Written Homework 04

Due: Wed 4 Nov 2015

Instructions:

- The assignment has to be uploaded to blackboard by the due date. NO assignment will be accepted after 11:59pm on that day.

- We expect that you will study with friends and often work out problem solutions together, but you must write up your own solutions, in your own words. Cheating will not be tolerated. Professors, TAs, and peer tutors will be available to answer questions but will not do your homework for you. One of our course goals is to teach you how to think on your own.

- We require that all homework submissions be neat, organized, and typeset. You may use plain text or a word processor like Microsoft Word or LaTeX for your submissions. If you need to draw any diagrams, however, you may draw them with your hand.

- To get full credit, show intermediate steps leading to your answers, throughout.

NOTE: In questions below give complete numerical answers (for example, ... = \(\binom{10}{2}\) is considered incomplete—you should write ... = \(\binom{10}{2}\) = 45. Similarly, you should write ... = \(2^7\) = 128).

Problem 1 [21 pts (5, 3, 5, 8)]: Words & Numbers

i. A palindrome is a word (or a number) which does not change when read from left or right. Disregarding the fact that some letter sequences may not have a meaning in English, how many 7 letter palindromes are there which consist of the small letters of the alphabet (the obtained words do not have to have a meaning).

Solution: Any 7 letter palindrome is symmetric with respect to its middle letter. Hence we can generate all such words by 1) choosing the middle letter, 2) choosing a 3 letter "half" of the palindrome and 3) reflecting the "half" from 2) about the middle letter from 1). For example,

\[\text{"rot" + "a" + "tor" = "rotator".}\]

Here a is the middle letter and rot is the "half" of the word which is reflected about a.

We can perform step 1), i.e. choose the middle letter, in 26 ways. The three letter "half" of the palindrome, i.e. the step 2), can be chosen in 26\(^3\) ways. With choices in 1), 2) made, the palindrome is determined. This gives

\[26 \times 26^3 = 26^4 = 456976 \]

7 letter palindromes.
ii. How many 9-digit numbers are there whose all digits are prime?

\textbf{Solution:}

Prime \textit{digits} are the numbers 2, 3, 5, 7. Hence we are asked how many 9-digit numbers are there whose every digit belongs to the set \( Q = \{2, 3, 5, 7\} \). Since \(|Q| = 4\), and a choice of one digit does not influence the choice of any other, we can form

\[4^9 = 262144\]

numbers with the required property.

iii. Consider products

\[a_1a_2...a_{10}\]

where \( a_i = x \) or \( a_i = y \) for some distinct real numbers \( x, y \). How many of such products simplify to \( x^4y^6 \)? For example, \( xyxyxyxyyy \) and \( xxyyyxyyxyy \) are two possibilities.

\textbf{Solution:} To generate all distinguishable products \( a_1a_2...a_{10} \) (not necessarily simplifying to \( x^4y^6 \)), we need to choose positions for the variable \( x \) in a row of 10 place holders. All other places are assumed to hold variable \( y \). For example, below we chose three places for \( x \) (\( y \) being assumed to occupy all others)

\[\square\square\square\X\square\square\X\X\square\]

This particular choice of \( x \)'s and \( y \)'s positions in the multiplication process would give us the simplified product \( x^3y^7 \).

Hence to generate all products simplifying to \( x^4y^6 \), we need to find the number of ways we can choose 4 places for the variable \( x \) in a row of 10 positions. Naturally, that number is equal to the number of 4 element subsets of a 10 element set, i.e. to:

\[\binom{10}{4} = 210\]

\textbf{Note.} One can obtain all distinguishable products of length 10 (not necessarily simplifying to \( x^4y^6 \)) of variables \( x \) and \( y \) we could multiply out the expression

\[(x + y)^{10} = \frac{(x + y)(x + y)(x + y)(x + y)(x + y)(x + y)(x + y)(x + y)(x + y)(x + y)}{(x + y)(x + y)(x + y)(x + y)(x + y)(x + y)(x + y)(x + y)(x + y)(x + y)}\]

Among all the obtained terms exactly \( \binom{10}{4} \) would simplify to \( x^4y^6 \), and hence that's the coefficient at \( x^4y^6 \) given by the binomial formula

\[(x + y)^{10} = x^{10} + \binom{10}{1}xy^9 + \binom{10}{2}x^2y^8 + \binom{10}{3}x^3y^7 + \binom{10}{4}x^4y^6 + ... + \binom{10}{9}x^9y + y^{10}\]

iv. How many distinguishable words can be formed from the word \( \text{CHIHUAHUA} \).

