We'll start @ 11:47

Now playing:

CAKE, "Short Skirt Long Jacket"

Maroon 5, "Harder to Breathe"

AC/DC, "Back in Black"

yes, we're enjoying the early 2000s today. :) (mostly)

↳ except for AC/DC :)

Get out:

- your notes
- a place to do your ICA for today (this is ICA 3)

Matrices and Vector Geometry

Felix has 2 cats and rates themself at 1 happiness

Krishna has 1 dog and 2 cats and rates herself at 2 happiness

$\frac{1}{2}$

Lacee has no dogs and 1 parakeet and rates herself at 0 happiness

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ 2 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 2 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

How happy do you think that Felix will be if they get 2 more cats? → 4?

2 more cats and 1 dog? → 3 happiness

Choux

Solving linear systems

ICA Question 1: for each matrix, write down the following:

a) is it in RREF?

b) if no, identify a specific reason why not

c) if yes, identify whether the system has:

- no solutions
- one unique solution
- many solutions

breakout rooms:
- camera
- talk to one another

$\rightarrow r_0' = r_0 - 2r_1$

A) $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 4 \end{array}\right]$ B) $\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 8 \end{array}\right]$ C) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 6 \end{array}\right]$ D) $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right]$

$\rightarrow$ no solutions $\rightarrow$ inconsistent

$\rightarrow$ 0 0 1

$\rightarrow$ many solutions

$$\left[\begin{array}{cccc|c} 1 & 0 & 4 & 5 & \xi \\ 0 & 1 & 6 & 12 & \xi \\ 0 & 0 & 0 & 0 & \xi \\ 0 & 0 & 0 & 0 & \xi \end{array}\right]$$

RREF
- zero rows at bottom
- leading coeffiencts are 1
- above/below the leading coef. are zero

Matrices & Vectors

- A **matrix** is an array of scalars
- Matrices have *shapes* rows by columns

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 8 \end{bmatrix} \quad 2 \times 3$$

- A **vector** is a matrix with 1 row or 1 column

column vector

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ shape: } 3 \times 1$$

row vector

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

shape: 1×5

Notation and convention (a bit)

- scalars - lowercase, not bold (e.g. $x = 2$)
- vectors - lowercase, **bold** (if typing) (e.g. **x**), arrow hat (if typing or by hand) (e.g. $\vec{x}$)

$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

- "truly 2 dimensional" matrix - uppercase (e.g. A)

↳ neither dim. is 1

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 9 \end{bmatrix}$

$A \in \mathbb{R}^{2 \times 3}$

↳ has shape 2×3 and the values are real numbers

Matrix operations

$C = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

- Matrix addition: $A + B + C$

$$\underbrace{\begin{bmatrix} 2 & 2 \\ 4 & 6 \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}}_{B} = \begin{bmatrix} 3 & 1 \\ 7 & 11 \end{bmatrix}$$

shape: 2×2

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Add corresponding entries of two matrices
- A and B (sometimes, always, ~~never~~) have the same shape

Matrix operations

- Scalar multiplication

$$\begin{bmatrix} 2 & 2 \\ 4 & 6 \end{bmatrix} * 3 = \begin{bmatrix} 6 & 6 \\ 12 & 18 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix} * -1.5 = \begin{bmatrix} -1.5 \\ -3 \end{bmatrix}$$

- Multiply every entry of the matrix by some scalar

- A (sometimes, always, never) is a vector

↳ there is no relationship between the shape of A and whether or not you can mult. by a scalar

Matrix operations

- Matrix multiplication

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 10 & 11 \end{bmatrix} = ?$$

$$\begin{bmatrix} 5 \\ 6 \end{bmatrix} = ?$$

$$\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = ?$$

- ...

