CS 2810 Day 9

Starting a few mins late!

Admin:

All python examples may now be run in browser
- You may find the python helpful to compute inverses in HW 3

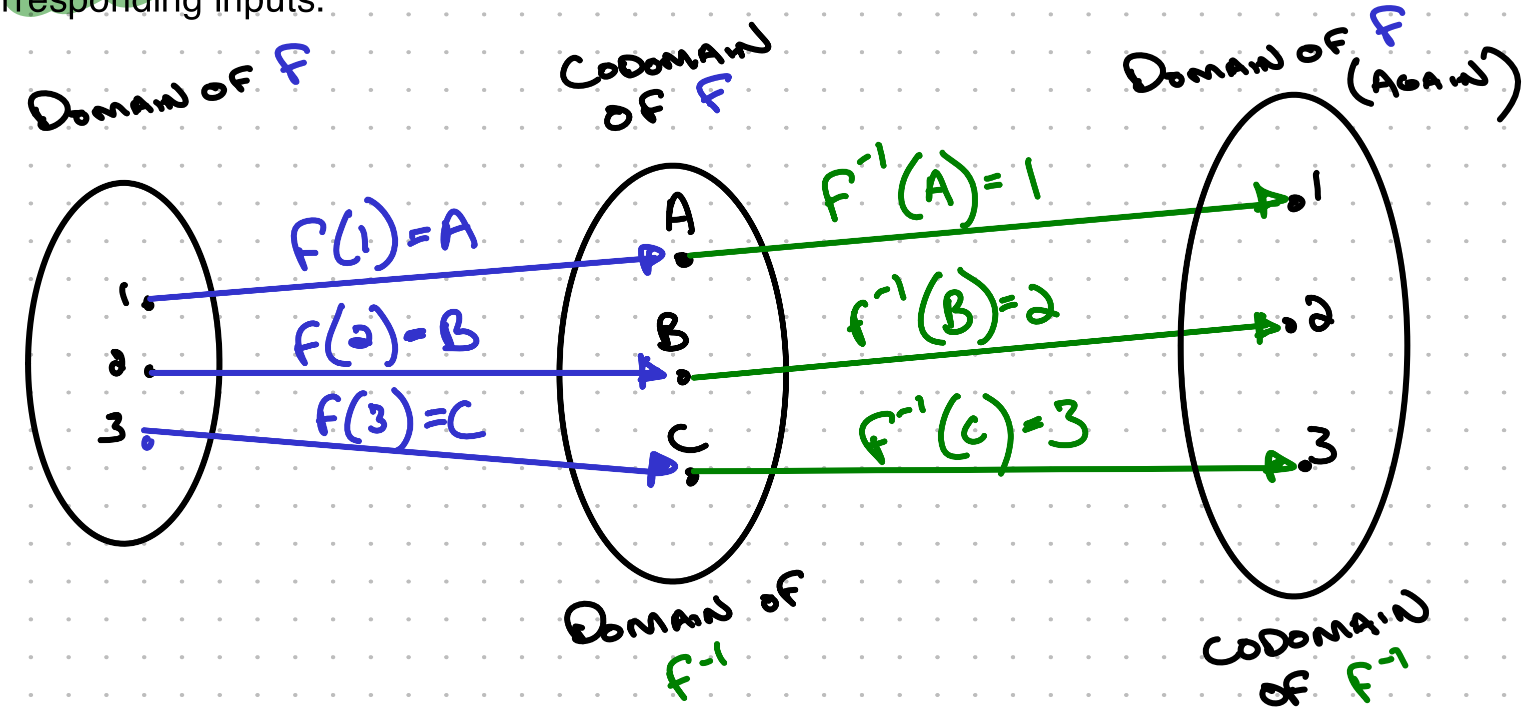
Quiz on Friday

Content:

