Written Homework 03 Solutions

Problem 1 [28 pts, (4,4,4,4,6,6)]**: Picture Arrangements**

The College of Computer and Information Science has 30 faculty members: 13 Full Professors, 11 Associate Professors, and 6 Assistant Profesors. Four of the professors are women and 26 are men. Among the professors are 1 Dean (full professor, male) and 2 Associate Deans (full professors, one male and one female). Photos of the faculty have been taken, and they will be displayed in a row along the entry hallway.

i. In how many ways can the photos be displayed so that all of the women are together?

Solution: $4! \cdot 26! \cdot 27 = 2.61332867 \times 10^{29}$.

There are 4! ways of arranging the women one behind the other. There are 26! ways of arranging the men. There are 27 places for inserting the block of 4 women among the 26 men.

ii. In how many ways can the photos be displayed so that all of the women are together and all of the men are together?

Solution: $4! \cdot 26! \cdot 2 = 1.93579901 \times 10^{28}$.

We can arrange the women in 4! ways, the men in 26! ways, and then arrange the block of men either ahead of or behind the block of women.

iii. In how many ways can the photos be displayed so that all of the women are together or all of the men are together (or both)?

Solution: Number of ways of arranging the professors such that all women are together is $4! \cdot 26! \cdot 27$.

Number of ways of arranging the professors such that all men are together is $4! \cdot 26! \cdot 5$.

Number of ways of arranging the professors such that all men are together *and* all the women are together is $4! \cdot 26! \cdot 2$.

By the principle of inclusion-exclusion, the total number of ways of arranging the professors so that all the men are together one behind the other, or all the women are together one behind the other (or both), is:

$$4! \cdot 26! \cdot 5 + 4! \cdot 26! \cdot 27 - 4! \cdot 26! \cdot 2 = 4! \cdot 26! \cdot (5 + 27 - 2) = 4! \cdot 26! \cdot 30 = 2.90369852 \times 10^{29}.$$

iv. In how many ways can the photos be displayed with the Dean appearing first, followed by the Associate Deans, the (remaining) Full Professors, the Associate Professors, and finally the Assistant Professors, in that order?

Solution: We can place the Dean in only 1 way, on the first position. We can arrange the two Associate Deans in 2! ways, the remaining $13 - 1 - 2 = 10$ Full Professors in 10! ways, the Associate Professors in 11! ways, and the Assistant Professors in 6! ways. The total number of ways of arranging the professors in that order is:

$$1 \cdot 2! \cdot 10! \cdot 11! \cdot 6! = 2.08584121 \times 10^{17}.$$

v. In how many ways can the photos be displayed if the Dean appears first, and the remaining photos are displayed in such a way that the Associate Deans are both together, the (remaining) Full Professors are all together, the Associate Professors are all together, and the Assistant Professors are all together?

We can think of each of the following as separate blocks, because their members are placed together: Associate Deans, remaining Full Professors, Associate Professors, Assistant Professors. There are 4 such blocks, therefore 4! ways of arranging them.

Using **iv**, the final result is:

$$4! \cdot (1 \cdot 2! \cdot 10! \cdot 11! \cdot 6!) = 5.0060189 \times 10^{18}.$$

vi. In how many ways can the photos be displayed so that no two women are adjacent?

Hint: Consider arranging the men's photos first and then placing the women's photos among the men's photos so as to ensure that no two women are adjacent.

Solution: The men can be arranged in 26! ways. Once the men are arranged, the four women have to be put in one of 27 places. So the first woman has 27 choices, the second has 26 choices, the third has 25 choices, and the last woman has 24 choices. The total number of arrangements is thus:

$$26! \cdot 27 \cdot 26 \cdot 25 \cdot 24 = 1.69866363 \times 10^{32}.$$

Problem 2 [16 pts, (3,5,8)]**: Exams**

Consider an exam consisting of 16 problems divided into two groups of 8 problems each. Students are required to solve 10 of the 16 problems.

i. In how many ways can a student choose 10 of the 16 problems?

<u>*Solution:*</u>

$$\binom{16}{10} = \frac{16!}{10! \cdot 6!} = 8008$$

ii. Suppose that students are required to solve 5 problems from the first group and 5 problems from the second group. In how many ways can a student choose the 10 problems to solve?