Vectors

- common, but *not universal* convention is that $\mathbb{R}^n$ is a column vector (rather than a row vector)

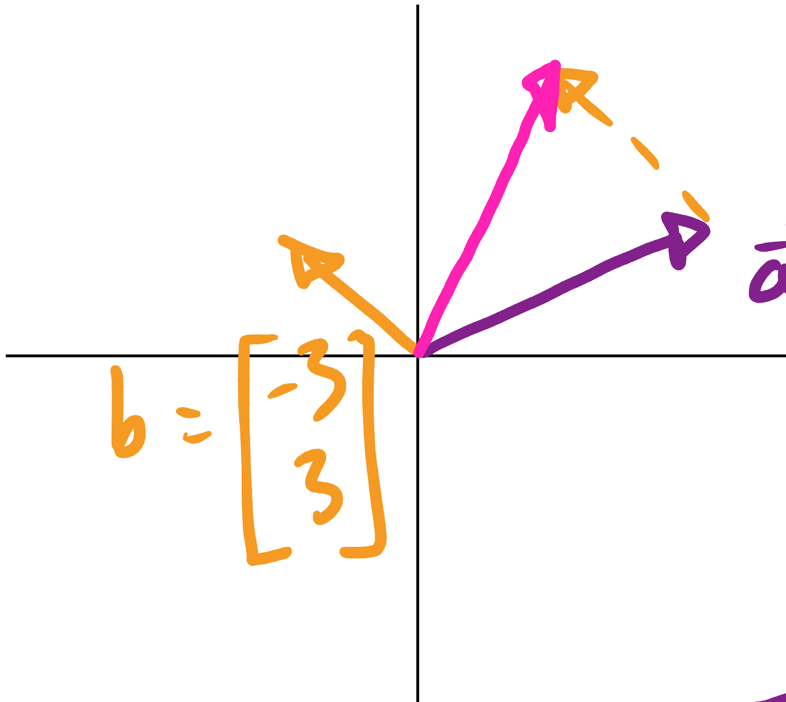
↳ assume $\mathbb{R}^{n \cancel{\times n}}$ is a column vector

$\vec{x} \in \mathbb{R}^4$ $\quad \vec{x} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$

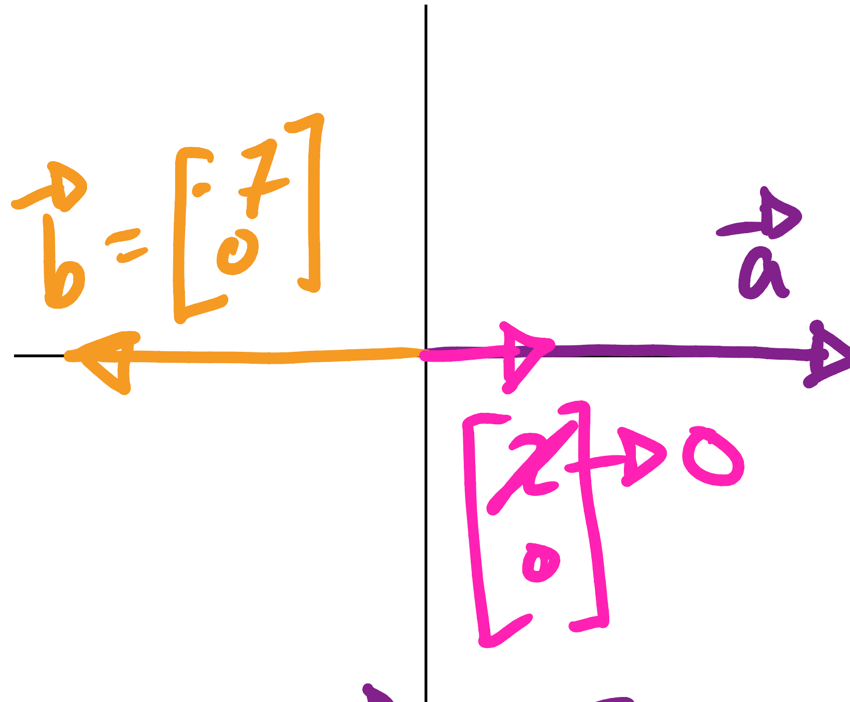
$y \in \mathbb{R}^3$ $\quad y = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$