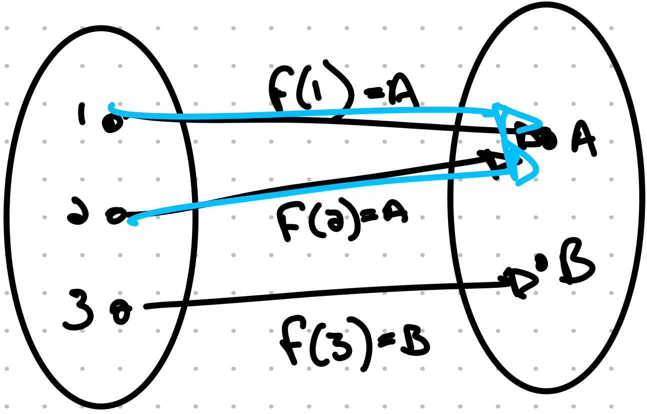
Inverses
- What are they?
- When do they exist (in general, for a matrix transform)
- Computing the inverse of a matrix transform

Change of basis (via Image Registration example)

The inverse of function f is another function, f^-1, which maps the outputs of f back to their corresponding inputs.



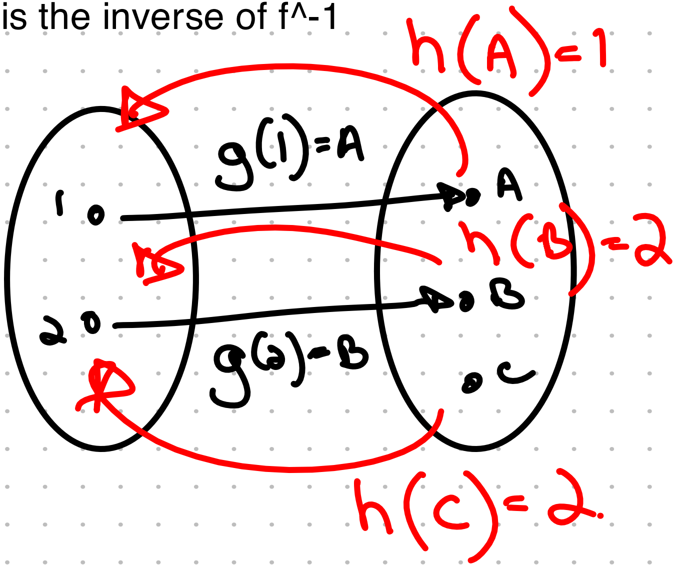
These function have no inverse, why?



$f^{-1}(A)=1$ OR $f^{-1}(A)=2$?