<u>*Solution:*</u>

$$\binom{8}{5} \times \binom{8}{5} = \frac{8!}{5! \cdot 3!} \times \frac{8!}{5! \cdot 3!} = 3136$$

iii. Now suppose that students are allowed to solve any 10 problems out of the 16 but no more than 6 from either group. In how many ways can a student choose the 10 problems to solve?

<u>*Solution:*</u>

They can solve 10 problems out of the 16 but no more than 6 from either group in one of the following ways: 6 problems from the first group and 4 from the second group, *or* 5 problems from the first group and 5 from the second group, *or* 4 problems from the first group and 6 from the second group:

$$\binom{8}{6} \times \binom{8}{4} + \binom{8}{5} \times \binom{8}{5} + \binom{8}{4} \times \binom{8}{6} = \frac{8!}{6! \cdot 2!} \times \frac{8!}{4! \cdot 4!} + \frac{8!}{5! \cdot 3!} \times \frac{8!}{5! \cdot 3!} + \frac{8!}{4! \cdot 4!} \times \frac{8!}{6! \cdot 2!} = 7056$$

Problem 3 [12 pts, (6 each)]**: Surveys**

A marketing firm has been hired to conduct a survey of automobile purchases in the greater Boston area. The survey consists of recording, for each household surveyed, the number of cars presently owned by that household among the following six major manufacturers: GM, Ford, Chrysler, Toyota, Honda, and Nissan. Thus, one household may have two GMs and a Ford, another household may have one Honda and one Nissan, etc.

i. Suppose that no household surveyed has more than four cars. In how many different ways can the survey sheet be filled out?

<u>*Solution:*</u> This is an instance of the "balls in bins" problem. Say that a household has n cars. Then there are $\binom{n+6-1}{n}$ ways that the n cars can be distributed among the six manufacturers. A household can have 0, 1, 2, 3 or 4 cars. Therefore, there are

$$\binom{5}{0} + \binom{6}{1} + \binom{7}{2} + \binom{8}{3} + \binom{9}{4} = 210$$

different ways the survey sheet can be filled out.

ii. There are approximately 1,200,000 households in the greater Boston area. Suppose that all households are surveyed, the resulting survey sheets are organized and tallied by type (e.g., two GMs and a Ford vs. one Honda and one Nissan, etc.), and the survey results are sorted by popularity (e.g., two GMs and a Ford is the most common, one Honda and one Nissan is the second most common, etc.). At a minimum, how large must be the count associated with the most popular survey result?

Solution: We have 1,200,000 surveys, and 210 possible survey results. According to the Pigeonhole Principle, the most popular result must have a count of at least $\lceil \frac{1{,}200{,}000}{210} \rceil = 5715$.

Problem 4 [18 pts, (3,3,6,6)]: Pseudorandom Number Generators

Pseudorandom number generators are programs that generate sequences of 0s and 1s that "look" random. These sequences have a wide range of applications, including in cryptography, cellular networks, and simulations of large physical systems. Many pseudorandom generators used in practice generate sequences with some specific properties, as discussed below.

Consider a pseudorandom number generator that generates 24-bit sequences; i.e., each sequence is of length 24 and consists of 0s and 1s. For each of the following questions, complete all the calculations and show all your work.

i. What is the total number of different 24-bit sequences?

Solution: Each bit in the 20-bit string has two choices – 0 or 1. By the product rule, the number of different 20-bit sequences is

$$2^{24}.$$

ii. One criterion that a sequence generator may desire is to have an equal number of 1s and 0s. What number of 24-bit sequences contain an equal number of 0s and 1s?

Solution: A 24-bit sequence has an equal number of 0s and 1s if it has exactly 12 0s. There are 24 positions in the sequence, and we need to choose 12 of these to be 0s. So the total number of 24-bit sequences that contain an equal number of 0s and 1s is

$$\binom{24}{12}.$$

iii. Another commonly-used criterion is limiting the length of *runs*. A run is a maximal contiguous sequence of identical bits. That is, a run of 0s is a contiguous sequence of 0s, which is preceded by a 1 or is at the start of the sequence, and succeeded by a 1 or at the end of the sequence. Similarly, a run of 1s is a contiguous sequence of 1s, which is preceded by a 0 or is at the start of the sequence, and succeeded by a 0 or is at the end of the sequence. For example, the sequence 0011111000011100000111 has a run of 0s of length 2, two runs of 0s of length 4, one run of 1s of length 5, and two runs of 1s of length 3.

