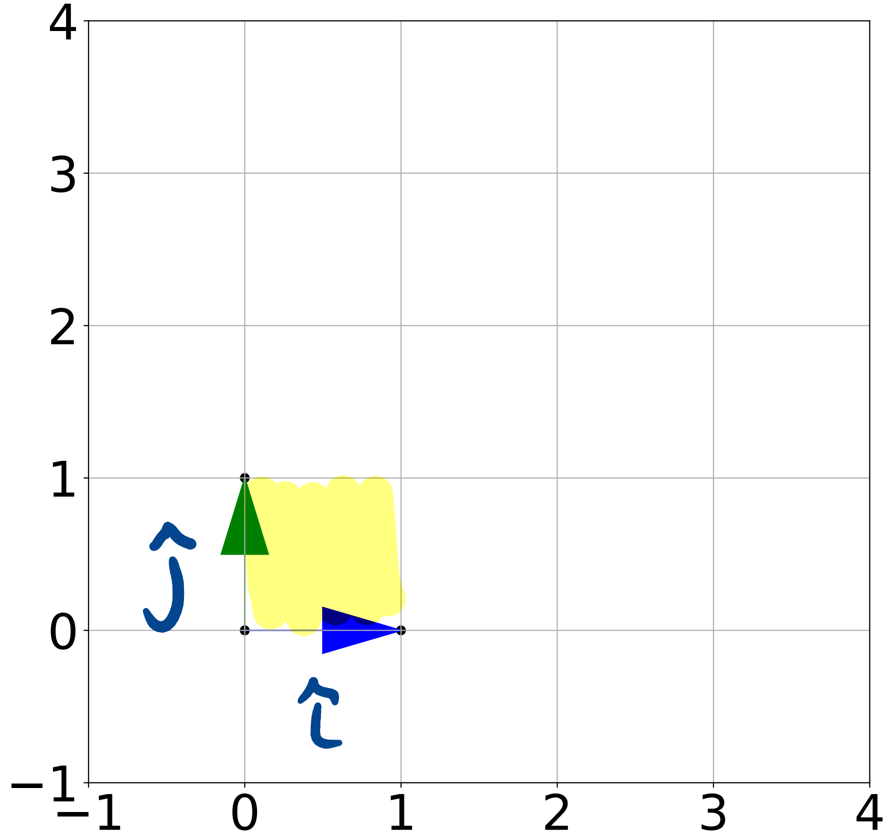
↳ content for HW 3 + HW 4

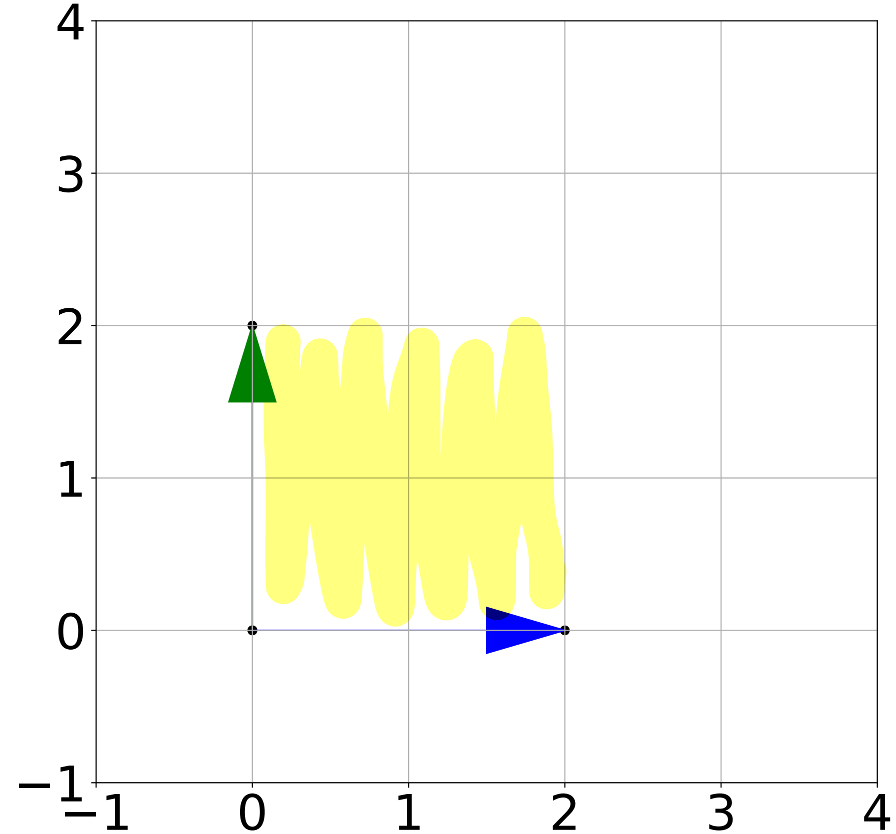
is on the website now

Determinants

det = 4

- Say that you have a transformation defined by the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$



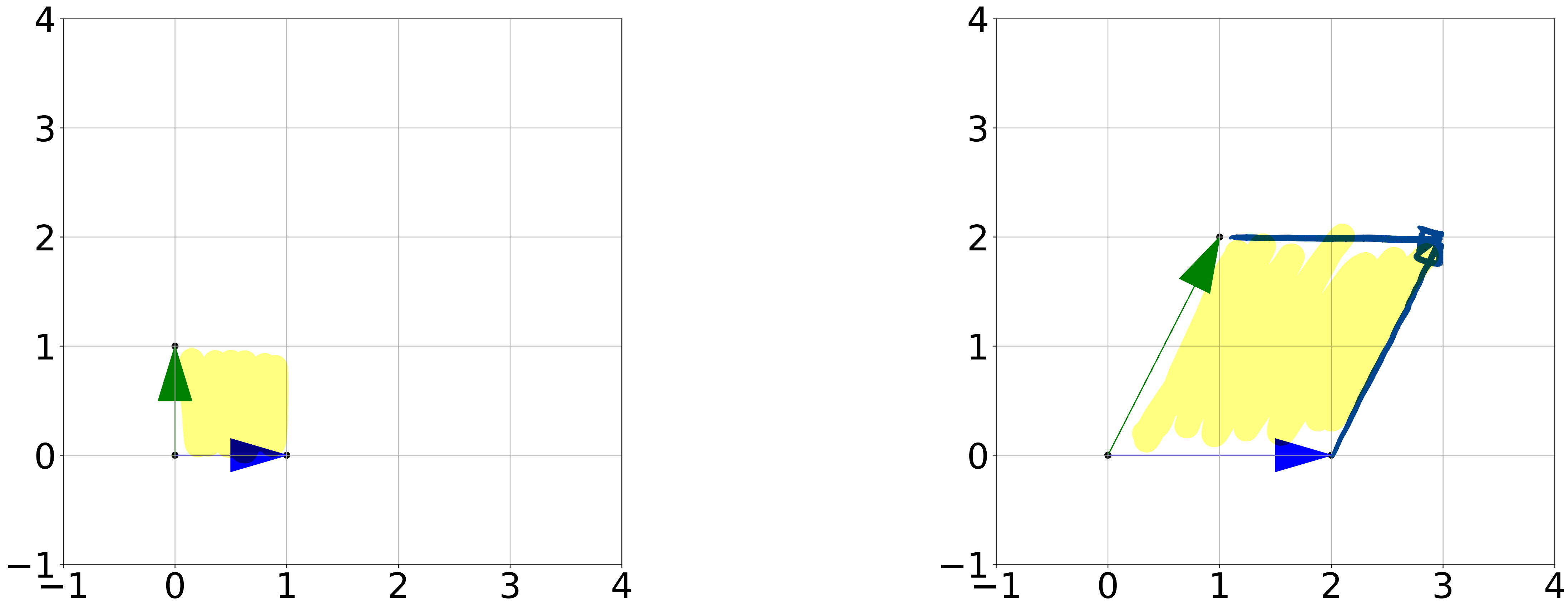


4 times bigger

Determinants

$\det = 4$

- Say that you have a transformation defined by the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$



4 times bigger + slanty!