\textbf{Solution:}
The above word contains 9 letters: 2 letters A, 1 letter C, 3 letters H, 1 letter I, and 2 letters U. Notice that swapping the second H with the fourth in the word CHIHUAHUA does not produce a new word. What distinguishes one word from another are the position sets of the letters. For example, a word in which H occupies positions \{2, 4, 7\} (the current case) will certainly be distinguishable from any word in which H is placed on positions \{1, 5, 9\}.

Hence to generate all distinguishable rearrangements of the letters in CHIHUAHUA we need to find out in how many ways we can perform the following 5 stage process:

1. choose 2 positions from the 9 available positions for the letter A
2. choose 1 position from the 7 remaining positions for the letter C
3. choose 3 positions from the 6 remaining for the letter H
4. choose 1 position from the 3 remaining for the letter I, and
5. choose 2 positions from the 2 remaining for the letter U.

Each stage asks for the number of $k$-element subsets of an $m$-element, set where pairs $(k, m)$ are $(2, 9)$, $(1, 7)$, $(3, 6)$, $(1, 3)$ and $(2, 2)$ respectively. Since the number of $k$-element subsets of an $m$-element set is $\binom{m}{k}$, the answer is

$$\binom{9}{2} \binom{7}{1} \binom{6}{3} \binom{3}{1} \binom{2}{2} = \frac{9!}{2!1!3!1!2!} = 5 \times 6 \times 7 \times 8 \times 9 = 15120$$

**Problem 2 [21 pts (5, 8, 8)]: Restaurant**

i. A restaurant received an order for delivery of 3 different appetizers, 3 different main courses and 3 different desserts. However the connection on the phone broke off and the caller did not give any additional information. In how many ways such a meal can be assembled if the restaurant offers 5 appetizers, 10 main courses, 7 desserts?

**Solution:**

Any particular choice $C = \{A_1, A_2, A_3, M_1, M_2, M_3, D_1, D_2, D_3\}$ of 3 appetizers from a set of 5, 3 main courses from a set of 10, and 3 desserts from a set of 7 fulfills the customer’s request. From the perspective of the restaurant, the order of the meals is irrelevant. Hence from their perspective they could create all requested meals by

1. first choosing a 3 element subset from the 5-element set of appetizers,
2. then choosing a 3 element subset from the 10-element set of main courses,
3. and finally, choosing a 3 element subset from the 7-element set of desserts.

Step 1) can be performed in $\binom{5}{3}$ ways, step 2) in $\binom{10}{3}$ ways, and step 3) in $\binom{7}{3}$ ways. Hence the total number of meals satisfying the customer’s (partial) specification is

$$\binom{5}{3} \binom{10}{3} \binom{7}{3} = 42000$$
ii. Assume that before breaking off, the caller managed to convey an additional restriction that one of the main courses must be vegetarian (no restriction on the others). In how many ways could the restaurant fulfill the order if out of the 10 main courses 4 are vegetarian?

Solution:
Here we can assemble the required meals following the process described in part i. except that we need to modify step 2) in order to accommodate the vegetarian requirement. Hence we assemble the meals as follows.

1. Choose a 3 element subset from the 5-element set of appetizers.
2. In how many ways can we choose a triple of distinct main courses containing at least one vegetarian? We first find the number of triples of distinct main courses which do not contain a vegetarian one. That number is \( \binom{6}{3} \) as we are restricted here to the 6 (out of 10) non-vegetarian main courses. Hence the number of triples of distinct main courses containing a vegetarian course is \( \binom{10}{3} - \binom{6}{3} = 120 - 20 = 100 \).
3. Choose a 3 element subset from the 7-element set of desserts.

Combining steps 1, 2, 3 gives
\[
\binom{5}{3} \left( \binom{10}{3} - \binom{6}{3} \right) \binom{7}{3} = 35000 \text{ ways.}
\]

iii. Consider the same problem (i.e. part i) from the perspective of 3 friends who visit the same restaurant and would like to eat 3 different appetizers, 3 different main courses and 3 different desserts. In how many distinguishable ways can they have their meal?