NO TWO INPUTS MAY BE MAPPED TO SAME OUTPUT

if f^-1 is the inverse of f then
f is the inverse of f^-1



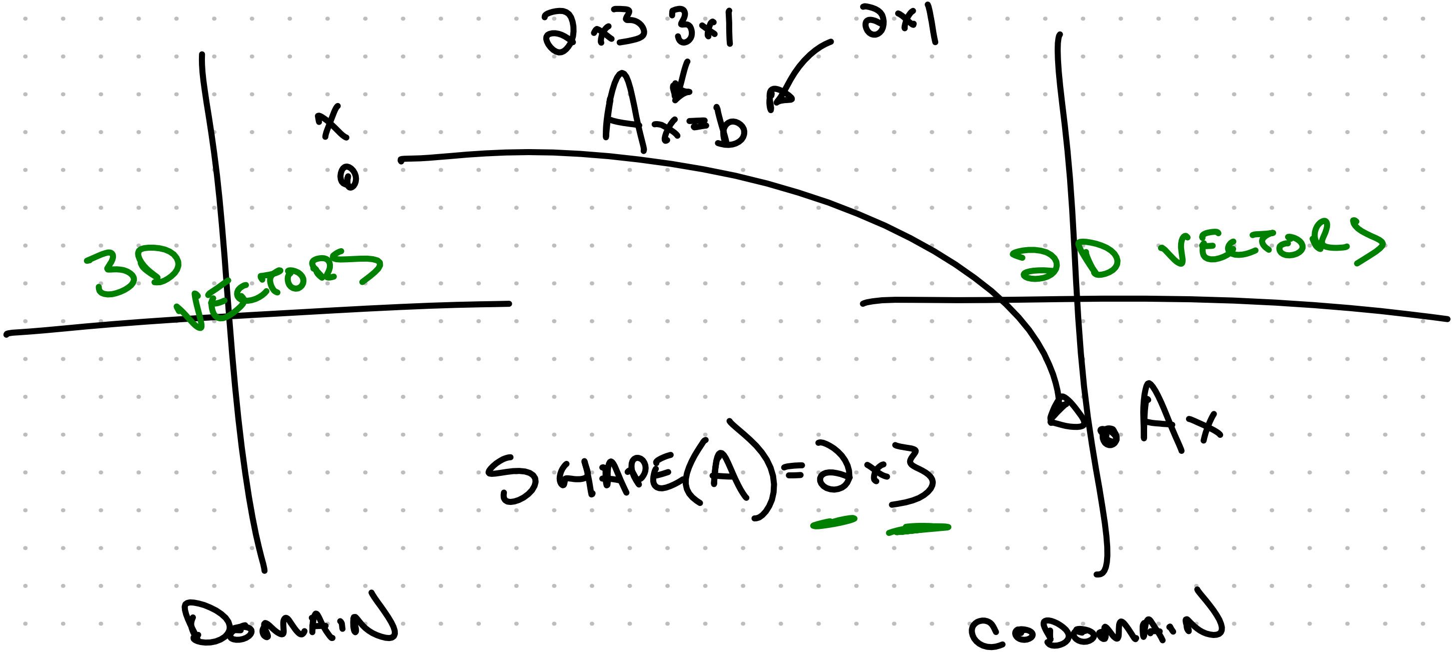
h "undoes" g

but g does not "undo" h

A function has an inverse when it pairs each input to exactly one output (bijective).

The input and output spaces must have the same number of items for this to be possible.

Reminder: How can I think of a matrix as a function?

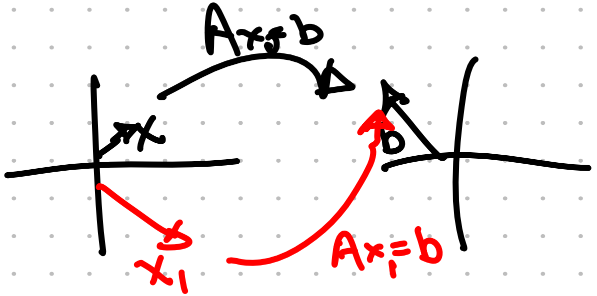


ICA0: When do matrix functions have inverses?

- For a matrix to have an inverse, input space and output space must have same number of elements. If matrix A has an inverse, what can we say about its shape?

$$\text{SHAPE}(A) = N \times N \qquad A \text{ IS SQUARE}$$

- For a matrix to have an inverse, each output is paired to exactly one input.
How many x are mapped to the same b via Ax = b
What can we say about the RREF of A?



$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

IDENTITY MATRIX

$$\text{RREF}(A) = I$$

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \xrightarrow[r_1' = r_1 - 2r_0]{} \begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix} \xrightarrow[r_1' = -1/3\, r_1]{} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow[r_0' = r_0 - r_1]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A matrix has an inverse if (and only if):
✓ - its square
✓ - its reduced row echelon form is the identity matrix:
    - (a diagonal of 1s, otherwise 0s)

COMPUTING INVERSES

SUPPOSE A^{-1} IS THE INVERSE OF A.