Vector addition

$$\vec{a} + \vec{b} = \vec{c}$$



$$b = \begin{bmatrix} -3 \\ 3 \end{bmatrix} \qquad \vec{a} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$



$$\vec{b} = \begin{bmatrix} -7 \\ 0 \end{bmatrix} \qquad \vec{a} = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$$

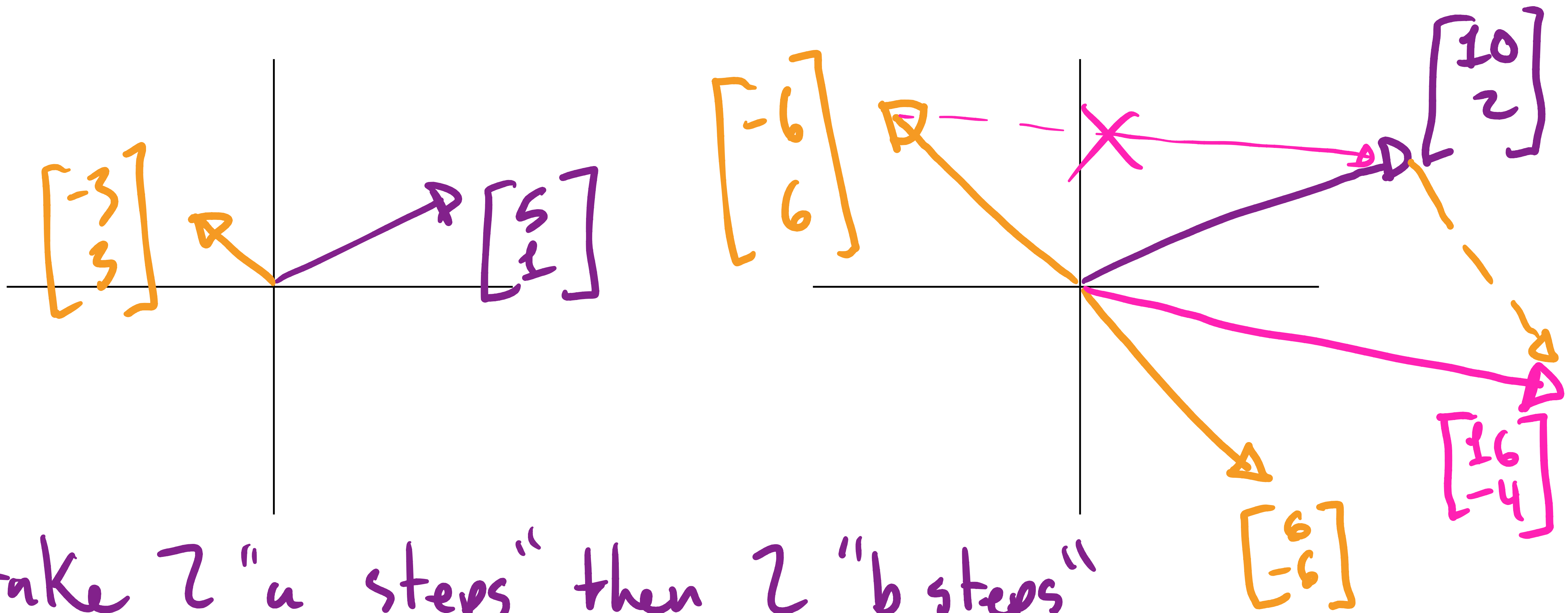
$$\begin{bmatrix} \cancel{2} \\ 0 \end{bmatrix} \to 0$$

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\vec{a} + \vec{b} = ?$$

Vector addition

$2a - 2b = ?$



- take 2 "a steps" then 2 "b steps" backwards

Expressing system solutions via vector

- If our system has one solution, the RREF looks like....

