CS 2810 April 5 Day 20

chi square "goodness of fit"

Admin:

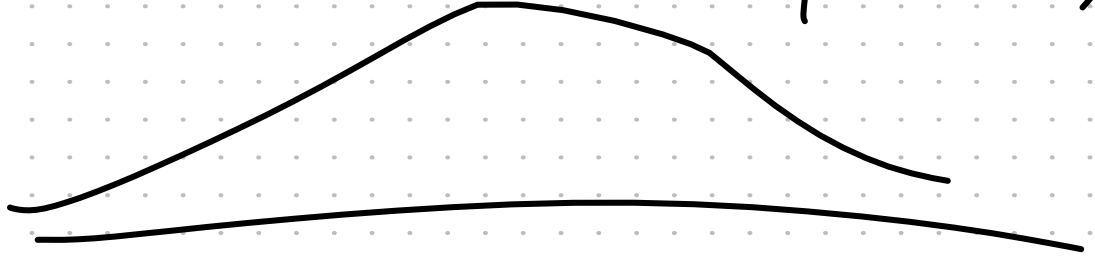
short class

Sec 3 Finals sched

"STANDARD" NORMAL DISTRIBUTION

$Z \sim N(\mu=0, \sigma^2=1)$

Z IS OFTEN USED AS NAME OF SAMPLE FROM THIS DISTRIBUTION



IF $X \sim N(\mu, \sigma^2)$ THEN $\frac{X-\mu}{\sigma} \sim N(\mu=0, \sigma^2=1)$

"Z-SCORE"

Let X be a normally distributed random variable with mean 7 and variance 10. Identify the linear function of X so that it has a "standard" normal distribution (mean 0 and variance 1).

$X \sim N(\mu = 7, \sigma^2 = 10)$

$X_7 = 11$

$Z = \frac{X - \mu}{\sigma} = \frac{X - 7}{\sqrt{10}} \sim N(0, 1)$

$\sim$: Distributed As

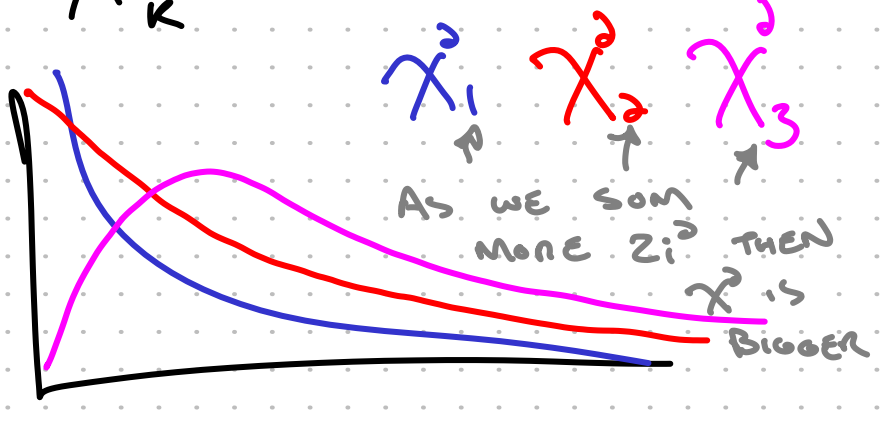
$\frac{11 - 7}{\sqrt{10}} = \frac{4}{\sqrt{10}} \approx 1.2$

Chi-Square Distribution

Let $Z_i \sim N(0,1)$ be iid standard normal

Chi-square is sum of k standard normal squared

$$\sum_{i=1}^{k} Z_i^2 \sim \chi^2_k$$



CHI-SQUARE "GOODNESS OF FIT" TEST

ARE THESE OUTCOMES OF A 6-SIDED DIE FAIR?

Do these N=36 outcomes come from a fair (uniform) 6-sided die?