WHAT IS $A^{-1}Ax = x$

$$A^{-1}A = I$$

COMPUTING INVERSES

LET $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$ AND $A^{-1} = \begin{bmatrix} x & z \\ y & w \end{bmatrix}$

THEN $AA^{-1} = \underset{\substack{\uparrow \\ \text{MULTIPLICATIVE} \\ \text{IDENTITY}}}{I} \Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ AND $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

ICA 1 SOLVE SYSTEMS BELOW TO BUILD A^{-1}

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & -1 & 1 \end{array}\right]$$

$$x + y = 1$$
$$2x - y = 0$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & -1 & 0 \end{array}\right] \xrightarrow[r_1' = r_1 - 2r_0]{} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -3 & -2 \end{array}\right] \xrightarrow[r_1' = -1/3 r_1]{} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2/3 \end{array}\right]$$

$$\xrightarrow[r_0' = r_0 - r_1]{} \left[\begin{array}{cc|c} 1 & 0 & 1/3 \\ 0 & 1 & 2/3 \end{array}\right] \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/3 & z \\ 2/3 & w \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & -1 & 1 \end{array}\right] \xrightarrow[r_1' = r_1 - 2r_0]{} \left[\begin{array}{cc|c} 1 & 1 & \\ 0 & -3 & \end{array}\right] \xrightarrow[r_1' = -1/3 r_1]{} \left[\begin{array}{cc|c} 1 & 1 & \\ 0 & 1 & \end{array}\right]$$

$$\xrightarrow[r_0' = r_0 - r_1]{} \left[\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1/3 & z \\ 2/3 & w \end{bmatrix}$$

COMPUTING INVERSES (WITHOUT THE REDUNDANCY)

Finding each column of the inverse (each subpart of previous ICA) can be achieved with the same row operations. Maybe there's a way to row reduce all columns at once ...

$$\left[\begin{array}{cc|cc} \multicolumn{2}{c|}{A} & \multicolumn{2}{c}{I} \\ \multicolumn{2}{c|}{\downarrow} & \multicolumn{2}{c}{\downarrow} \\ 1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{array}\right] \xrightarrow[r_1' = r_1' - 2r_0]{} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -3 & -2 & 1 \end{array}\right] \xrightarrow[r_1' = -\frac{1}{3} r_1]{} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{array}\right]$$

$$\xrightarrow[r_0' = r_0 - r_1]{} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{array}\right]$$

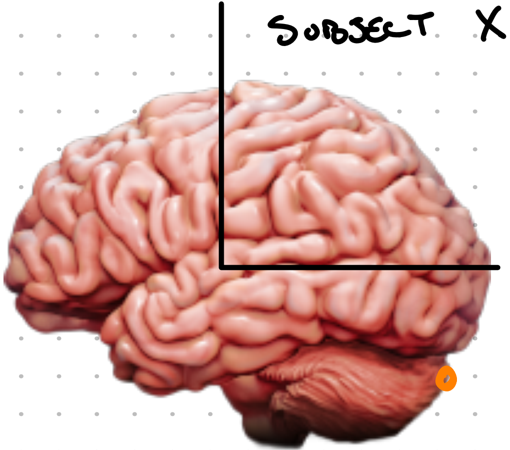
HOW TO COMPUTE INVERSE

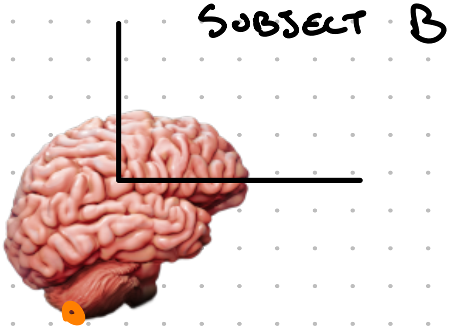
$$\left[\underset{\substack{\uparrow\\\text{GIVEN}\\\text{MATRIX}}}{A}\;\middle|\;\underset{\substack{\uparrow\\\text{IDENTITY}}}{I}\right]\xrightarrow[\text{RREF}]{\sim\sim}\left[\underset{\uparrow}{I}\;\middle|\;\underset{\substack{\uparrow\\\text{INVERSE}}}{A^{-1}}\right]$$

Students on the right side of the classroom are smarter than students who sit on the left side of the classroom

Same anatomical landmark has different representation in subject X than subject B
(subject B has smaller brain and faced opposite & rotated a bit during brain scan)