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \Rightarrow \begin{array}{l} x = 5 \\ y = 2 \\ z = 3 \end{array} \rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$$

Expressing system solutions via vector

- If our system has many solutions, the RREF looks like....

$$\begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow \begin{array}{l} x + z = 3 \\ y + 2z = 4 \\ z = z \end{array} \Rightarrow \begin{array}{l} x = 3 - z \\ y = 4 - 2z \\ z = 0 + z \end{array}$$

$0 = 0$

- for every 0 on the diagonal, add corresponding ~~reflexive~~ identity equality

$$= \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} z$$

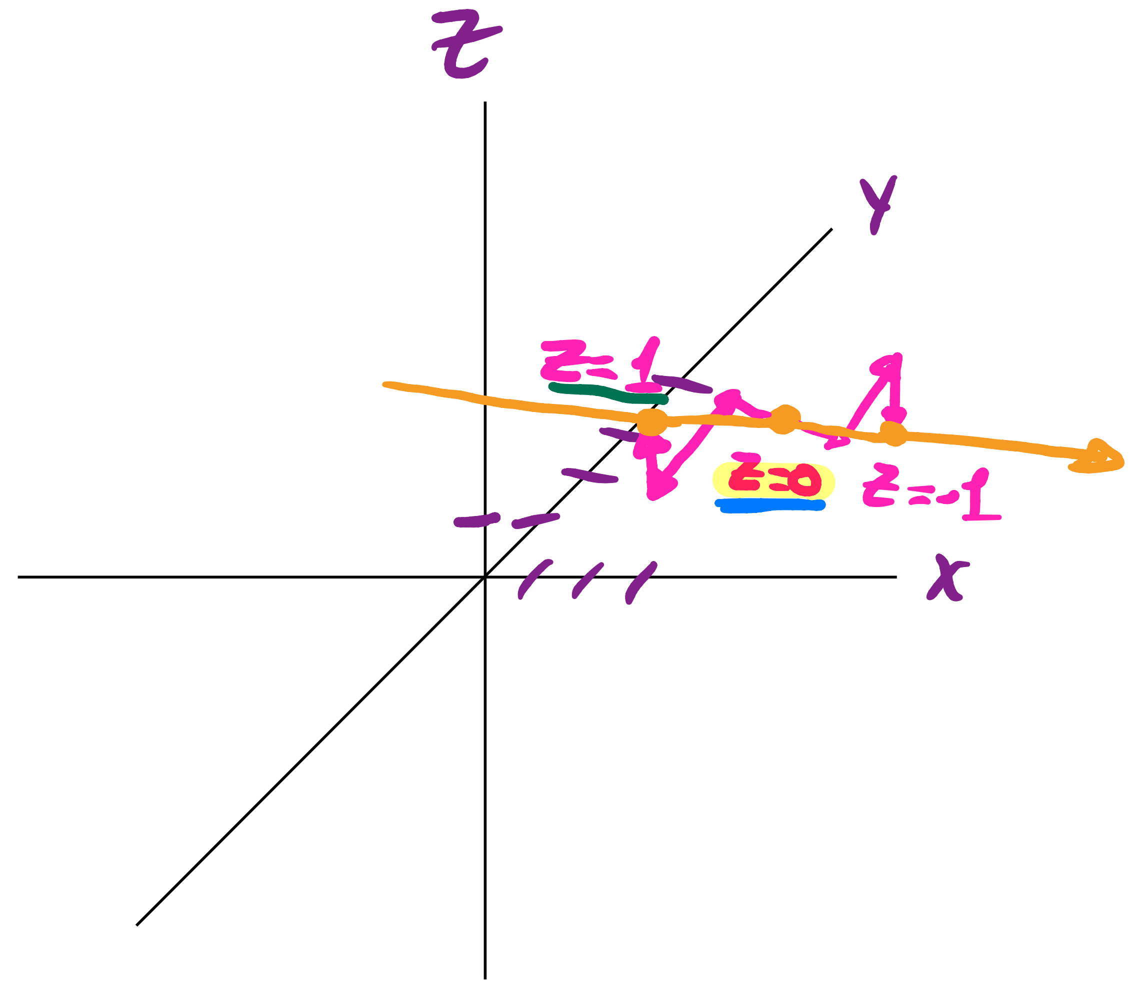
Visualizing our solutions space (many solutions)

what are $\begin{bmatrix}x\\y\\z\end{bmatrix}$ that satisfy:

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}3\\4\\0\end{bmatrix} + \begin{bmatrix}-1\\-2\\1\end{bmatrix} z$$

$\begin{bmatrix}3\\4\\0\end{bmatrix}$ → starting point

$\begin{bmatrix}-1\\-2\\1\end{bmatrix}$ → "adding" z steps



$z=0 \quad \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}3\\4\\0\end{bmatrix}$

$z=1 \quad \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}3\\4\\0\end{bmatrix} + \begin{bmatrix}-1\\-2\\1\end{bmatrix} = \begin{bmatrix}2\\2\\1\end{bmatrix}$

Visualizing our solutions space (many solutions)

ICA Question 2: come up with a system of 3 linear equations for which the solution space is a **plane**. Write this system down as an augmented matrix.

hints: think of previous ex. as having 1 "degree of freedom", you want 2!

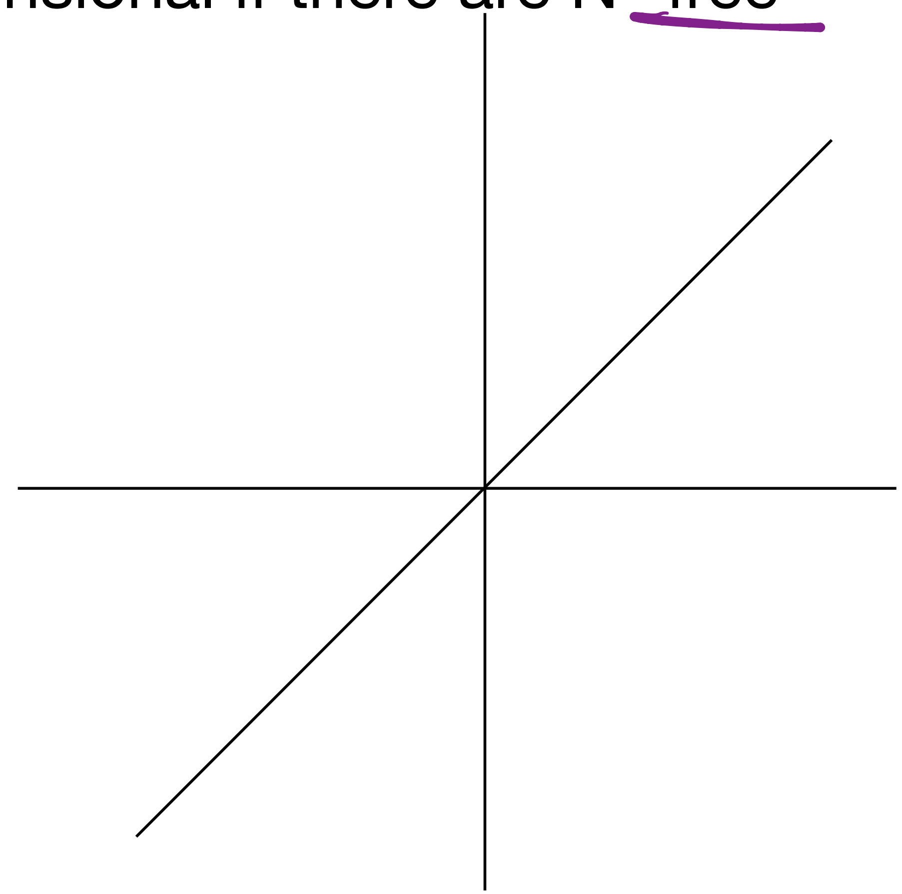
We'll return to this as warm-up on thurs