Determinants

- Why? → a scalar
- Tell us how this matrix would squish or stretch space when applied to vectors

- A determinant of >1 would mean that space was stretched
- A determinant of $0-1$ would mean that space was squished

(0, 1) → same size

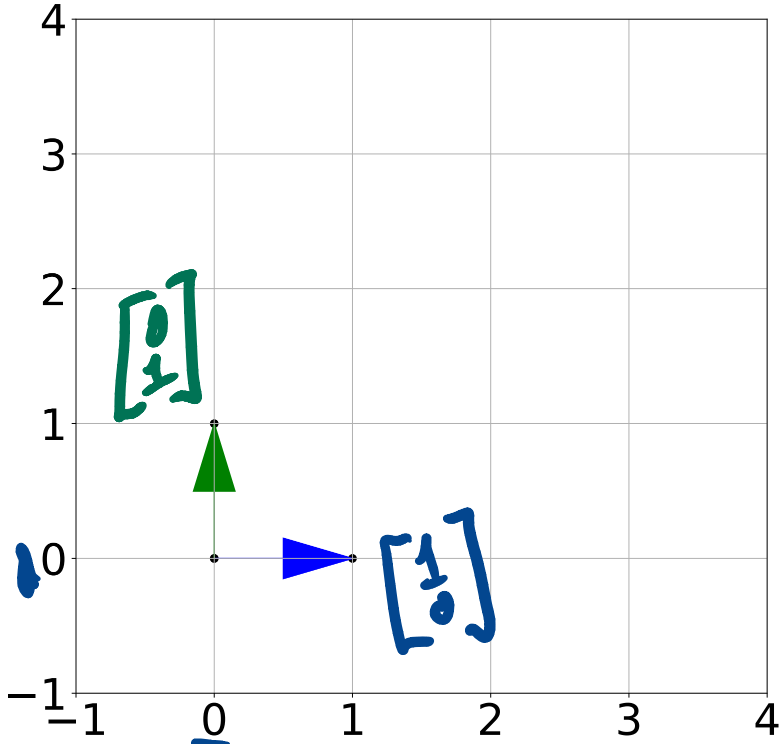
special!

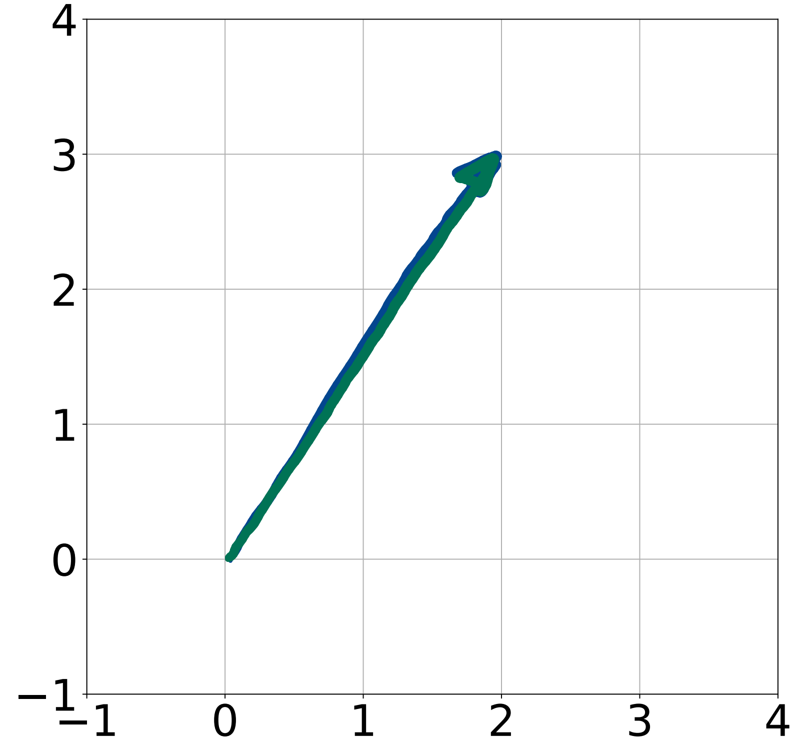
Determinants - ICA Question 1

Say you have this matrix: $A = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$

First, draw the resulting space as applied to $\hat{i}$ and $\hat{j}$.

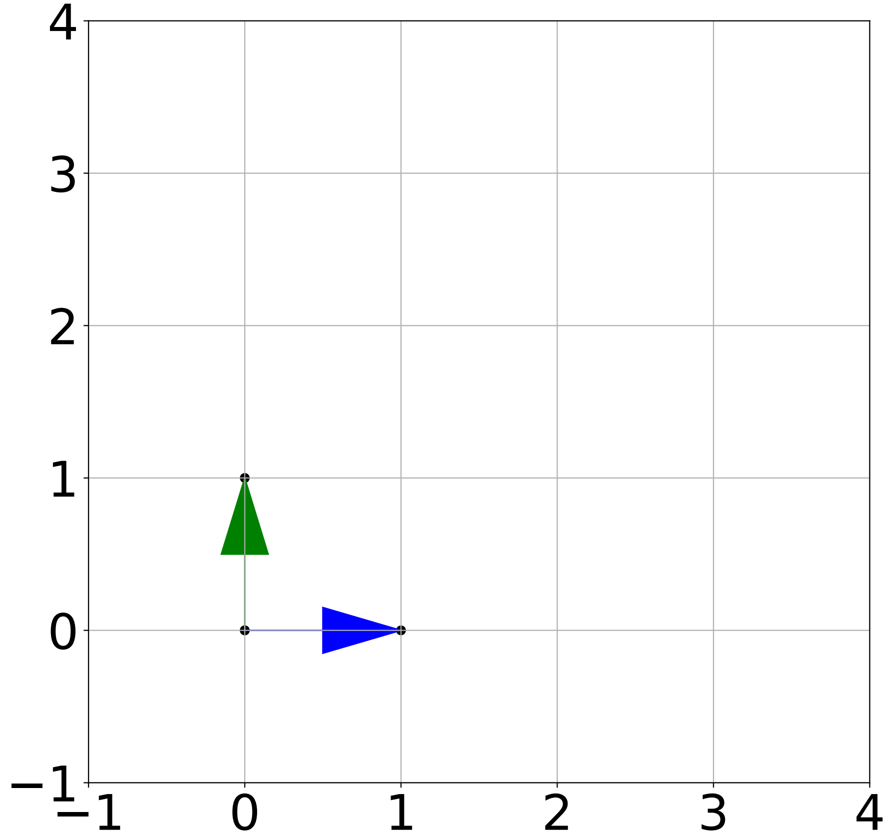
Then, make a proposal for what the determinant should be. → det of 0