How do we identity corresponding anatomical structures? (Image Registration)

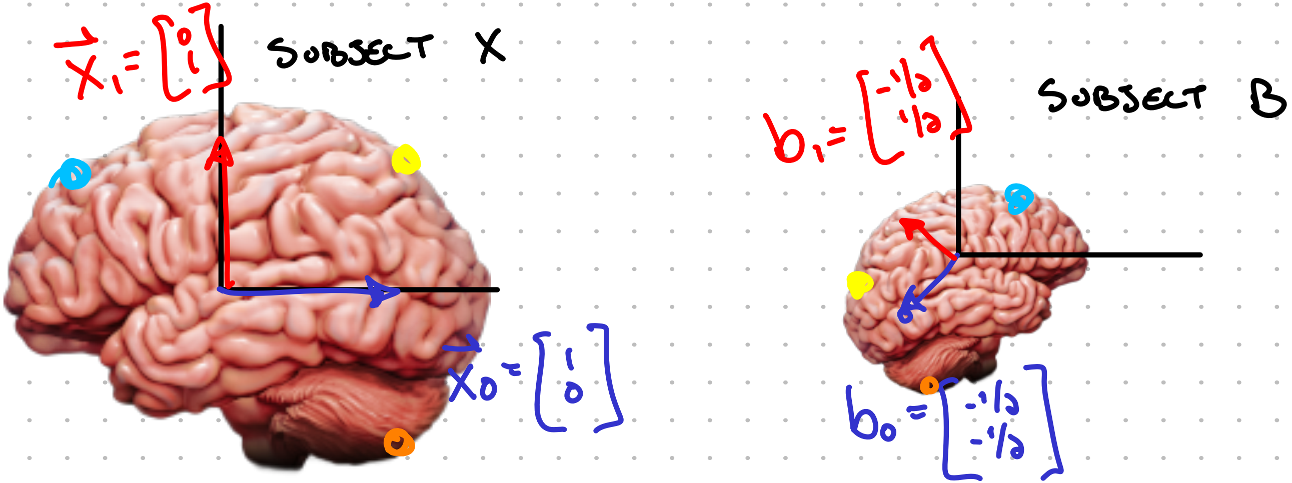




ICA 2:

Assume that vectors x_i and b_i correspond to the same anatomical location

- Find the matrix A which maps an anatomical landmark's representation in subject x to subject b
- What is b representation of the anatomical location whose x representation is x = [-2, -2]^T
  (verify that this makes sense visually)
- Find the matrix which maps an anatomical landmark's representation in subject b to subject x
- What is the x representation of the anatomical location whose b representation is b = [0, 1]^T?
  (verify that this makes sense visually)