H_0: PROB OF EACH OUTCOME IS 1/6

H_1: NO ITS NOT

(we sort outcomes below so they're easier to work with):

1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 4, 4, 5, 5, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6

8	9	5	2	7	5
O_1	O_2	O_3	O_4	O_5	O_6

O_i HOW MANY TIMES DID WE OBSERVE OUTCOME i?

Assuming H_0 (Die is fair in this Ex) what is expected count of each outcome?

$$E_i = N \cdot p_i = 36 \cdot \frac{1}{6} = 6 \quad \text{for all outcomes}$$

Total outcomes (36 in this Ex)

Prob of each outcome under H_0

Contingency Table + χ^2 Test Statistic

	1	2	3	4	5	6
Observed (O_i)	8	5	9	2	7	3
Expected (E_i)	6	6	6	6	6	6
$\frac{(O_i - E_i)^2}{E_i}$	$\frac{(8-6)^2}{6}$	$\frac{(5-6)^2}{6}$	$\frac{9}{6}$	$\frac{(2-6)^2}{6}$	$\frac{(7-6)^2}{6}$	$\frac{1}{6}$

$$\chi^2 = \sum_i \frac{(O_i - E_i)^2}{E_i}$$

ICA 1:
- Does there exist a minimum or maximum chi-squared statistic?
- Describe what kind of O_i and E_i achieve this min or max chi-squared statistic.
- Which values of the chi-squared stat are most typical of the null hypothesis? Justify your response with one or two sentences.

chi-square = 0 happens only when each outcome is observed as many times as its expected

	1	2	3	4	5	6
OBSERVED (O_i)	8	5	9	3	7	5
EXPECTED (E_i)	6	6	6	6	6	6
$\frac{(O_i - E_i)^2}{E_i}$	$\frac{4}{6}$	$\frac{1}{6}$	$\frac{9}{6}$	$\frac{16}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$\chi^2 = \sum_i \frac{(O_i - E_i)^2}{E_i}$

$\chi^2 \approx 5.3$

chi-square = 0 happens only when each outcome is observed as many times as its expected

the chi-square stat closer to zero are more consistent with the null hypothesis

Modeling Null Hypothesis

Assuming the null hypothesis (die is fair) then the chi square statistic follows a chi square distribution with k = size of sample space - 1 degrees of freedom (df=6-1=5 in this example)

STATISTIC

DISTRIBUTION

$$\chi^2 = \sum_i \frac{(E_i - O_i)^2}{E_i} \sim \chi^2_K$$

K = # CATEGORIES - 1

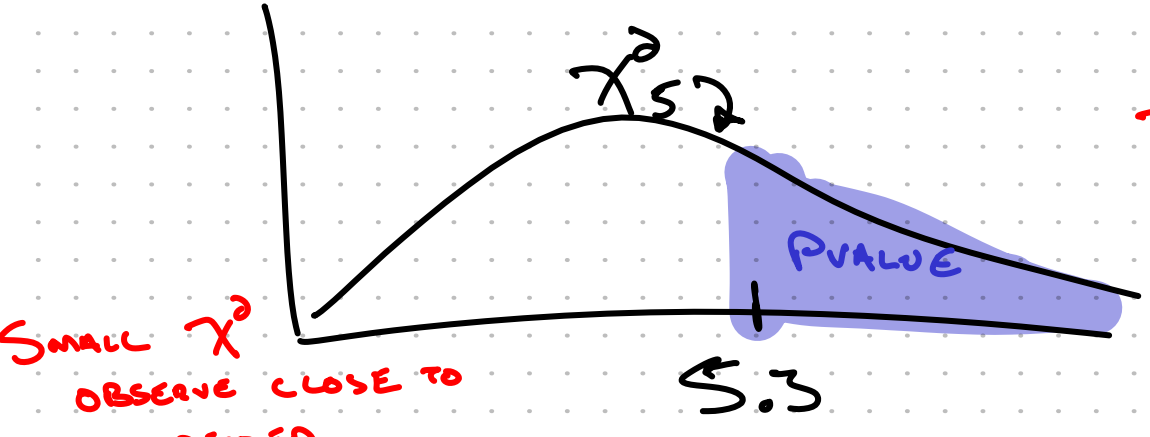
IN THIS EXAMPLE:

K = 6 - 1 = 5

Computing P-value with Chi-Squared Goodness of Fit

Remember: P-value is prob of all outcomes which are less consistent with null hypothesis

ASSUMING H_0, $\chi^2 = 5.3$ IS χ^2 DISTRIBUTED WITH DF=5



$$\chi^2 = \sum_i \frac{(O_i - E_i)^2}{E_i}$$

SMALL χ^2
OBSERVE CLOSE TO EXPECTED
more consistent with H_0

LARGE χ^2
OBSERVE FAR FROM EXPECTED
less consistent with H_0

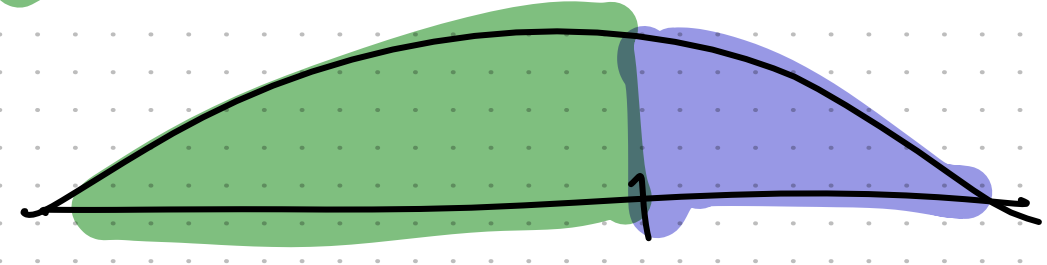
Finalize Chi2 example

NOT PVAL

CDF(5.3, K=5)

PVALUE

1-CDF(5.3, K=5)



PVAL = .38

5.3

$P\text{VAL} > \alpha = .05$ ← SMALL

DON'T REJECT H_0

... DIE COULD BE FAIR

$P\text{VAL} < \alpha = .05$

REJECT H_0, CLAIM H_1

... DIE IS NOT FAIR

ICA 2

A "silly die" is supposed to roll higher outcomes more often than others:

OUTCOME	1	2	3	4	5	6
PROB	1/21	2/21	3/21	4/21	5/21	6/21

If a die is observed to roll:

1, 1, 1, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6

3 3 4 7 4 12

perform a hypothesis test to (potentially) claim the die is not "silly".

H_0: Die is silly

H_1: Die not silly

OUTCOME	1	2	3	4	5	6
PROB	$1/21$	$2/21$	$3/21$	$4/21$	$5/21$	$6/21$
E_i EXPECTED # OUTCOMES	$\frac{1}{21}\cdot 33$	$\frac{2}{21}\cdot 33$	$\frac{3}{21}\cdot 33$	$\frac{4}{21}\cdot 33$	$\frac{5}{21}\cdot 33$	$\frac{6}{21}\cdot 33$
O_i OBSERVED # OUTCOMES	3.	3	4	7	4	12
$\chi^2 = \sum_i \frac{(E_i - O_i)^2}{E_i}$	1.29	.006	.108	.081	1.89	.701

$3+3+4+7+4+12 = 33$ TOTAL TRIALS

$\chi^2 = 4.08 \quad P = 1 - \text{CHI2.CDF}(4.08, DF=5) = .538$

DON'T REJECT H_0

CHI-SQUARE "BINNING"

χ^2 REQUIRES A FINITE SAMPLE SPACE

WE CAN "BIN" A DISTRIBUTION:

EXPECTED $N \cdot \int_{x_1}^{x_2} P(x)\,dx$

TOTAL OBSERVATIONS — PROB IN BIN (1)

OBSERVED ARE SPLIT INTO BINS

CHOOSING BINS IMPACTS ANALYSIS SENSITIVITY