How many 24-bit sequences contain a run of 0s of length 12 or a run of 1s of length 12 (or both)?

Solution: Let us first focus on a run of 0s of length 12, since counting runs of 1s will be similar. Note that there can be no more than one run of 0s of length 12. This run can be in one of 13 positions: the first 0 of the run can be in position 1 of the sequence, position 2 of the sequence, . . . , or position 13 of the sequence.

Consider the case where the run is at the very beginning of the sequence. Then, the sequence has the form:

0000000000001???????????

where the "?"'s represent bits that can be either a 0 or a 1. Note that we need to have a 1 in the 13th position to stop the run, since we are considering a run of length exactly 12. So we have 2^{11} such sequences.

Now, consider the case where the run starts at the second position. Then, the sequence has the form:

10000000000001??????????

In this case we need to have a 1 at either side of the run, since we are considering a run of length exactly 12. So we have 2^{10} such sequences.

Continuing this reasoning, all 13 cases look like this:

0000000000001???????????

10000000000001??????????
?10000000000001?????????
??10000000000001????????
???10000000000001???????
????10000000000001??????
?????10000000000001?????
??????10000000000001????
???????10000000000001???
????????10000000000001??
?????????10000000000001?
??????????10000000000001

???????????1000000000000

The first and last have 11 bits that can vary and the middle only have 10. So the total number of sequences with a run of 0s of length 12 is

$$2 \cdot 2^{11} + 11 \cdot 2^{10}$$

Now consider sequences with a run of 1s of length 12—similar reasoning shows that the number of these is exactly the same:

$$2 \cdot 2^{11} + 11 \cdot 2^{10}$$

We're computing the number of sequences with a run of 1s of length 12 *or* a run of 0s of length 12. The term "or" indicates a sum, but we need to be careful and subtract the number of sequences which have *both a run of 12 0s and a run of 12 1s*, since we've counted these twice. There are only two such sequences:

0000000000011111111111
1111111111100000000000

So the total number of sequences with a run of length 12 is

$$2 \cdot \left(2 \cdot 2^1 1 + 11 \cdot 2^1 0\right) - 2 = 30718$$

iv. How many 24-bit sequences contain at least one run of 1s of length 11 ?

<u>*Solution:*</u> This is similar to the previous problem, except we aren't interested in runs of 0s and there might be multiple runs of 1s of length 11 in the same string.

Which strings have one or more runs of 1s of length 11? If a run starts at the beginning then it needs to be terminated by a 0 on the right, if it starts in the middle then it needs to be terminated by 0s on both sides, and if it ends at the end then it needs to be terminated by a 0 on the left. That gives us the following 14 possibilities:

111111111110????????????

0111111111110???????????
?0111111111110??????????
??0111111111110?????????
???0111111111110????????
????0111111111110???????
?????0111111111110??????
??????0111111111110?????
???????0111111111110????
????????0111111111110???
?????????0111111111110??
??????????0111111111110?
???????????0111111111110

????????????011111111111

The first and last have 12 bits that can vary and the middle 12 only have 11, so the number of sequences we've counted so far is

$$2 \cdot 2^{12} + 12 \cdot 2^1 1$$

and this accounts for strings with multiple runs of 1s of length 11. But we're overcounting—each of the 3 strings with multiple runs matches exactly two of the patterns above.

$$11111111111011111111110$$
$$11111111111001111111111$$
$$01111111111101111111111$$

After correcting for this overlap, the number of sequences with at least one run of 1s of length 11 is

$$2 \cdot 2^{12} + 12 \cdot 2^1 1 - 3 = 32765$$

Show your work.

Problem 5 [26 pts, (6,3,6,3,4,4)]: The World Series

The New York Yankees are playing the Philadelphia Phillies in a seven game World Series that ends when one team has won four games. Suppose we record the outcome of a game with a **W** for a Yankees win and an **L** for a Yankees loss. Then, the outcome of a series is a sequence that ends as soon as we get exactly four **W**s or exactly four **L**s. Thus, for example, **WWWW**, **WLWLWLW**, **WWLLLL**, and **LLLWL** are all possible outcomes of the series, while **LLWLLL** and **WWWLWLW** are not possible outcomes of the series.