Determinants

- A determinant of ___0___ would mean that space was ___reduced in dimensionality___

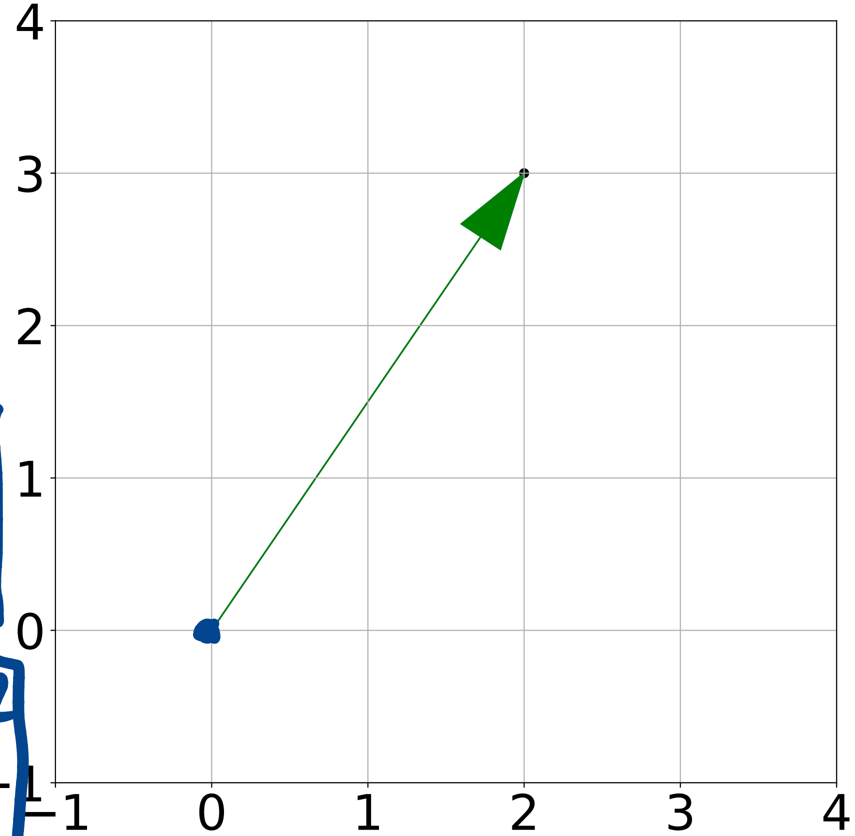


$$A = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

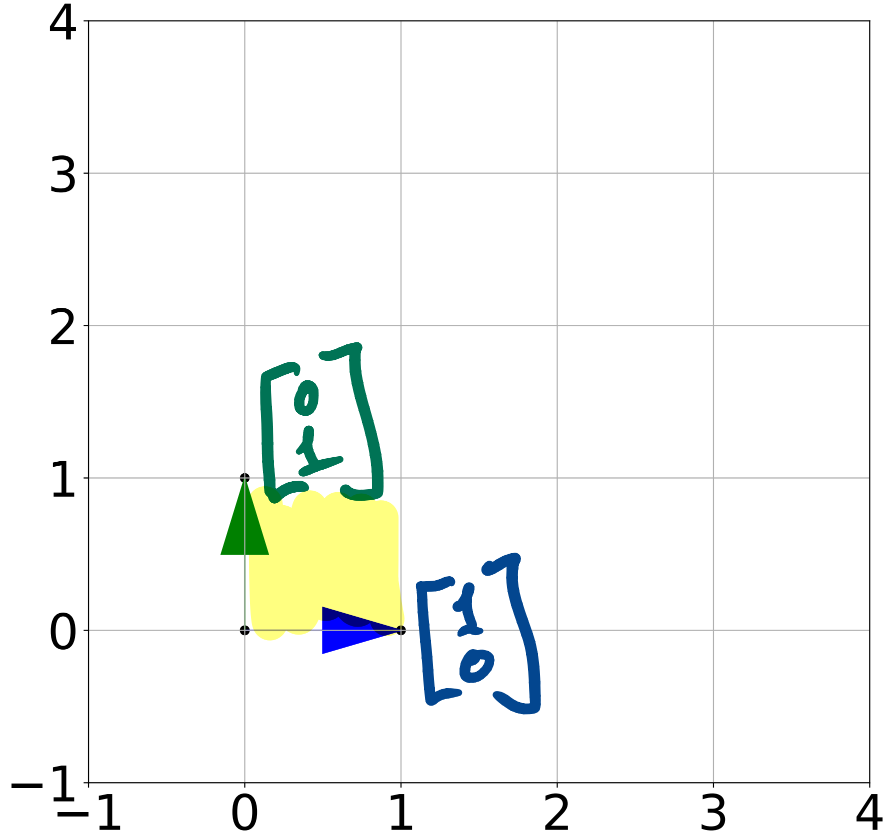
$$D = \begin{bmatrix} -12 & -12 \\ 25 & 25 \end{bmatrix}$$