Visualizing our solutions space (many solutions)

- In general, solutions spaces are N-dimensional if there are N "free parameters"

$z = z$

$y = y$



$$\begin{bmatrix} v \\ a \\ r \\ i \\ a \\ b \\ l \\ e \\ s \end{bmatrix} = \begin{bmatrix} s \\ t \\ a \\ r \\ t \end{bmatrix} + \begin{bmatrix} c \\ o \\ e \\ f \\ \\ v \\ e \\ c \end{bmatrix} f_1 \ldots$$

for all free variables

Homogenous Systems

- A system is **homogenous** if the augment is all zeroes

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 0 \end{array}\right]$$

- The **corresponding** homogenous system to any linear system is:

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 3 & 4 & 10 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 0 \end{array}\right]$$

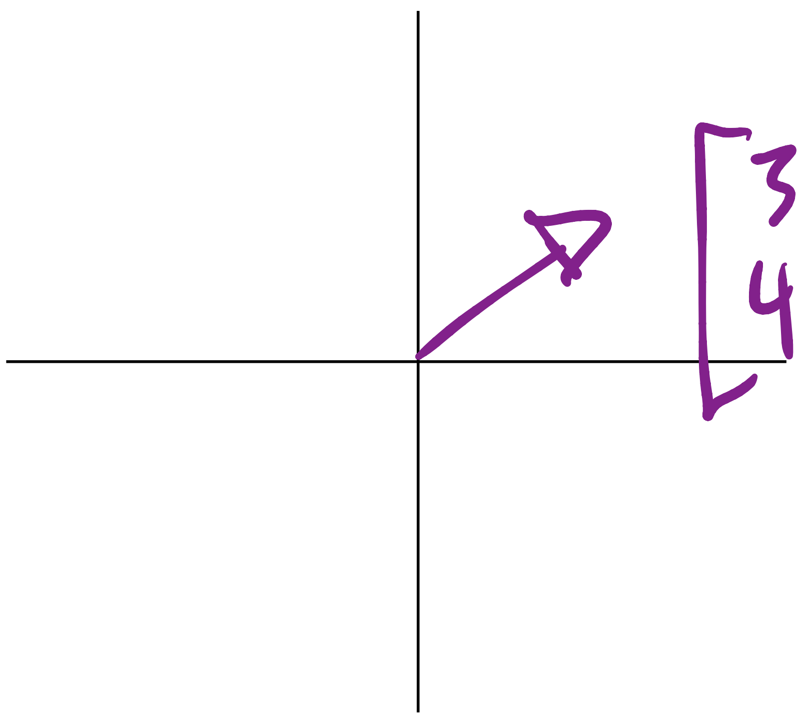
Vector Geometry: Length

- The length of a vector is: $||\overrightarrow{x}|| = \sqrt{\sum_i x_i^2}$

$$= \sqrt{(3^2+4^2)}$$

$$= \sqrt{25}$$

$$= 5$$



$$\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Vector Geometry: dot product