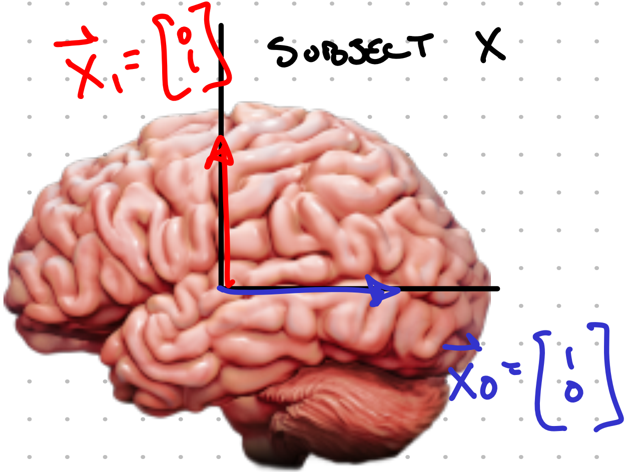


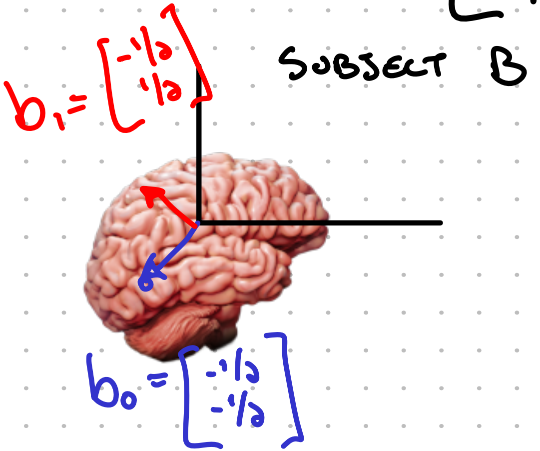
ICA 2.1 Assume that vectors x_i and b_i correspond to the same anatomical location

- Find the matrix A which maps an anatomical landmark's representation in subject x to subject b

$$Ax = b \qquad \begin{bmatrix} | & | \\ a_0 & a_1 \\ | & | \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \end{bmatrix} = a_0 \cdot 1 + a_1 \cdot 0 = a_0$$

$$A = \begin{bmatrix} -1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$



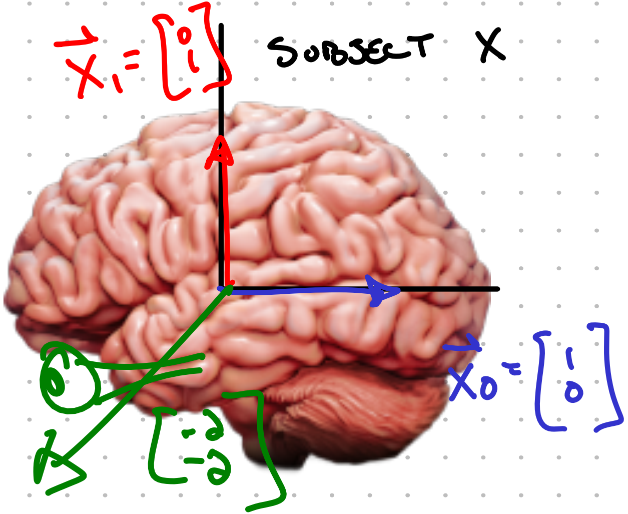


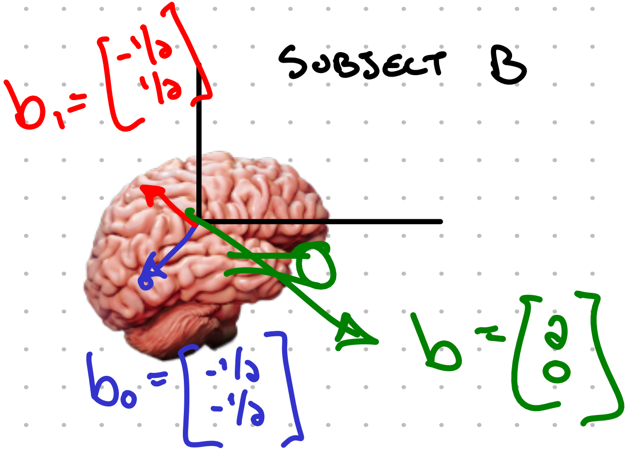
$$\begin{bmatrix} | & | \\ a_0 & a_1 \\ | & | \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/6 \\ 1/2 \end{bmatrix} = a_0 \cdot 0 + a_1 \cdot 1 = a_1$$

ICA 2.2 Assume that vectors x_i and b_i correspond to the same anatomical location
- What is b representation of the anatomical location whose x representation is x = [-2, -2]^T
(verify that this makes sense visually)

$Ax=b$

$$b = Ax = \begin{bmatrix} -1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} -1/2 \\ -1/2 \end{bmatrix} - 2 \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$