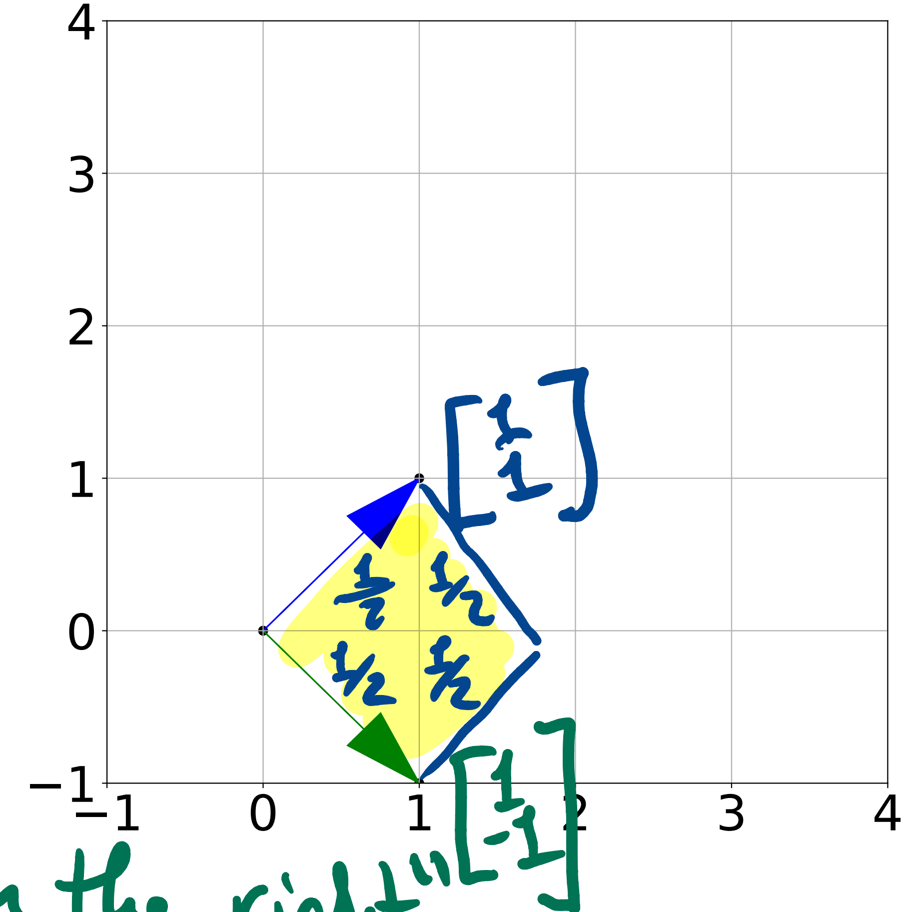


Determinants - inverting space

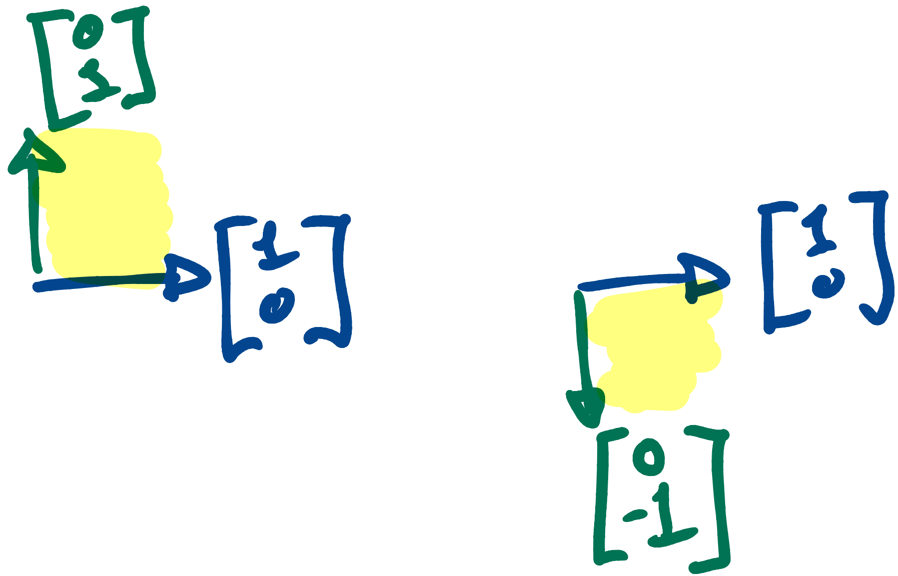
$(1 \cdot -1) - (1 \cdot 1) = -2$

- Say that you have a transformation defined by the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$





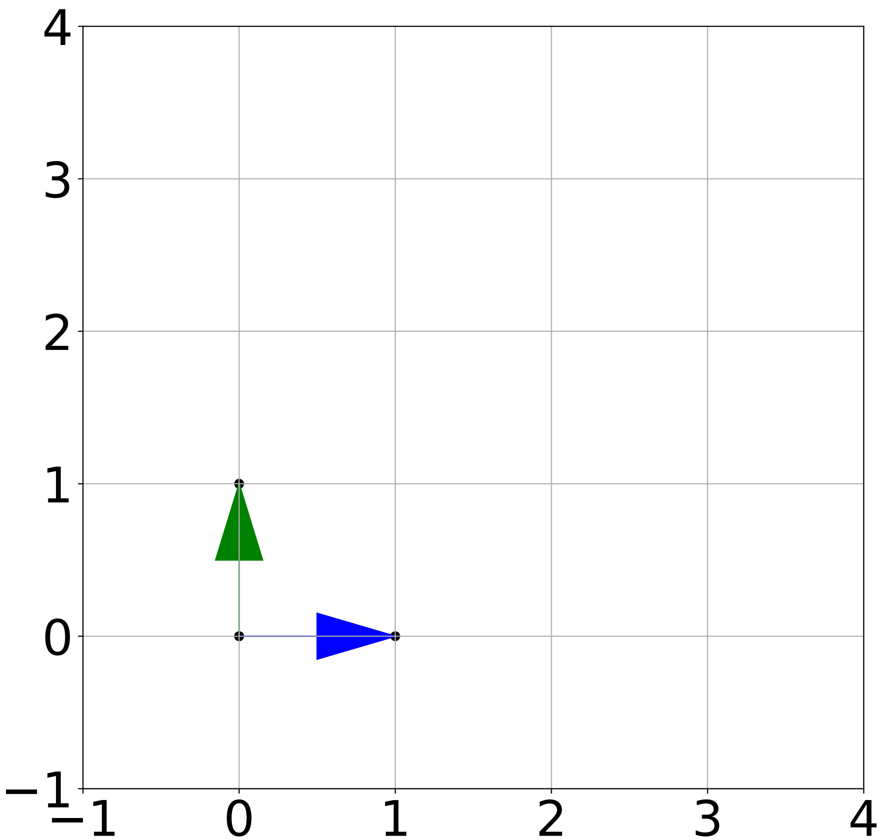
green is now "on the right"

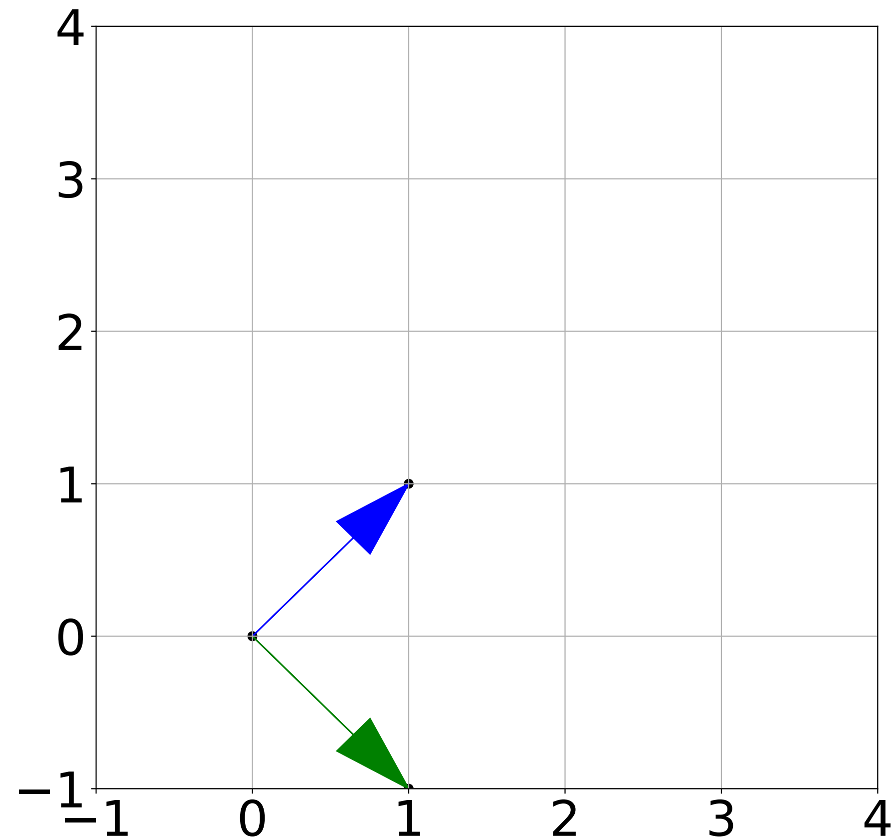


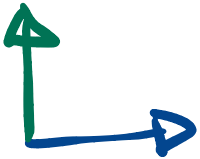
$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (1 \cdot -1) - (0 \cdot 0) = -1$$