Solution:
We are asked for the numbers of ways in which we can generate assignments of the friends \( \{F_1, F_2, F_3\} \) to their 3 appetizers \( \{A_1, A_2, A_3\} \), 3 main courses \( \{M_1, M_2, M_3\} \), and 3 desserts \( \{D_1, D_2, D_3\} \). Observe that here the same set of appetizers \( \{A_1, A_2, A_3\} \) can be assigned in \( 3! = 6 \) distinguishable ways to the friends \( \{F_1, F_2, F_3\} \).

Consider part i. of this problem again. We can construct all possible assignment of the friends to their appetizers, main courses, and desserts by

1. first generating a 3 appetizer-3 main course-3 dessert meal exactly as we did in part i.
   This can be done in 42000 ways,
2. having a particular meal we can go about assigning the dishes to the friends:
   (a) we can assign the appetizers to the friends in \( 3! = 6 \) ways
   (b) similarly, we can assign the main courses to the friends in \( 3! = 6 \) ways
   (c) finally, we can assign the three desserts to the friends in \( 3! = 6 \) ways

Hence there are
\[42000 \times 6 \times 6 \times 6 = 9072000\]
ways in which the friends could eat a meal consisting of 3 appetizers, 3 main courses, and 3 desserts.
Problem 3 [8 pts]: Symbols

In how many ways can you arrange the symbols

\[ \bullet \quad \blacksquare \quad \blacktriangle \quad \blacktriangledown \quad \blacklozenge \quad \blackdiamond \]

in a row under the restriction that the symbol \( \bullet \) cannot appear at the beginning or at the end of
the sequence.

Solution:

Consider the following three sets

\[ A = \text{the set of all arrangements of the above symbols (i.e., without restriction on the first and the last item)} \]

\[ B = \text{the set of all rearrangements which start with } \bullet \]

\[ C = \text{the set of all rearrangements which end with } \bullet \]

The question asks for the cardinality (i.e. the number of elements) of the set \( A \setminus (B \cup C) \). We have

\[ |A| = 6! = 720 \]

\[ |B| = 5! = 120 \text{ (the first element is fixed; we’re dealing with rearrangements of 5 objects in a row)} \]

\[ |C| = 5! = 120 \text{ (the last element is fixed; we’re dealing with rearrangements of 5 objects again)} \]

Note that each symbol can occupy only one position in a given arrangement. Hence there are no
arrangements in which the symbol \( \bullet \) appears at the beginning and at the end. That means that
\( B \cap C = \emptyset \).

Since \( B \) and \( C \) are disjoint, \( |B \cup C| = |B| + |C| = 5! + 5! = 240 \). Hence

\[ \# \text{ of asked arrangements} = |A \setminus (B \cup C)| = |A| - |B \cup C| = 720 - 240 = 480 \]

Problem 4 [28 pts (5, 5, 5, 8, 5)]: Probabilities

i. Suppose that you are choosing a 6-digit number at random ( numbers that are < 100,000 are considered 6-digit; the first digits are assumed to be 0s). What is the probability that the chosen number consists of distinct digits? (Note. the number 12345 is represented here as 012345 and consists of 6 distinct digits.)

Solution:

The sample space \( S \) is here the set of all 6 digits numbers (in the above sense). Let the event \( E \subset S \) consists of the numbers whose digits are distinct. We have

\[ |S| = 10^6 = 1000000 \]
because $S$ can be viewed as the set of sequences of length 6 whose elements belong to the 10-element set \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.

We also have
\[ |E| = \frac{10!}{4!} = 6! \binom{10}{6} = 151200 \]
because $E$ can be viewed as the set of sequences of length 6 whose elements are distinct and belong to a 10-element set \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}—that is, the set of 6-element permutations of a 10-element set.

Hence
\[ P(E) = \frac{151200}{1000000} = 15.12\% \]

**ii.** In a lottery, 6 balls are drawn from a pool of 49 balls painted with numbers 1 through 49. What is the probability that the numbers on the chosen balls are all squares of integers?