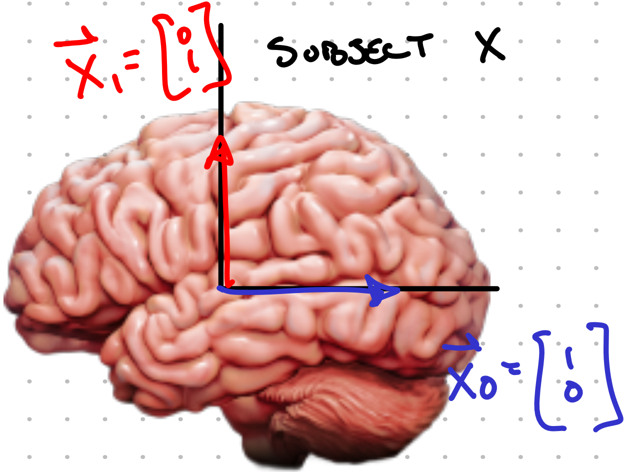


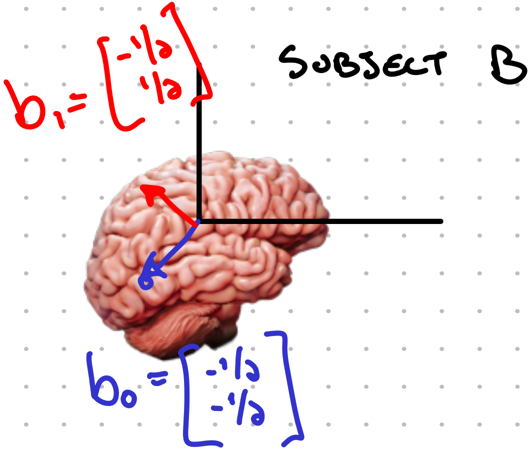
ICA 2.3 Assume that vectors x_i and b_i correspond to the same anatomical location
- Find the matrix which maps an anatomical landmark's representation in subject b to subject x

```
a = np.array([[-.5, -.5], [-.5, .5]])
np.linalg.inv(a)
```

$$A^{-1} b = x$$

$$A^{-1} = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$$



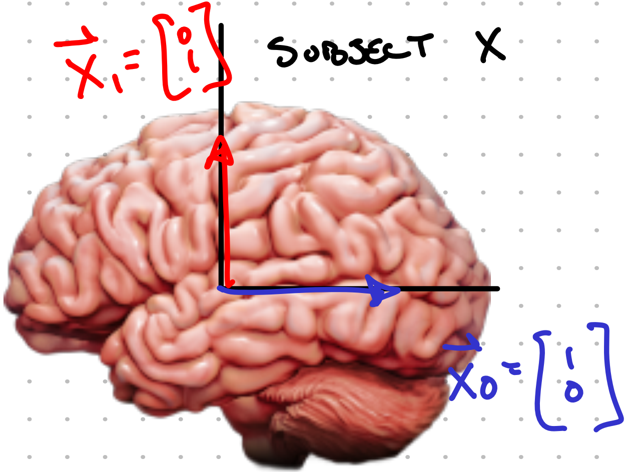


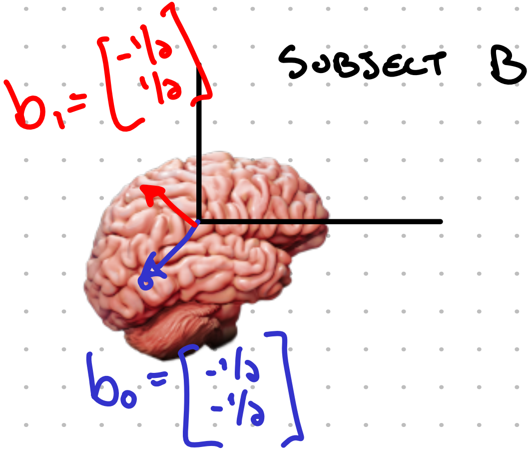
ICA 2.4 Assume that vectors x_i and b_i correspond to the same anatomical location

-What is the x representation of the anatomical location whose b representation is b = [0, 1]^T?
(verify that this makes sense visually)

$$A^{-1}b = x$$

$$\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = x$$





Change of basis: swapping between different coordinate systems

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{bmatrix}$$