Determinants - inverting space

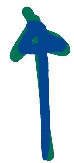
- A determinant that is negative means that space has been flipped



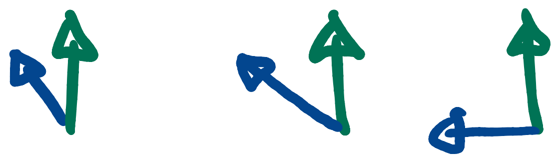




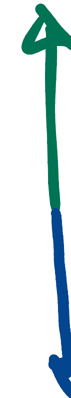
det → approaching 0



det : 0



det → approaching -1

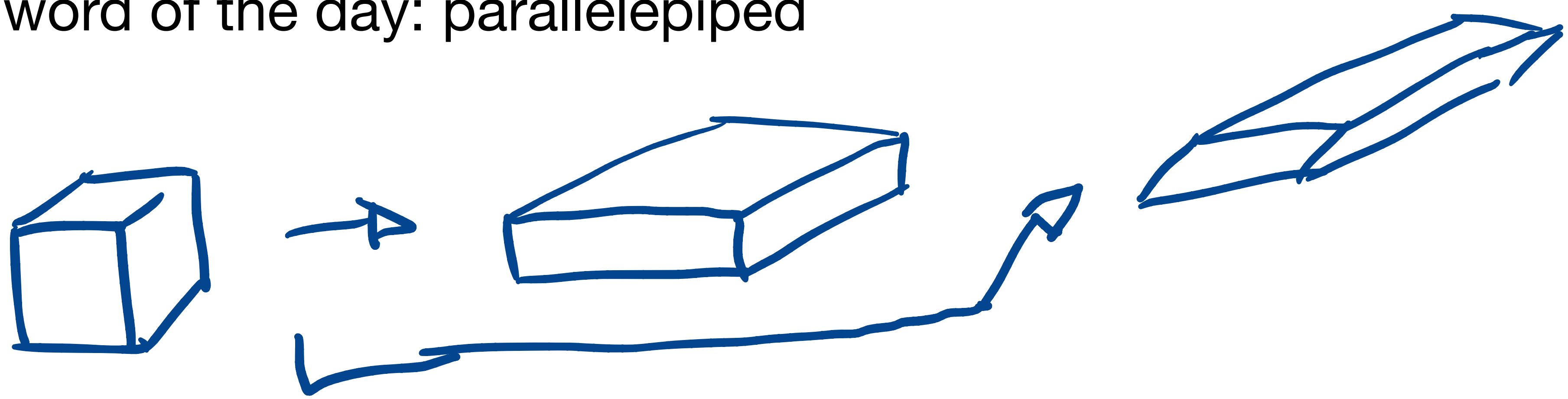


det : 0

Determinants in 3 dimensions

change to 2 dimensions
→ det of 0

- Same idea as 2 dimensions, but this time they tell us about how much volume (instead of area) gets scaled.
- The cube defined by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ might become a taller cube, a wider cube, or a slanty and squished cube!
- Fun word of the day: parallelepiped
-



Calculating determinants

- With a 2 x 2 matrix, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is: ad - bc

$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} = (2 \cdot 3) - (2 \cdot 3) = 6$

$\begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix} = (1 \cdot 2) - (-3 \cdot 4)$

$= 14$

- For a 3 x 3 matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ ... this is "complicated"

↳ combined scaled det. of 2x2 matrices

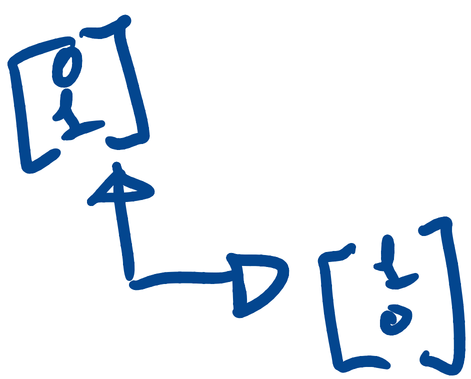
Calculating determinants

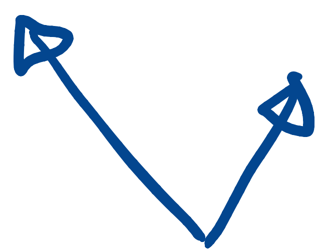
- But really just ask python/your computer to do this for you: np.linalg.det(matrix)

In [16]:
```python
import numpy as np

# getting the determinant of a matrix
A = np.array([[1, 2, 3], [0, 5, 0], [1, 2, -2]])
print(np.linalg.det(A))
```

```
-24.999999999999996
```

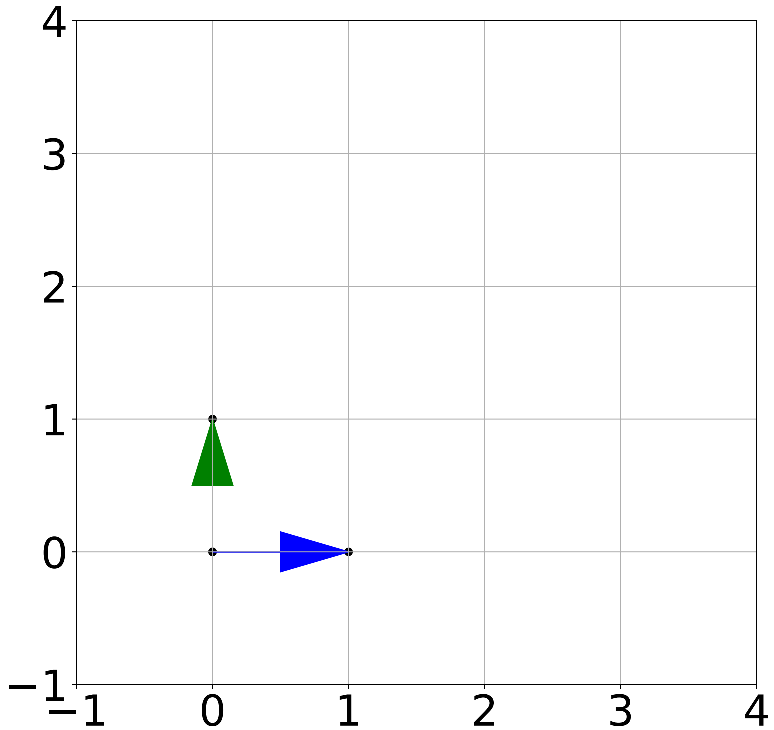




Inverses

- We saw these last lecture!
- We know that $A^{-1}A = I$ where I is the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- But what is happening is space?