For each of these parts, show all your work.

i. How many possible outcomes of the series are there?

<u>*Solution:*</u> Begin by considering just the case where the Yankees win the series. We want to count all of the strings that represent four wins (**W**). Depending on how many times the Yankees lose, these strings might be of lengths 4, 5, 6, or 7. In each case, the string will have 4 **W**'s and the rest of the letters will be **L**. Also, each string must end in **W** to signal the final win and the end of the series, so we just need to count the number of ways to place 3 **W**'s in strings of length 3, 4, 5, or 6—arranging the **W**'s determines where we place the **L**'s, so no decisions remain. There are

$$\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \binom{6}{3} = 35$$

ways to do this, and thus 35 ways for the Yankees to win the series. If we simply take this argument and swap our **L**'s and **W**'s, then we see that there are also 35 ways for the Phillies to win the series. Thus, there are

$$2\left[\binom{3}{3} + \binom{4}{3} + \binom{5}{3} + \binom{6}{3}\right] = 70$$

possible outcomes in total.

Alternatively, we can simplify the problem by representing any outcome with a string of length 7. Again start with the case where the Yankees win, and if the series ends before game 7, extend the string that represents it with enough **L**'s to make it 7 characters long. Now we've represented any Yankees-winning series as a string of 4 **W**'s and 3 **L**'s, and our counting task is easier. To construct one of these strings, we can either choose where to put the 4 **W**'s or where to put the 3 **L**'s, so there are

$$\binom{7}{4} = 35 = \binom{7}{3}$$

ways to do this. Again repeating our logic to count the outcomes where the Phillies win, we see that there are

$$2\binom{7}{4} = 70 = 2\binom{7}{3}$$

possible outcomes in total.

ii. How many series would have to be played to be sure that some series outcome happens twice?

Solution: If we take each occurrence of an outcome to be a pigeon, and each type of outcome to be a hole, then the pigeonhole principle says that we need to play

$$70 + 1 = 71$$

series to be guaranteed that some outcome occurs twice.

iii. The first two and the last two games of the series are scheduled to be held in New York while the middle three are scheduled to be held in Philadelphia. How many possible outcomes are there in which New York wins the series yet loses at least two of its home games? (New York has already lost Game 1 at home.)

Solution: The Yankees win the series, yet lose two of its home games. So the series has to be of length either 6 or 7.

If the series is of length 6, then the Yankees lose two of its home games and wins all the rest. There are 3 home games played in New York in a 6 game series – the first, second, and the sixth. The Yankees have to win the sixth game. So, in this case, there is only 1 possible sequence: LLWWWW.

If the series is of length 7, then the Yankees lose two of its four home games and one of the three away games. The Yankees have to win the seventh game – last home game. So the Yankees lose game 1 and one of games 2, and 6, and one of games 3, 4, and 5. There are $\binom{2}{1}$ ways of selecting the former, and $\binom{3}{1}$ ways of selecting the latter, to yield a total of 6 ways, in this case.

There is also the case where the Yankees lose all three of games 1, 2, and 6 at home.

Adding the cases gives 8 as the total number of different ways for The Yankees to win the series, yet lose at least two of its home games.

The World Series uses a 2-3-2 home-away-home format, as described above. Let's denote this format HHAAAHH. Another popular playoff format is 2-2-1-1-1, i.e., HHAAHAH.

i. How many formats are there consisting of four home games and three away games?

Solution: This amounts to choosing when in the series the four home games will be played. There are $\binom{7}{4}$ ways to do that. Alternatively, you could choose when in the series the three away games are played. There are $\binom{7}{3}$. Both ways give you the same answer: 35.

ii. How many formats are there consisting of four home games and three away games where the series format must start and end with a home game? (This is a common constraint.)

Solution: Since the first and the last games are fixed, there are five slots left to fill. Similar to the previous problem, you can choose when in the series to place the three away games, which is $\binom{5}{4}$, or you could choose when to place the two remaining home games, which is $\binom{5}{2}$. Both ways give you 10.

iii. How many formats are there consisting of four home games and three away games where the four home games cannot be consecutive? (This is another common constraint.)

Solution: There are exactly four ways that the home games could be consecutive. HHHHAAA, AHHHHAA, AAHHHHA, and AAAHHHH. So we subtract four from our answer to part one and get 35-4=31.