**Solution:**
The numbers
\[ 1, 4, 9, 16, 25, 36, 49 \]
are the only squares of integers among numbers 1, 2, ..., 49. Hence in order for all the drawn balls be numbered by squares, they have to be chosen from the above set (we’re identifying a ball with its number). Hence consider the following sets:

1. \[ S = \{\{x_1, x_2, x_3, x_4, x_5, x_6\} | x_i \text{ are distinct integers and } 1 \leq x_i \leq 49\} \]
   that is, the set of all choices of 6 balls from a 49 element set.

2. \[ E = \{\{x_1, x_2, x_3, x_4, x_5, x_6\} | x_i \text{ are distinct and } x_i \in \{1, 4, 9, 16, 25, 36, 49\}\} \]
   that is, the set of all 6 elements subsets of the set \{1, 4, 9, 16, 25, 36, 49\}. In probabilistic language this set is the event whose probability we are looking for.

Hence in order to solve the problem we have to find the number of elements in set $S$ and $E$. However, $S$ is the set of 6-element combinations of a 49-element set, and $E$ the set of 6-element combinations of a 7-element set. Hence
\[ P(E) = \frac{|E|}{|S|} = \binom{7}{6} \frac{\binom{49}{6}}{\binom{49}{6}} = \frac{7}{13983816} \approx 5 \times 10^{-7} \]

**iii.** What’s the probability that all the chosen numbers in **ii.** are composite (since 1 is not considered prime, we will view 1 as a composite number for the purpose of this question).

**Solution:**
We will need to know the number of composite positive integers which are \leq 49.

\[
\# \text{ composite integers } \leq 49 = 49 - (\# \text{ prime integers } \leq 49) = 49 - |\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47\}| = 49 - 15 = 34
\]
The sample space $S$ is the same as in part ii. The event $E$ is the set of all choices of 6 numbers from the above set of 34 numbers, i.e., the set of 6-element combination of a 34-element set. Thus

$$P(E) = \frac{|E|}{|S|} = \frac{\binom{34}{6}}{\binom{49}{6}} = \frac{29 \times 30 \times 31 \times 32 \times 33 \times 34}{44 \times 45 \times 46 \times 47 \times 48 \times 49} = \frac{15283}{158907} \approx 9.6\%$$

iv. Consider a set $S$ of 9x9 matrices of 0’s and 1’s in which is each row contains exactly one 1. What is the probability a randomly chosen matrix from $S$ has the property that there is exactly one 1 in each of its columns?

**Solution:**

Consider a particular matrix $M \in S$. Assign to it an ordered 9-tuple of numbers $(p_1, p_2, \ldots, p_9)$, where

$p_i$ is the position of the number 1 in i-th row

For example the diagonal matrix (the diagonal starting in the upper left corner) would be assigned the 9-tuple $(1, 2, 3, 4, 5, 6, 7, 8, 9)$. This assignment is one-to-one. Hence

1. the set $S$ corresponds to the set of all 9-tuples whose elements belong to the set $\{1, 2, 3, \ldots, 9\}$,
2. the set $E$ consisting of all matrices with one 1 in a column corresponds to the set of all 9-tuples in which all numbers are distinct.

We have

$|S| = 9^9$ because $|S|$ is the number of all sequences of length 9 with entries belonging to a 9-element set.

$|E| = 9!$ because $|E|$ is the number of 9-element permutations of a 9-element set.

All that gives us

$$P(E) = \frac{|E|}{|S|} = \frac{9!}{9^9} = \frac{4480}{4782969} \approx 0.0009 = 0.09\%$$

v. Suppose you tossed a coin 6 times. What is the probability that you obtained at least 1 tails?

**Solution:**

We can identify a sequence of 6 coin tosses with a a $\{H, T\}$ sequence of length 6. Hence the set $S$ of all such tossings has $|S| = 2^6 = 64$ elements. Instead of considering the event that there is at least one tails in a tossing sequence, consider the complementary event $\bar{E}$ that there are no tails in a sequence of 6 tosses. The event $\bar{E}$ consists of only one series of tosses, namely $(H, H, H, H, H, H)$. Thus

$$P(E) = 1 - P(\bar{E}) = 1 - \frac{|\bar{E}|}{|S|} = 1 - \frac{1}{2^6} = 63/64$$
Problem 5 [22 pts (5, 5, 2, 10)]: Events & Bayes Theorem

Suppose that you are throwing a die twice. Consider the following events

\[ A = \text{the event that you get an odd number in the first throw} \]
\[ B = \text{the event that you get an odd number in the second throw} \]

i. What is the event \( A \cap B \)? Calculate its probability.