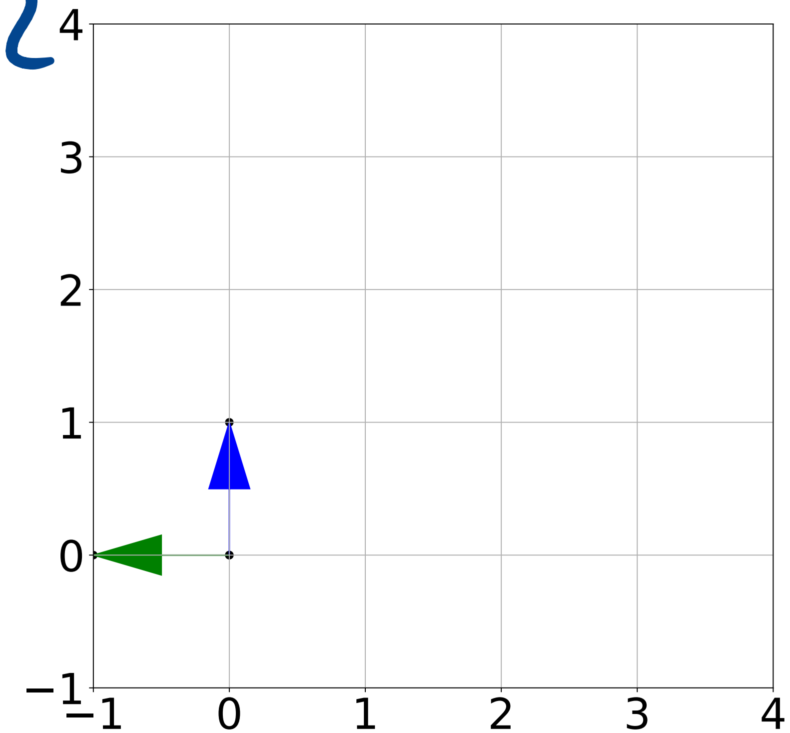
#1



$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Counter clockwise

#2

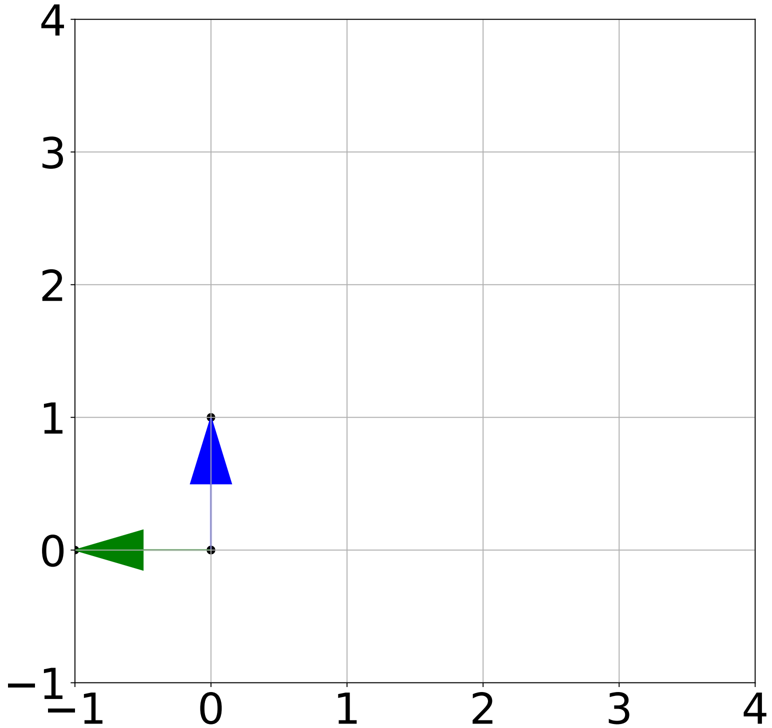


Inverses

- Spatially, how do we get "back to" our original vectors after applying A?

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

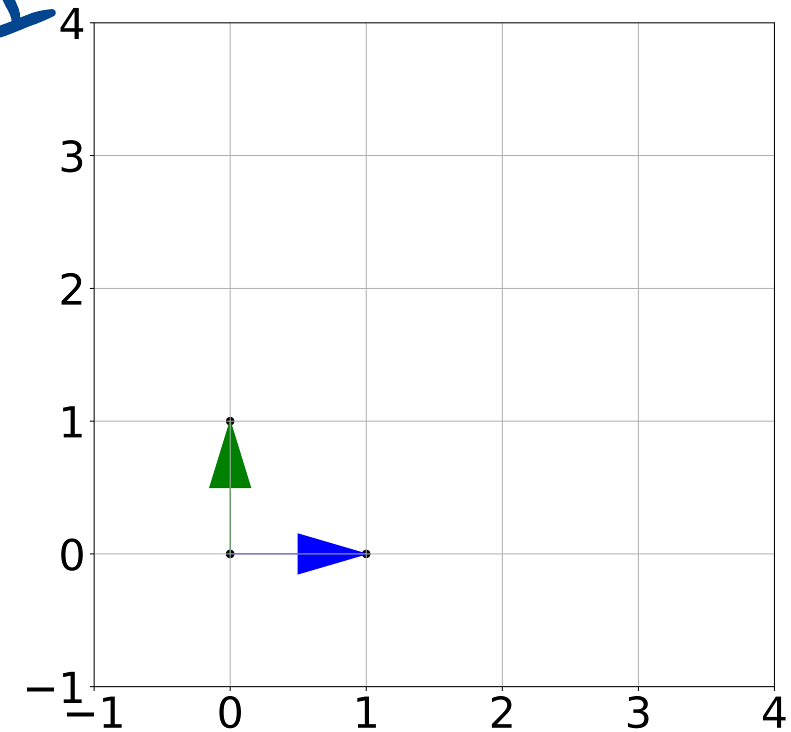
#2



$$A^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

a clockwise rotation

#1



Inverses

- Inverses have some neat properties!
- if $Ax = b$ then $x = A^{-1}b$
- $A^{-1}A = I$

Inverses

- In practice (in the real world), you'll ask your computer for the inverse of a matrix when needed

you will also need to know how to do this by hand

In [17]:

```
import numpy as np

# get the inverse of a matrix
B = np.array([[0, 1], [-1, 0]])
B_inv = np.linalg.inv(B)
print(B_inv)

# get the inverse of a matrix
C = np.array([[1, 2], [4, 5]])
C_inv = np.linalg.inv(C)
print(C_inv)
```

```
[[-0. -1.]
 [ 1.  0.]]
[[-1.66666667  0.66666667]
 [ 1.33333333 -0.33333333]]
```

Inverses & Determinants

- How are inverses related to determinants?
- if $det(A) \neq 0$, then the inverse exists
- if $det(A) = 0$, then there is no inverse

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $det(A) = 1$

Inverses

- Inverses have some neat properties!

- Say that you have a system of equations that we can write as:

matrix $\qquad\qquad$ augment

- $$\underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 10 \\ 2 \\ 3 \end{bmatrix}}_{b}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 10 \\ 1 & 1 & 0 & 2 \\ 1 & -1 & 1 & 3 \end{array}\right]$$

$$Ax = b \qquad x = A^{-1} b$$

Inverses - calculating - ICA question 2

Say you have the matrix $A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$ and we want to find A^{-1}.