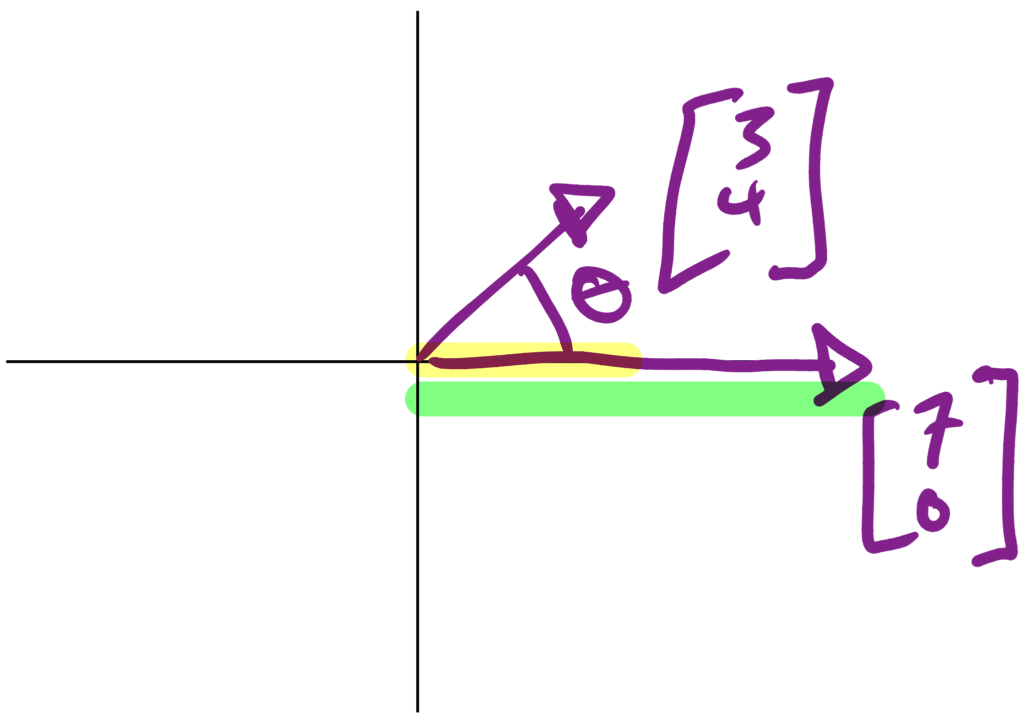
- written as $\vec{x} \cdot \vec{y}$ and answers the question "how much of $\vec{x}$ is in the direction of $\vec{y}$?"

$$= \text{comp. of } \vec{x} \text{ in the dir. of } \vec{y} * \|\vec{y}\|$$

$$= \|\vec{x}\| \cos\theta * \|\vec{y}\|$$

$$= 3 * 7$$

$$= 21$$



Vector Geometry: dot product

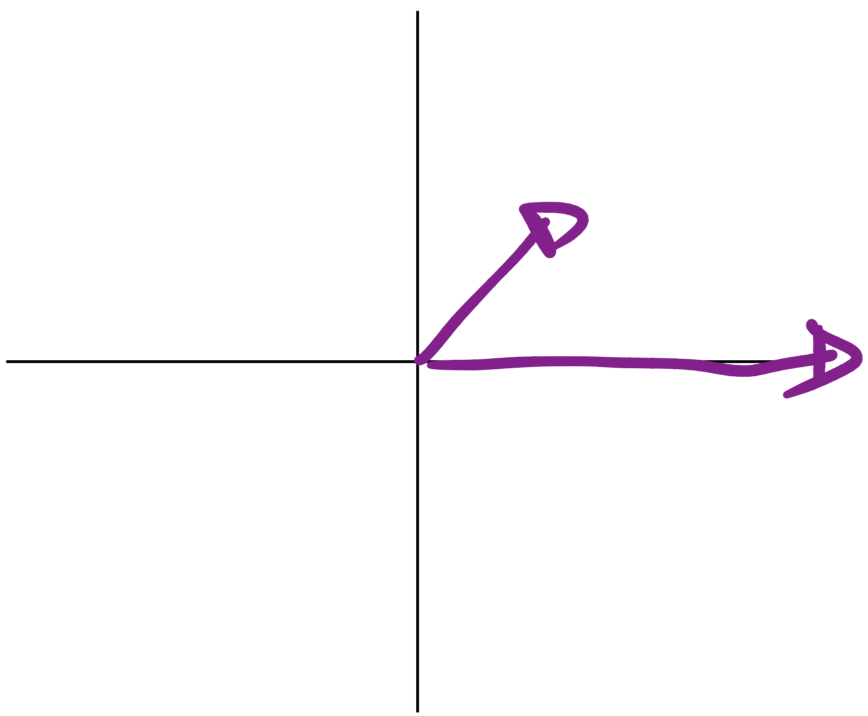
- Also (more commonly) written as: $\overrightarrow{x} \cdot \overrightarrow{y} = \sum_i x_i * y_i$