Solution:

\( A \cap B \) is the event that in both rolls you obtain an odd number. The sample space \( S \) of all possible double throws consists of all pairs \( (a, b) \) where \( a, b \in \{1, 2, 3, 4, 5, 6\} \). Hence \( |S| = 6 \times 6 = 36 \) as \( a \) and \( b \) can take 6 values and the value taken by any of them does not influence the value of the other. The event \( |E| \) of getting odd numbers in both throws can identified with the set of all pairs \( (a, b) \) where this time \( a, b \in \{1, 3, 5\} \). Since \( a \) is does not influence \( b \), we have \( |E| = 3 \times 3 = 9 \). This gives

\[ P(A \cap B) = \frac{9}{36} = \frac{1}{4} \]

ii. Explain what is the event \( A \cup B \) and calculate its probability.

Solution:

\( A \cup B \) is the event that you obtained an odd number (i.e., 1, 3, 5) in the first or the second (or both) rolls. It’s probability can be calculated in several ways. Let’s use the formula

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]

which tells us that in order to answer the question we need to find \( P(A) \), \( P(B) \) and \( P(A \cap B) \). Naturally, \( P(A) = \frac{3}{6} = \frac{1}{2} \) and \( P(B) = \frac{3}{6} = \frac{1}{2} \). From part i we have \( P(A \cap B) = \frac{1}{4} \), so

\[ P(A \cup B) = \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4} \]

iii. Verify that \( P(A \cap B) = P(A)P(B) \) (i.e. that the events \( A \) and \( B \) are independent).

Solution:

In parts i. and ii. we have found that \( P(A) = \frac{1}{2} \) and \( P(A \cap B) = \frac{1}{4} \) Thus

\[ P(A \cap B) = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = P(A)P(B) \]

as required.

iv. In a certain town where all cars are either green or blue., there are 3 times more green cars than the blue ones. Studies indicate that at a particularly dimly lighted pedestrian crossing at night the probability that a person correctly identifies the color of a green car is 0.68. The probability that they correctly identify the color of a blue car is 0.45. You were just hit by a car at that crossing and believe that the car was green. What is the probability that you were actually hit by a green car?

Solution:

Denote
$I_G$ — the event that when a seeing a car at the crossing at night a person identifies it as green
$G$ — the event that the car seen at the crossing is green
$I_B$ — the event that when a seeing a car at the crossing at night a person identifies it as blue
$B$ — the event that the car seen at the crossing is blue

We can naturally assume that the cars pass through the crossing randomly. Hence

$$P(G) = \frac{3}{4}, P(B) = \frac{1}{4}$$

According to the problem we can write

$$0.68 = P(\text{the seen car was identified correctly}|G) = P(\text{the seen car was identified as green}|G) = P(I_G|G)$$

What we are asked for is $P(G|I_G)$. Bayes Theorem states that

$$P(G|I_G) = \frac{P(I_G|G)P(G)}{P(I_G)}$$

Hence in order to solve the problem we need to find $P(I_G)$ (the values of all other terms we already know). But


The only term we’re not given is

$$P(I_G|B) = P(\text{the probability that the car was identified as green given that it was blue})$$

We can find that probability as follows

$$P(I_G|B) =$$

$$= P(\text{the car was identified incorrectly}|B)$$

$$= 1 - P(\text{the car was identified correctly}|B)$$

$$= 1 - P(I_B|B)$$

$$= 1 - 0.45$$

$$= 0.55$$

We can now complete the calculations

$$P(I_G) = P(I_G|G)P(G) + P(I_G|B)P(B) = 0.68 \times 0.75 + 0.55 \times 0.25 = 0.6475$$

$$P(G|I_G) = \frac{P(I_G|G)P(G)}{P(I_G)} = \frac{0.68 \times 0.75}{0.6475} \approx 0.79$$