We know that A^{-1} is of shape 2 x 2 and that $\underset{A}{\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right]$$

→ row operations so that the left side is I

↳ the right side becomes A^{-1}

(Also check your answer by calculating $A^{-1}A$)

Inverses - calculating - ICA question 2

$$\begin{array}{c} r_1 \\ r_2 \end{array}\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right] \rightarrow \quad r_2' = r_2 - 2r_1 \quad \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$\downarrow$$

$$r_1' = r_1 + 2r_2$$

$$\left[\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & \end{array}\right.\underbrace{\boxed{\begin{matrix} -3 & 2 \\ -2 & 1 \end{matrix}}}_{A^{-1}}\left.\vphantom{\begin{matrix}1\\1\end{matrix}}\right]$$

$$\hookrightarrow r_1' = -3r_1 + 2r_2$$

1) solve for a, b, c, d

2) use a formula: $\dfrac{1}{\det}$ [some switching w/ negatives]

Inverses & python

- And with python....

In [21]:

```python
import numpy as np

# get the inverse
A = np.array([[1, 2, -1], [1, 1, 0], [1, -1, 1]])
A_inv = np.linalg.inv(A)
print(A_inv)

# multiply the constants by the inverse to get
# the values of x, y, z
v = np.array([[10, 2, 3]])
print(A_inv @ v.T)
```

```
[[ 1. -1.  1.]
 [-1.  2. -1.]
 [-2.  3. -1.]]
[[ 11.]
 [ -9.]
 [-17.]]
```

x
y
z

Inverses

- Wait, aren't there other ways to calculate inverses?
  - Yes!
- See the resources at the end of the lecture for descriptions of other ways to do this!

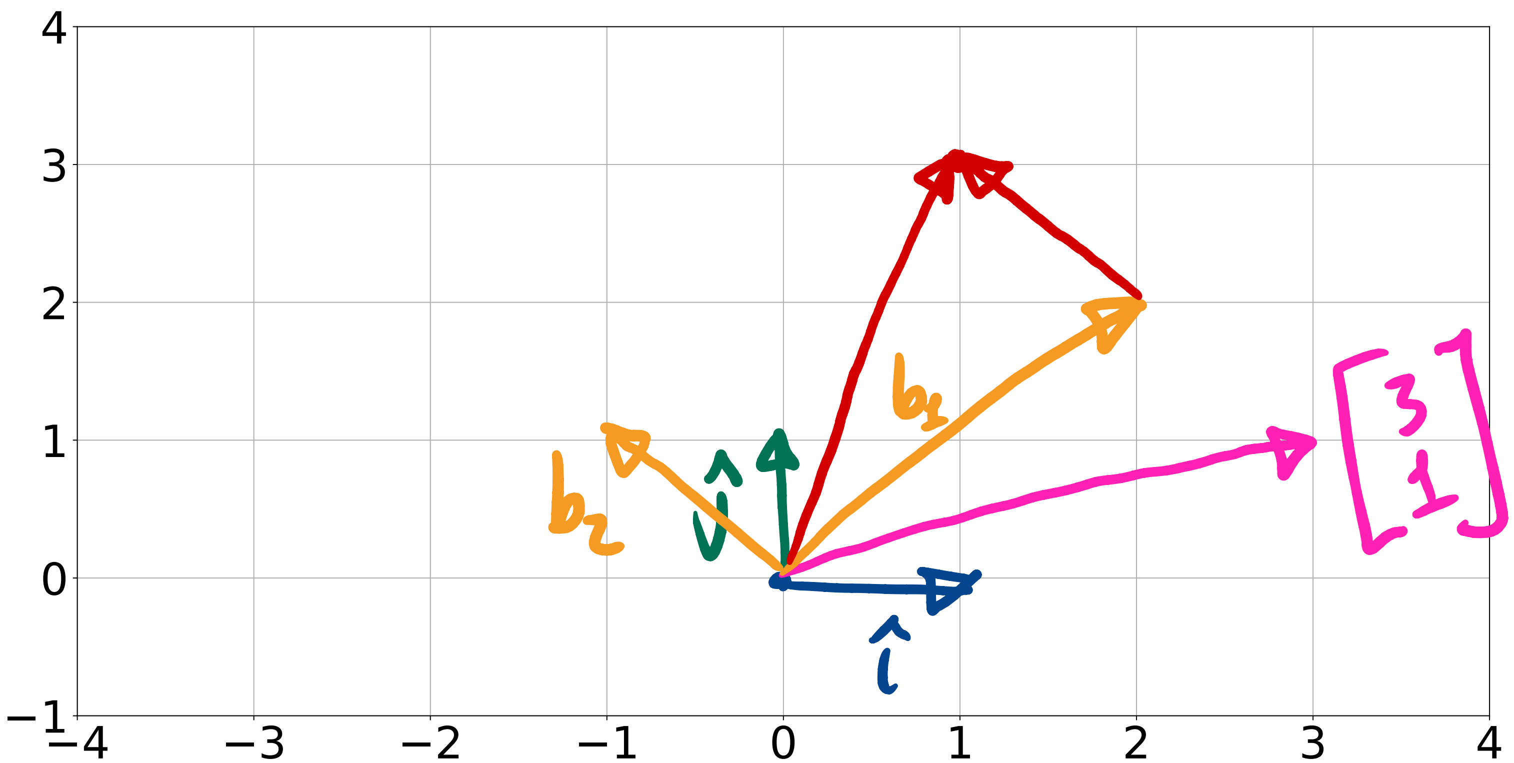
Change of basis

- Recall: when we learned about span we learned that our default **basis vectors** are $\hat{i}$ and $\hat{j}$.

- However, we may want to translate coordinates to/from system with different basis vectors.

$\begin{bmatrix}1\\1\end{bmatrix}$

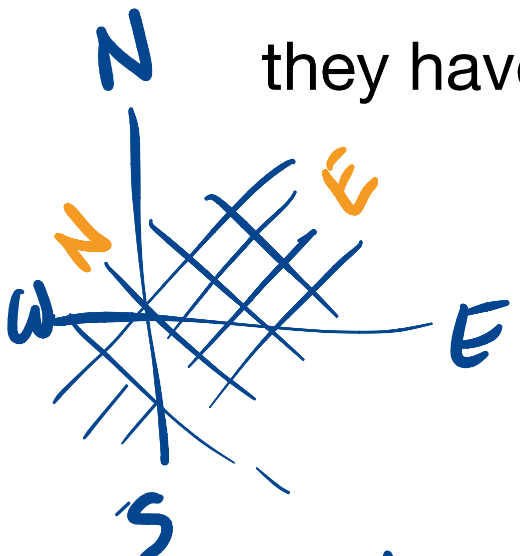
$\begin{bmatrix}1b_1\\1b_2\end{bmatrix}$



$\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}3\hat{i}\\1\hat{j}\end{bmatrix}$