mult. corresponding elements and sum them

$$= \begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 0 \end{bmatrix}$$

$$= 3*7 + 4*0$$

$$= 21$$



Vector Geometry: dot product

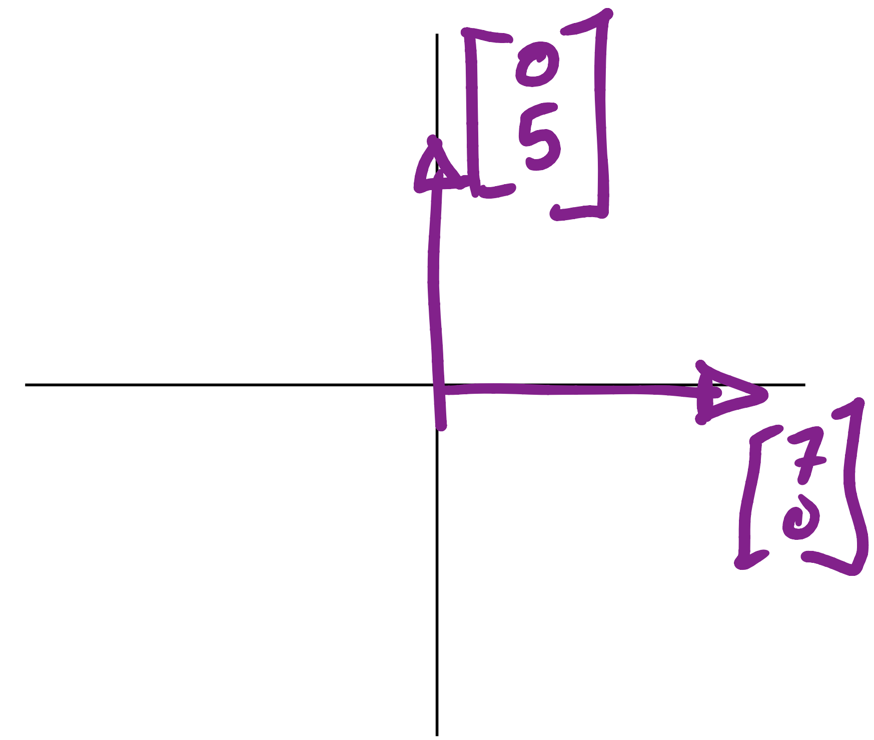
- $\overrightarrow{x} \cdot \overrightarrow{y} = \sum_i x_i * y_i = || \overrightarrow{x} || * || \overrightarrow{y} || \cos \theta$

- What is the angle between $\overrightarrow{x}$ and $\overrightarrow{y}$ here?

90°

- What about the cosine of this angle?

0



- dot product $\to$ 0

Vector Geometry: dot product

- Wait, so why do we care about dot products?
  - Extends intuition of angles to more (and less) than 2D/3D space
  - Allows us to identify vectors at right angles with one another

- Alert! A dot product give us a scalar!

Vector Geometry: dot product

- Side note: in modern Natural Language Processing, we use dot products to measure how semantically similar two words are.
  - How close is "dog" to "cat" what about to "bumblebee"?

Other vocab

- A system is **inconsistent** if it has no solutions

Schedule

ICAs graded on effort

Turn in ICA 3 on Gradescope

We are remote until Feb 5th

instr. for OH will be released later today w/ HW1

HWs post on course website, we'll include a link on canvas too though

Mon	Tue	Wed	Thu	Fri	Sat	Sun
January 24th Lecture 3 - Matrices & vector geometry	**Felix OH Calendly**		Lecture 4 - ML, linear perceptron **Felix OH Khoury Office Hours**			
January 31st Lecture 5 - Linear Perceptron	**Felix OH Calendly**	**HW 1 due @ 11:59pm**	Lecture 4 - matrix multiplication, transforms **Felix OH Khoury Office Hours**			