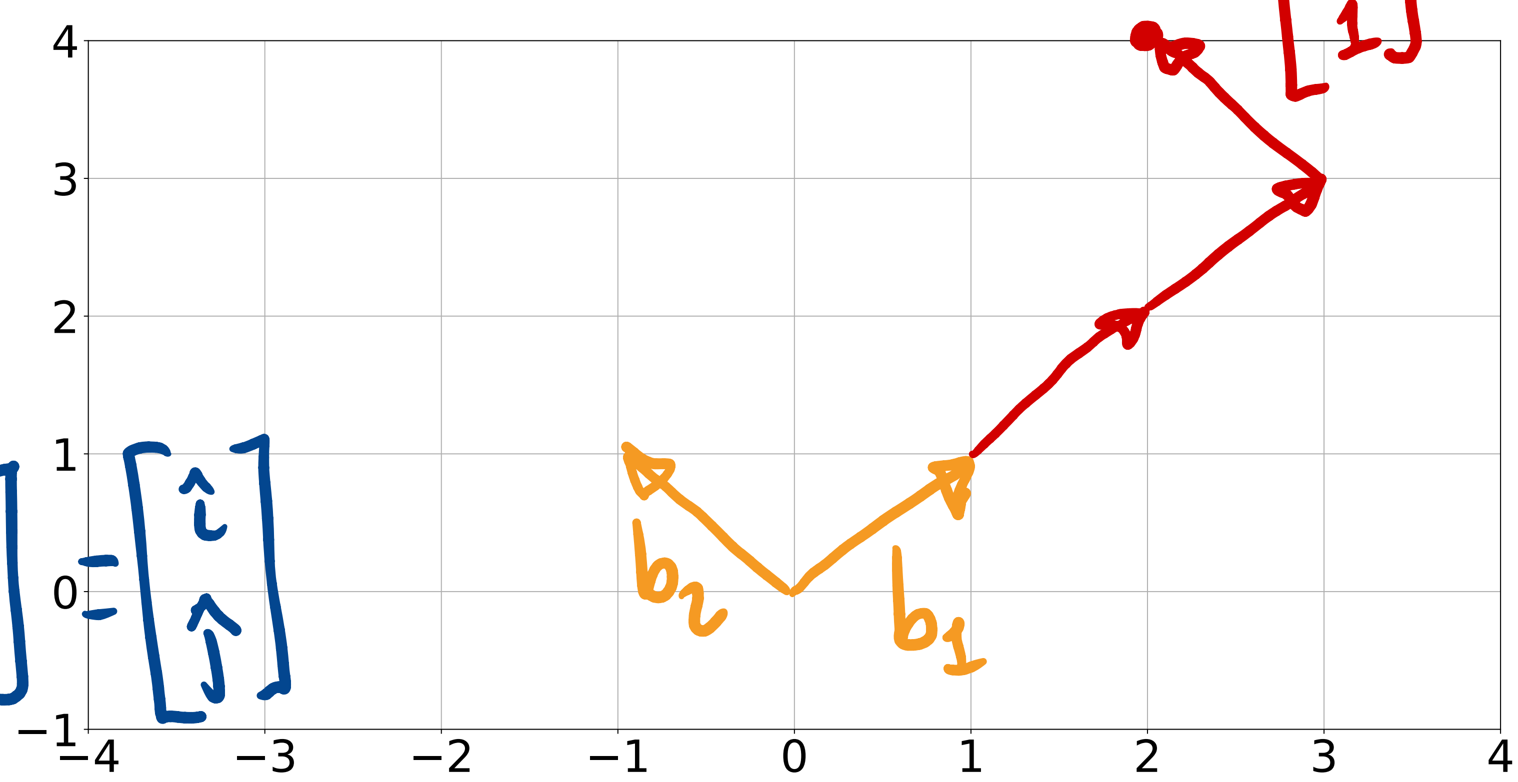
Change of basis

- Our friend tells us that the pizza shop is at $\begin{bmatrix}3\\1\end{bmatrix}\begin{matrix}\text{east}\\\text{north}\end{matrix}$, however, we know that they have redefined "east" to be northeast and "north" to be "northwest"

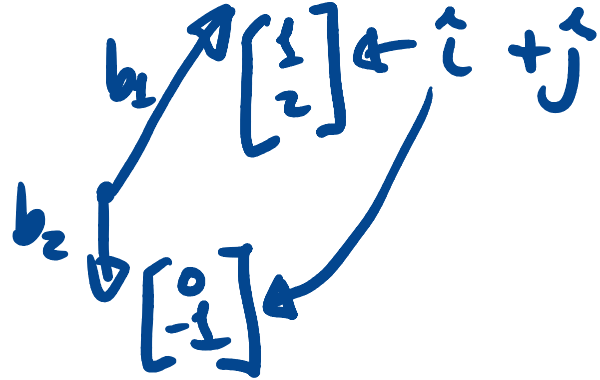


$$Ax = b$$

$$\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}\hat{i}\\\hat{j}\end{bmatrix}$$



$$\begin{bmatrix}3\\1\end{bmatrix} \quad \begin{bmatrix}?\,\hat{i}\\?\,\hat{j}\end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \text{b-based} \\ \text{coord} \end{bmatrix} = \begin{bmatrix} \hat{i} \\ \hat{j} \end{bmatrix}$$

$$A \qquad x \qquad = b$$

$$x = A^{-1} b$$

Change of basis

- To translate any vector from another basis to our basis:
-
$$A = \begin{bmatrix} \text{their basis vectors} \\ \text{according to us} \end{bmatrix} \longleftarrow \hat{i} + \hat{j}$$

$$\text{their co-ord} = \begin{bmatrix} x_t \\ y_t \end{bmatrix} \qquad \text{our co-ord} = \begin{bmatrix} x_u \\ y_u \end{bmatrix}$$

$b_1 + b_2$ $\qquad\qquad$ $\hat{i} + \hat{j}$

$$A \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} x_u \\ y_u \end{bmatrix} \quad \text{from their coord} \rightarrow \text{ours}$$

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = A^{-1} \begin{bmatrix} x_u \\ y_u \end{bmatrix}$$ → translate from us to them

- is a supp. video "linked" at the end

- MUST agree on the origin

Schedule

quiz I/A 9 → "cat"

Turn in ICA 9 on **Canvas**

HW 3 is due on Sunday

Quiz-test 1 is in class on Thursday

notice these due dates!

Mon	Tue	Wed	Thu	Fri	Sat	Sun
February 14th Lecture 9 - determinants, basis, inverses	**Felix OH Calendly**		Lecture 10 - **QUIZ 1 (HW 1 - 2), in class** **Felix OH Calendly**			**HW 3 due @ 11:59pm**
February 21st → don't come to class President's day! Asynchronous lecture to be done before class Thursday, Eigenvectors, dynamical systems	**Felix OH Calendly**		Lecture 12 - intro prob. and stats **Felix OH Calendly**			**HW 4 due @ 11:59pm**

More recommended resources on these topics

- Youtube: "The determinant | Chapter 6, Essence of Linear Algebra" 3Blue1Brown
- Youtube: "Inverse matrices, column space, and null space | Chapter 7, Essence of Linear Algebra" 3Blue1Brown
- Finding the Inverse of a Matrix: https://courses.lumenlearning.com/ivytech-collegealgebra/chapter/finding-the-inverse-of-a-matrix/
- Youtube: "Change of basis | Chapter 13, Essence of Linear Algebra" 3Blue1Brown

Quiz-test 1

- in person on Thurs @ 11:45
- Snell Eng. 108 (here!)
- no calculators
- edit the collaborative study guide on piazza!
- Just on HW 1 + 2 material